Convection in Superposed Fluid and Porous Layers in the Presence of a Vertical Magnetic Field

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Abstract: - A linear stability analysis is applied to a system consisting of a horizontal fluid layer overlying a layer of porous medium saturated with the same fluid, with uniform heating from below in the presence of a vertical magnetic field. The flow in porous medium is assumed to be governed by Darcy’s law. The Beavers-Joseph condition is applied at the interface between the two layers. Numerical solutions are obtained using expansion of Chebyshev polynomials. This spectral method has a strong ability to solve multi-layered problems and allows us to obtain a highly accurate eigenvalues in a very efficient manner. Numerical results are obtained for different values of the parameter \( \hat{d} (= \text{depth of fluid layer/depth of porous layer}) \) and different values of the magnetic parameter Q. The effect of magnetic field is investigated and it is found that the linear stability curves for the onset of convection motion are bimodal even for higher values of \( \hat{d} \).

Key-Words: - Superposed porous and fluid layers- Thermal convection- Darcy's law- Magnetic field- Stationary convection, Chebyshev tau method.

1 Introduction
The onset of convection of a horizontal fluid layer superposed a porous layer when the system is heated from below has been considered, firstly, by Sun [1] who showed that the critical Rayleigh number in the porous layer decreases continuously as the thickness of the fluid layer is increased. He used a shooting method to obtain numerical solutions for the linear stability equations. Nield [2] formulated the problem with surface-tension effects and obtained asymptotic solutions for small wave numbers for a constant heat-flux boundary condition. Experimental observations of the onset of convective motion in bed of inductively heated particle when a liquid layer exists over the bed have been obtained by Rhee et al. [3]. Their observations showed that increasing the depth of the liquid layer over the bed tended to lower the critical internal Rayleigh number at which the onset of convection occurred. Sun [1] and Nield [2] used Darcy’s law in formulating equations of porous layer while Somerton and Catton [4] used the Brinkman’s extension of Darcy’s law to study the thermal instability of superposed porous and fluid layers when internal heating is allowed in the porous medium. He showed that large Darcy number leads to less stable fluid layer and that is due to the increased freedom for fluid motion in the porous layer allowed by the increase in permeability.

A numerical study has been produced by Poulikakos et al. [5] to illustrate the occurrence of convection in a fluid layer which floats on top of a fluid-saturated porous medium. Their study provide a view of the main features of the flow in the convective regime, i.e. at Rayleigh numbers larger than the critical value needed for the onset of convection.

Chen and Chen [6] produced a classical paper in which they studied thermal convection in a two-layer system composed of a porous layer saturated with fluid over which was a layer of the same fluid. The layer was heated from below and they considered the bottom of the porous layer, as well as the upper surface of the fluid, to be fixed. They employed the fundamental model developed by Nield [2] and they showed that the linear instability curves for the onset of convective motion, i.e. the Rayleigh number against wave number curves, may be bimodal in that the curves possess two local minima. They interpreted their finding by showing that for \( \hat{d} (= \text{depth of fluid layer/depth of porous layer}) \) small (\( \leq 0.13 \)) the instability was initiated in the porous medium, whereas for \( \hat{d} \) larger than this the mechanism changed and instability was controlled by the fluid layer.
These results were qualitatively and quantitatively verified by experimental work of Chen and Chen [7]. Chen and Chen [8] extended the work of Chen and Chen [6] by assuming that the motion of the fluid in porous layer is governed by Darcy’s equation with the Brinkman terms for viscous effects and the Forchheimer term for inertial effects. Numerical results were obtained using a combination of Galerkin and finite-difference method.


Thermal instability theory has been enlarged by the interest in hydrodynamic flows of electrically conducting fluids in the presence of magnetic field. The presence of such field in an electrically conducting fluid usually has the effect of inhibiting the development of instabilities. Thompson [21], Chandrasekhar [22], Aggarwal & Verma [23] and others have examined the effect of magnetic field for a layer of fluid. However in the literature, so far no research have been done to discuss the effect of a magnetic field on the thermal convection of a two-layer system consists of a horizontal fluid layer overlying a layer of porous medium saturated with the same fluid with uniform heating from below.

The implementation of Chebyshev tau method is highly useful in obtaining accurate eigenvalues for one and two layer problems. The application of a $D^2$ Chebyshev tau methods to a variety of subject areas is discussed by Abdullah & Lindsay [24-26], Abdullah [27], Al-Aidrous & Abdullah [28], Straughan [12-13], Carr, and Straughan [14], Carr [15] and others. Bukhari [9] used a first order Chebyshev tau method to obtain results for multi-layered continua and he showed that the first order Chebyshev tau method is more accurate than the $D^2$ Chebyshev tau method.

Convective heat transfer in porous media has been a subject of great interest for the last several decades. This interest was motivated by numerous thermal engineering applications in various disciplines. The enormous volume of work devoted to this field is well documented in the book of Nield and Bejan [29] and recent work in this field is considered by Cheng [30] and Riahi [31].

The problem of convection in a porous-fluid system has many industrial and geophysical applications. A thorough understanding of the physics of buoyancy-driven flow in fluids overlying porous media is essential when designing thermal installation systems, thermal energy storage systems and grain stores. There are numerous environmental and engineering circumstances in which fluid layers and fluid-saturated porous layers are heated together from below(e.g. horizontal layers of fibrous insulation, grain storage installations, post-accident cooling of nuclear reactors). In connection with the problem of post-accident cooling of nuclear reactors Somerton & Catton [4] and Rhee et al. [3] report the linear stability analysis of natural convection in a bed of heat generating particles cooled from above by a layer of fluid. The thermal circulation in lakes, shallow coastal waters and other reservoirs is influenced by the interaction between the body of water and the other water-saturated substrate.

The purposes of this paper are twofold: one is to study the effect of magnetic field on the thermal convection of a two-layer system consists of a horizontal fluid layer overlying a layer of porous medium saturated with the same fluid with uniform heating from below and the other is to derive a first order Chebyshev tau method to yield eigenvalues accurately for the two layer problem. This problem is similar to that discussed by Chen and Chen [6] but here the effect of magnetic field is included and the linear stability equations are solved numerically using first order Chebyshev tau method.

2 Mathematical Formulation
Consider two infinite horizontal layers $L_1$ and $L_2$ confined between two parallel horizontal boundaries such that the top of layer $L_2$ touches the bottom of layer $L_1$. The plane interface between the two layers is $x_3 = 0$, the upper boundary of $L_1$ is $x_3 = d_f$, and the lower boundary of $L_2$ is $x_3 = -d_m$, where the subscripts $f$ and $m$ denote fluid layer and porous medium layer respectively.

Suppose that the upper layer $L_1$ is filled by an incompressible thermally and electrically conducting viscous fluid and is governed by Navier-Stokes equations, whereas the lower layer $L_2$ is occupied by a porous medium permeated by the same fluid which is governed by Darcy’s law. Both layers are subjected to
a constant vertical magnetic field \( H \). Gravity \( g \) acts in the negative direction of \( x_1 \) and the porous medium is heated at its lower boundary (see Fig. 1).

\[
H
\]

\[
\begin{align*}
x_1 &= d_f, \\
x_1 &= 0, \\
x_1 &= -d_m
\end{align*}
\]

\( L_f \) \hspace{1cm} \text{ Fluid layer } \hspace{1cm} \text{ L_m } \hspace{1cm} \text{ Porous medium layer } \hspace{1cm} \text{ g}

Fig. 1: Schematic diagram of the problem.

The Boussinesq approximation has been applied to both layers, as the density is constant everywhere except in the body force term. Let \( T \) be the Kelvin temperature of the fluid, \( T_0 \) be a constant reference Kelvin temperature, \( \rho_f \) be the density of the fluid at \( T_0 \) and \( \alpha \) be the coefficient of volume expansion of the fluid (constant), then the fluid density \( \rho_f \) is proportionate to \( T \) such that

\[
\rho_f = \rho_0 [1 - \alpha (T - T_0)];
\]

(1)

The governing equations of the fluid layer are

\[
\begin{align*}
\rho_f \left( \frac{\partial V_f}{\partial t} + (V_f \cdot \nabla)V_f \right) &= -\nabla P_f + \mu \nabla^2 V_f + \rho_f g + \frac{\mu_f}{4\pi} (H_f \cdot \nabla)H_f, \\
\left( \rho_0 c_p \right) \left( \frac{\partial T_f}{\partial t} + (V_f \cdot \nabla)T_f \right) &= k_f \nabla^2 T_f,
\end{align*}
\]

(2)

\[
\begin{align*}
\frac{\partial H_f}{\partial t} &= (H_f \cdot \nabla) V_f - (V_f \cdot \nabla)H_f + \eta_f \nabla^2 H_f,
\end{align*}
\]

where \( V_f, H_f \) and \( p_f \) are velocity, magnetic field and pressure in fluid layer and \( \mu, \mu_f, k_f, \eta_f, c_p \) are kinematic viscosity, magnetic permeability, thermal conductivity, electrical resistivity and specific heat at constant pressure in fluid layer. The governing equations of the porous medium layer \( L_m \) are

\[
\begin{align*}
\rho_0 \frac{\partial V_m}{\partial t} &= -\nabla P_m - \frac{\mu_m}{K} V_m + \rho_f g + \frac{\mu_m}{4\pi} (H_m \cdot \nabla)H_m, \\
\left( \rho_0 c_p \right) \left( \frac{\partial T_m}{\partial t} + (V_m \cdot \nabla)T_m \right) &= k_m \nabla^2 T_m,
\end{align*}
\]

(3)

\[
\begin{align*}
\frac{\partial H_m}{\partial t} &= (H_m \cdot \nabla) V_m - (V_m \cdot \nabla)H_m + \eta_m \nabla^2 H_m,
\end{align*}
\]

where \( V_m, H_m \) and \( p_m \) are velocity, magnetic field and pressure in porous medium layer and \( K, \mu_m, \eta_m, \phi \) are permeability of porous medium, magnetic permeability, electrical resistivity, and porosity in the porous medium layer and where

\[
X' = \phi X_f + (1 - \phi) X_m
\]

where in (3) \( X \) is replaced by \( k \) or \( \rho_0 c_p \).

### 3 The Boundary Conditions

Suppose that \( x_3 = d_f \) and \( x_3 = -d_m \) are rigid and maintained at constant temperature \( T_1 \) and \( T_l \) respectively. In terms of \( w_f \) and \( w_m \), the axial velocity components of the fluid in \( L_f \) and \( L_m \), these requirements lead to the following conditions

\[
w_f(d_f) = 0, \quad \frac{\partial w_f}{\partial x_3}(d_f) = 0, \quad T_f(d_f) = T_1, \quad H_f(d_f) = 0, \quad (4)
\]

on the top boundary of \( L_f \) and the conditions

\[
w_m(-d_m) = 0, \quad T_m(-d_m) = T_l, \quad H_m(-d_m) = 0, \quad (5)
\]

on the lower boundary of \( L_m \) where \( H_f \) and \( H_m \) are the normal components of the magnetic field in \( L_f \) and \( L_m \) respectively and the subscripts \( u \) and \( l \) denote upper and lower boundaries respectively. The fluid/porous medium interface boundary conditions are based on the assumption that temperature, heat flux and normal fluid velocity are continuous. Thus at the interface plane \( x_1 = 0 \) we have

\[
\begin{align*}
w_f(0) &= w_m(0), \\
T_f(0) &= T_m(0), \quad (6)
\end{align*}
\]

\[
k_f \frac{\partial T_f}{\partial x_3}(0) = k_m \frac{\partial T_m}{\partial x_3}(0), \quad -P_f(0) + 2\mu_f \frac{\partial w_f}{\partial x_3}(0) = -P_m(0).
\]

These interfacial conditions mean respectively that the normal fluid velocity, temperature, heat flux, and normal component of the stress tensor are continuous. It is important to remark that when using the Darcy equation, the continuity of normal stress does not include the viscous contribution in the porous region. This leaves three final conditions to be specified on \( x_1 = 0 \), two of these are related to the magnetic field in \( L_f \) and \( L_m \) respectively which are
\[ \partial H_f(0) \quad \frac{\partial H_m(0)}{\partial x_3} = 0, \quad \frac{\partial H_m(0)}{\partial x_3} = 0 \]  

(7)

and the final condition is due to Beavers & Joseph [32] which has the form

\[ \frac{\partial u_f}{\partial x_3} = \frac{\alpha_m}{\sqrt{K}} (u_f - u_m), \quad \frac{\partial v_f}{\partial x_3} = \frac{\alpha_m}{\sqrt{K}} (v_f - v_m), \]  

(8)

where \( u_f, v_f, u_m \) and \( v_m \) are the limiting tangential components of the fluid velocity as the interface is approached from the layers \( L_1 \) and \( L_2 \) respectively, and \( \alpha_m \) is Beavers & Joseph constant.

The steady state solution, in whose stability we are interested, is one for which there is no fluid motion in either layer, the magnetic field is constant in the vertical direction in both layers and the temperatures at the upper and lower boundaries are held at fixed constant temperatures \( T_m \) and \( T_f \) respectively with \( T_f > T_m \). Equations (2) and (3) have a steady state solution of the form

\[ V_f = 0, \quad V_m = 0, \quad H_f = (0,0,H_f), \quad H_m = (0,0,H_m) \]

\[ T_f = T_0 - (T_0 - T_m) \frac{x_3}{d_f}, \quad 0 \leq x_3 \leq d_f, \]

\[ T_m = T_0 - (T_0 - T_m) \frac{x_3}{d_m}, \quad -d_m \leq x_3 \leq 0, \]

where \( T_0 \) is the temperature at the interface, which is found by requiring continuity of temperature and heat flux as

\[ T_0 = \frac{k_f d_m T_m + k_d d_d T_d}{k_f d_m + k_d d_d}. \]

Since the pressures are not required explicitly in the ensuing instability analysis, we do not give them.

### 4 The Perturbation Equations

Let the equilibrium solution be perturbed by the following linear perturbation quantities

\[ V_f = 0 + v_f, \quad V_m = 0 + v_m, \quad H_f = H_f e_3 + h_f, \quad T_f = T_0 - (T_0 - T_m) \frac{x_3}{d_f} + \theta_f, \]

\[ H_m = H_m e_3 + h_m, \quad T_m = T_0 - (T_0 - T_m) \frac{x_3}{d_m} + \theta_m, \]

and derive linearized equations for these quantities in the usual manner. To obtain the non-dimensional equations for the fluid layer \( L_1 \) we choose length to be \( d_f \), time to be \( d_f^2 / \lambda_f \), velocity to be \( \lambda_f / d_f \), pressure to be \( \mu \lambda_f / d_f \), temperature to be \( |T_0 - T_f| \) and magnetic field to be \( H_f / \eta_f \). For the porous layer \( L_2 \) we choose length to be \( d_m \), time to be \( d_m^2 / \lambda_m \), velocity to be \( \lambda_m / d_m \), pressure to be \( \mu \lambda_m / K \), temperature to be \( |T_1 - T_0| \) and magnetic field to be \( H_m / \eta_m \), where \( \lambda_f = k_f / (\rho c_f) \) is the thermal diffusivity of the fluid phase. Thus the linearized version of equations (2) and (3) can be written in the non-dimensional form

\[ \frac{\partial v_f}{\partial t_f} = P \left( -\nabla p_f + \nabla h_f + Ra_f \theta_f e_3 + Q_f \frac{\partial h_f}{\partial x_3} \right), \]

\[ \frac{\partial \theta_f}{\partial t_f} + \beta w_f = \nabla \theta_f, \]

\[ P_{a_f} \frac{\partial h_f}{\partial t_f} = \frac{\partial v_f}{\partial x_3} + \nabla h_f, \]

\[ \frac{Da}{\phi} \frac{\partial v_m}{\partial t_m} = P_m \left( -\nabla p_m - v_m + Ra_m \theta_m e_3 + Da Q_m \frac{\partial h_m}{\partial x_3} \right), \]

\[ G_m \frac{\partial \theta_m}{\partial t_m} - \beta w_m = \nabla \theta_m, \]

\[ P_{a_m} \frac{\partial h_m}{\partial t_m} = \frac{\partial v_m}{\partial x_3} + \nabla h_m, \]

where

\[ \beta = \frac{(T_f - T_m)}{(T_0 - T_f)} = \frac{(T_0 - T_m)}{(T_0 - T_f)}, \]

-1, when heating from above,

1, when heating from below.

Notice that \( P_f, P_m, Q_f \) and \( Ra_f \) are non-dimensional numbers which denote respectively, the viscous Prandtl number, magnetic Prandtl number, Chandrasekhar number and Rayleigh number of the fluid layer and \( P_{a_f}, P_{a_m}, Q_m \) and \( Ra_m \) are the corresponding non-dimensional numbers in the porous medium layer. These non-dimensional numbers are given by

\[ P_f = \frac{v}{\lambda_f}, \quad P_m = \frac{v}{\lambda_m}, \quad Q_f = \frac{\mu_f H_f^2 d_f^4}{4 \pi \rho_f v \eta_f}, \quad Ra_f = \frac{g \alpha d_f (T_0 - T_f)}{v \lambda_f}, \]

\[ P_m = \frac{v}{\lambda_m}, \quad P_m = \frac{v}{\lambda_m}, \quad Q_m = \frac{\mu_m H_m^2 d_m^4}{4 \pi \rho_m v \eta_m}, \quad Ra_m = \frac{g \alpha K d_m (T_1 - T_0)}{v \lambda_m}. \]
and \( Da = \frac{K}{d_m} \) is the Darcy number, 
\[ G_a = \left( \rho_a c_p \right) \frac{\partial P}{\partial x} \] and \( \lambda_m = k'/(\rho_a c_p) \) is the thermal diffusivity of the porous medium phase. We shall discuss the problem in the case of heating the porous medium layer from below, so we shall assume that \( \beta = 1 \). We now apply the normal mode expansion of the form 
\[ \psi(x,t) = \psi(x_1) \exp \left[ i (\epsilon x_1 + qx_1) + \sigma t \right]. \] (12)

For the functions \( w_f, \theta_f, h_f, w_m, \theta_m, h_m \). Thus the fluid and porous layers equations become
\[
\begin{align*}
\sigma_f \left( D_f^2 - a_f^2 \right) w_f - \sigma_f Q P_{a_f}^{-1} D_f h_f &= (D_f^2 - a_f^2) w_f, \\
\sigma_f \theta_f &= w_f + (D_f^2 - a_f^2) \theta_f, \\
\sigma_f P_{a_f}^{-1} h_f &= D_f w_f + (D_f^2 - a_f^2) h_f, \\
\end{align*}
\] (13)

\[
\begin{align*}
\frac{Da}{P_m \phi} \left( D_m^2 - a_m^2 \right) w_m + \sigma_m D_m Q P_{a_m}^{-1} D_m h_m &= \\
(D_m^2 - a_m^2) w_m + a_m^2 R_a \theta_m + Da Q_m D_m w_m, \\
G_m \sigma_m \theta_m &= w_m + (D_m^2 - a_m^2) \theta_m, \\
\sigma_m P_{a_m}^{-1} h_m &= D_m w_m + (D_m^2 - a_m^2) h_m,
\end{align*}
\] (14)

where \( a_f = \sqrt{q_f^2 + q_0^2} \) and \( a_m = \sqrt{q_m^2 + q_0^2} \) are non-dimensionalised wave numbers in the fluid layer and porous medium layer respectively, \( \sigma_f \) and \( \sigma_m \) are the growth rates and
\[
\begin{align*}
a_f &= \frac{d a_m}{dx_m}, \quad \sigma_f = \frac{d^2}{k} \sigma_m, \quad D_f = \frac{\partial}{\partial x_1} x_1 \in (0,1), \\
D_m &= \frac{\partial}{\partial x_3} x_3 \in (-1,0).
\end{align*}
\]

The boundary conditions in the final form are
\[ w_f = 0, \quad D_f w_f = 0, \quad \theta_f = 0, \quad h_f = 0, \quad \text{on } x_3 = 1, \] (15)

\[
\psi f(x) = \psi f(x_1) \exp \left[ i (\epsilon x_1 + qx_1) + \sigma t \right].
\]

We now discuss the problem in the case of heating the porous medium layer from below, so we shall assume that \( \beta = 1 \). We now apply the normal mode expansion of the form 
\[ \psi(x,t) = \psi(x_1) \exp \left[ i (\epsilon x_1 + qx_1) + \sigma t \right]. \] (12)

We notice that
\[
\begin{align*}
\hat{d} &= d_f / d_m, \quad \hat{k} = k_f / k_m, \quad \epsilon_f = \frac{T_f - T_m}{T_0 - T_m} = \frac{k_f}{\hat{k}}.
\end{align*}
\]

The complete eigenvalue system to be solved comprises (13)-(17). This system consists of an eighth order ODE in the fluid layer and a sixth order ODE in the porous medium layer, with 14 boundary conditions. The method of solution used to solve this system is the first order Chebyshev tau method. The implementation of this method is important since it is a highly accurate method and we are able to calculate as many eigenvalues as we need which is useful in cases when the eigenvalues of interest are changing in parameter space. We can also calculate the eigenfunctions easily.

We first map \( x_1 \in (0,1) \) and \( x_3 \in (-1,0) \) into the Chebyshev domain \((-1,1)\) by the transformations \( z = 2x_1 - 1 \) and \( z = 2x_3 + 1 \) respectively, then we suppose that
\[
y_r(z) = \sum_{i=0}^{14} \alpha_i T_i(z), \quad 1 \leq r \leq 14, \quad z \in (-1,1),
\]

to be the Chebyshev expansion of the variables \( y_r \), which are defined by.
\[ y_1 = w_f, \quad y_2 = D w_f, \quad y_3 = D^2 w_f, \quad y_4 = D^3 w_f, \]
\[ y_5 = \theta_f, \quad y_6 = D \theta_f, \quad y_7 = h_f, \quad y_8 = Dh_f, \]
\[ y_9 = w_m, \quad y_{10} = D w_m, \quad y_{11} = \theta_m, \quad y_{12} = D \theta_m, \]
\[ y_{13} = h_m, \quad y_{14} = Dh_m. \]

then equations (13) and (14) can be written as a system of 14 first order ordinary differential equations of the form \( \frac{dY}{dz} = Ey + \sigma FY \), to yield an eigenvalue problem of the form \( Ax = \sigma Bx \). This generalized matrix eigenvalue problem is solved using the QZ algorithm which we employed via the NAG routine F02BJF.

5 Results and Discussion
The numerical results obtained show that the onset of instability is in the form of stationary convection even in the presence of magnetic field. This explained the assumption of Chen and Chen [6] from the beginning that the principle of exchange of instabilities holds.

Numerical results and stability curves are obtained for the present problem with thermal conductivity ratio \( \hat{k} = 0.7 \), \( Da = 4 \times 10^4 \), \( \alpha_{zy} = 0.1 \), \( \phi = 1 \), \( P_m = 1 \), \( P_n = 1 \), \( G_m = 1 \), and for variety of reciprocal depth ratio \( \hat{d} \) ranging from 0.001 to 1 and for various values of \( Q_m \), ranging from 0 to \( 10^5 \).

Fig.2 shows the relation between \( Q_m \) and the critical Rayleigh number in the porous medium layer, \( Ra_m \), for different values of \( \hat{d} \) (see Table 1.a and Table 1.b). It is clear from the figure that \( Ra_m \) increases continuously as \( Q_m \) increases which means that the magnetic field has a stabilizing effect in this system. Moreover, \( Ra_m \) decreases continuously as the thickness of the fluid layer increases and this result is in agreement with the result obtained by Sun [1] in the absence of magnetic field.

The marginal stability curves at a number of depth ratios are shown in Fig. 3 and Fig. 4 for \( Q_m = 0 \) and \( Q_m = 50000 \) respectively. The results in Fig.3, \( Q_m = 0 \), are consistent with those of Chen and Chen [6] in which he showed that the marginal stability curves are bimodal, exhibiting two relative minima when \( \hat{d} \) is less than a certain critical value. In the presence of magnetic field, as in the present problem, the marginal stability curves are also bimodal but for large depth ratios. Fig.4 shows that when \( \hat{d} \geq 0.33 \) and \( Q_m = 50000 \) the marginal stability curves are bimodal.

In fact \( Q_m \) must exceeds a certain critical value in order to have two minimum values for \( Ra_m \). As an example, this value has been calculated numerically when \( \hat{d} = 0.33 \) and we found that it should be greater than 7000.

Moreover in Fig.3, \( Q_m = 0 \), for \( \hat{d} \leq 0.1 \) the long wave branch is the most unstable whereas for \( \hat{d} > 0.1 \) the short wave branch is the most unstable. In the presence of magnetic field as in Fig.4, \( Q_m = 50000 \) for \( \hat{d} \leq 0.33 \) the short wave branch is the most unstable.

Figure (5) shows the relation between \( \hat{d} \) and critical wave number of porous medium layer \( a_m \) for different values of \( Q_m \). It is clear from this figure that as \( \hat{d} \) increases there is a rapid change of the critical wave number \( a_m \).

Figure (6) shows the relation between \( \hat{d} \) and critical \( Ra_m \) for different values of \( Q_m \). It is clear from this figure that there is a precipitation drop of the values of the critical \( Ra_m \) as \( \hat{d} \) increases. This precipitation decreases as \( Q_m \) increases. The numerical results related to this figure is listed in table (2).

The critical Rayleigh number in the fluid layer, \( Ra_f \), can be calculated from equation (18) and the relations between \( \hat{d} \) and the critical thermal Rayleigh numbers in the porous and fluid layers when \( Q_m = 0, 50000 \) are displayed in Fig. 7. As \( \hat{d} \) increases \( Ra_f \) increases and it quickly exceeds the critical value and convection ensues. In fact when \( \hat{d} \) is large the problem reduces to a horizontal layer of fluid heated from below and the critical Rayleigh number of \( Ra_f \) approaches the value 1100 which is the value of the critical Rayleigh number with one rigid and one free boundary in the classical Benard problem. Here the porous interface behaves more like a free boundary than a rigid one. Similarly in the presence of magnetic field, when \( \hat{d} \) is large the problem reduces to a horizontal layer of fluid heated from below and subject to a vertical magnetic field and the critical Rayleigh number \( Ra_{f} \) approaches the value of \( Q_m \). This figure shows also the precipitous drop of the critical Rayleigh number \( Ra_{m} \) with increasing \( \hat{d} \). The
numerical results related to this figure is listed in table (3).

The precipitous drop of the critical Rayleigh number $Ra_m$ with increasing $\dot{d}$ and the rapid change of the critical wave number $a_m$ are shown in Fig. 8 when $Q_m = 0.50000$.

The marginal stability curves for different values of $a_{bj}$ when $\dot{d} = 0.04, 0.13, 0.5$ are shown in Fig. 9-11, respectively when $Q_m = 0$. It appears from the figures that $Ra_m$ is sensitive to $a_{bj}$ for small values of $\dot{d}$ ($\dot{d} \leq 0.13$) when $a_m$ exceeds a certain value, and this value of $a_m$ is decreased when $a_{bj}$ is increased. Generally $Ra_m$ increases by increasing $a_{bj}$. Moreover if $\dot{d} > 0.13$, $Ra_m$ is no longer sensitive to $a_{bj}$.

In the presence of magnetic field, the marginal stability curves for different values of $a_{bj}$ when $\dot{d} = 0.04, 0.13, 0.5$ are shown in Fig. 12-14 respectively when $Q_m = 50000$. It appears from the figures that $Ra_m$ is sensitive to $a_{bj}$, and that the effect of $a_{bj}$ is different when magnetic field is applied. Here $Ra_m$ increases as $a_{bj}$ increases for small values of $a_m$ but when $a_m$ exceeds a certain value $Ra_m$ decreases as $a_{bj}$ increases.

The problem is also solved when the upper surface is free. The conditions on the upper layer is replaced by the following conditions

$$w_y = 0, \quad D_y w_y = 0, \quad \theta_y = 0, \quad D_y h_y = 0, \quad \text{on} \quad x_3 = 1 .$$

Numerical results are obtained for the same values of the parameters suggested in the case when the upper boundary is rigid. The effects of the parameters in this case are similar to the case when the upper layer is rigid but the numerical results are quantitatively less.

### 6 Conclusion

In this paper we have implemented a detailed linear stability analysis in a superposed fluid and porous layers in the presence of a vertical magnetic field. The effect of magnetic field has been investigated in detail. A wide range of parameters have been used to allow for the possibility of overstability and the numerical results showed that instability occurs only via stationary convection.

We have shown that even in the presence of magnetic field the onset of convection had a bimodal nature in which convection may be dominated by the porous medium or the fluid layer depending on the value of the depth ratio $\hat{d}$.

The first order Chebyshev tau method is used to obtain the numerical solutions of the problem. This method is important for this class of problems since it is highly accurate. Several numerical results are obtained relating the critical Rayleigh numbers of fluid and porous layers, the wave number, the depth ratio, the magnetic parameter and the Beavers and Joseph constant.

#### Table 1.a: The relation between $Q_m$ and critical $Ra_m$.

<table>
<thead>
<tr>
<th>$Q_m$</th>
<th>Critical $Ra_m$ at $\dot{d}$ =</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>0</td>
<td>39.418</td>
</tr>
<tr>
<td>100</td>
<td>39.426</td>
</tr>
<tr>
<td>10000</td>
<td>40.203</td>
</tr>
<tr>
<td>100000</td>
<td>46.976</td>
</tr>
</tbody>
</table>

#### Table 1.b: The relation between $Q_m$ and critical $Ra_m$.

<table>
<thead>
<tr>
<th>$Q_m$</th>
<th>Critical $Ra_m$ at $\dot{d}$ =</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.12</td>
<td>0.13</td>
</tr>
<tr>
<td>0</td>
<td>10.572</td>
</tr>
<tr>
<td>100</td>
<td>10.876</td>
</tr>
<tr>
<td>500</td>
<td>12.058</td>
</tr>
<tr>
<td>1000</td>
<td>13.477</td>
</tr>
<tr>
<td>5000</td>
<td>18.664</td>
</tr>
<tr>
<td>10000</td>
<td>19.117</td>
</tr>
<tr>
<td>100000</td>
<td>23.470</td>
</tr>
</tbody>
</table>

#### Table 2. The relation between $\dot{d}$ and critical $Ra_m$.

<table>
<thead>
<tr>
<th>$\dot{d}$</th>
<th>Critical $Ra_m$ at $Q_m$ =</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>36.754</td>
</tr>
<tr>
<td>0.04</td>
<td>24.822</td>
</tr>
<tr>
<td>0.1</td>
<td>19.121</td>
</tr>
<tr>
<td>0.12</td>
<td>10.572</td>
</tr>
<tr>
<td>0.13</td>
<td>7.782</td>
</tr>
<tr>
<td>0.33</td>
<td>0.214</td>
</tr>
<tr>
<td>0.5</td>
<td>0.043</td>
</tr>
<tr>
<td>1</td>
<td>0.003</td>
</tr>
</tbody>
</table>
Table 3. The relation between \( \dot{d} \) and critical \( Ra_f \).

<table>
<thead>
<tr>
<th>( \dot{d} )</th>
<th>critical ( Ra_f ) at ( Q_m = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.188</td>
</tr>
<tr>
<td>0.04</td>
<td>32.420</td>
</tr>
<tr>
<td>0.1</td>
<td>975.578</td>
</tr>
<tr>
<td>0.11</td>
<td>1101.306</td>
</tr>
<tr>
<td>0.12</td>
<td>1118.478</td>
</tr>
<tr>
<td>0.13</td>
<td>1133.998</td>
</tr>
<tr>
<td>0.33</td>
<td>1296.833</td>
</tr>
<tr>
<td>0.5</td>
<td>1355.766</td>
</tr>
<tr>
<td>1</td>
<td>1430.448</td>
</tr>
</tbody>
</table>

Fig. 2: The relation between \( Q_m \) and \( Ra_m \).

Fig. 3: The relation between \( a_m \) and \( Ra_m \) \( (Q_m = 0) \).
Fig. 7: The relation between \( \hat{d}, R_{a_m} \) and \( R_{a_f} \).

Fig. 8: Variation of \( R_{a_m} \) and \( a_n \) with \( \hat{d} \).

Fig. 9: The relation between wave number \( a_n \) and Rayleigh number in the porous layer \( R_{a_m} \) for different values of \( \alpha_B \), when \( \hat{d} = 0.04 \) and \( Q_m = 0 \).

Fig. 10: The relation between wave number \( a_n \) and Rayleigh number in the porous layer \( R_{a_m} \) for different values of \( \alpha_B \), when \( \hat{d} = 0.13 \) and \( Q_m = 0 \).

Fig. 11: The relation between wave number \( a_n \) and Rayleigh number in the porous layer \( R_{a_m} \) for different values of \( \alpha_B \), when \( \hat{d} = 0.5 \) and \( Q_m = 0 \).

Fig. 12: The relation between wave number \( a_n \) and Rayleigh number in the porous layer \( R_{a_m} \) for different values of \( \alpha_B \), when \( \hat{d} = 0.04 \) and \( Q_m = 50000 \).
Fig. 13: The relation between wave number $a_m$ and Rayleigh number in the porous layer $Ra_m$ for different values of $\alpha_{RJ}$ when $d = 0.13$ and $Q_m = 50000$.

Fig. 14: The relation between wave number $a_m$ and Rayleigh number in the porous layer $Ra_m$ for different values of $\alpha_{RJ}$ when $d = 0.5$ and $Q_m = 50000$.

References


