Convection in Superposed Fluid and Porous Layers in the Presence of a Vertical Magnetic Field

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Abstract:- A linear stability analysis is applied to a system consisting of a horizontal fluid layer overlying a layer of porous medium saturated with the same fluid, with uniform heating from below in the presence of a vertical magnetic field. The flow in porous medium is assumed to be governed by Darcy's law. The Beavers-Joseph condition is applied at the interface between the two layers. Numerical solutions are obtained using expansion of Chebyshev polynomials. This spectral method has a strong ability to solve multi-layered problems and allows us to obtain a highly accurate eigenvalues in a very efficient manner. Numerical results are obtained for different values of the parameter $\hat{d}$ (= depth of fluid layer/depth of porous layer) and different values of the magnetic parameter Q. The effect of magnetic field is investigated and it is found that the linear stability curves for the onset of convection motion are bimodal even for higher values of $\hat{d}$.

Key-Words: -Superposed porous and fluid layers- Thermal convection- Darcy's law- Magnetic field- Stationary convection, Chebyshev tau method.

1 Introduction

The onset of convection of a horizontal fluid layer superposed a porous layer when the system is heated from below has been considered, firstly, by Sun [1] who showed that the critical Rayleigh number in the porous layer decreases continuously as the thickness of the fluid layer is increased. He used a shooting method to obtain numerical solutions for the linear stability equations. Nield [2] formulated the problem with surface-tension effects and obtained asymptotic solutions for small wave numbers for a constant heat-flux boundary condition. Experimental observations of the onset of convective motion in bed of inductively heated particle when a liquid layer exists over the bed have been obtained by Rhee et al. [3]. Their observations showed that increasing the depth of the liquid layer over the bed tended to lower the critical internal Rayleigh number at which the onset of convection occurred. Sun [1] and Nield [2] used Darcy's law in formulating equations of porous layer while Somerton and Catton [4] used the Brinkman's extension of Darcy's law to study the thermal instability of superposed porous and fluid layers when internal heating is allowed in the porous medium. He showed that large Darcy number leads to less stable fluid layer and that is due to the increased freedom for fluid motion in the porous layer allowed by the increase in permeability.

A numerical study has been produced by Poulikakos et al. [5] to illustrate the occurrence of convection in a fluid layer which floats on top of a fluid-saturated porous medium. Their study provide a view of the main features of the flow in the convective regime, i.e. at Rayleigh numbers larger than the critical value needed for the onset of convection.

Chen and Chen [6] produced a classical paper in which they studied thermal convection in a two-layer system composed of a porous layer saturated with fluid over which was a layer of the same fluid. The layer was heated from below and they considered the bottom of the porous layer, as well as the upper surface of the fluid, to be fixed. They employed the fundamental model developed by Nield [2] and they showed that the linear instability curves for the onset of convective motion, i.e. the Rayleigh number against wave number curves, may be bimodal in that the curves possess two local minima. They interpreted their finding by showing that for $\hat{d}$ (= depth of fluid layer/depth of porous layer) small (≤ 0.13) the instability was initiated in the porous medium, whereas for $\hat{d}$ larger than this the mechanism changed and instability was controlled by the fluid layer.

These results were qualitatively and quantitatively verified by experimental work of Chen and Chen [7]. Chen and Chen [8] extended the work of Chen and Chen [6] by assuming that the motion of the fluid in porous layer is governed by Darcy's equation with the Brinkman terms for viscous effects and the Forchheimer term for inertial effects. Numerical results were obtained using a combination of Galerkin and finite-difference method.

The problem of Chen and Chen [6] has been solved numerically by Bukhari [9] using first and second order Chebyshev tau method. McKay [10] studied a similar porous-fluid layer problem to Chen and Chen [6] but he allowed chemical reactions in the layers. Valencia-Lopez and Ochoa-Tapia [11] presented a comparison of two models, Darcy and Brinkman, to study the buoyancy-driven-flow in a confined fluid overlying a porous layer. Straughan [12,13] used a D^2 Chebyshev tau method to obtain results for the surface-tension-driven convection and for the effect of property and modeling respectively in a fluid overlying a porous layer. Further references to work on the porous-fluid convection problem can be found in [14-20].

Thermal instability theory has been enlarged by the interest in hydrodynamic flows of electrically conducting fluids in the presence of magnetic field. The presence of such field in an electrically conducting fluid usually has the effect of inhibiting the development of instabilities. Thompson [21], Chandrasekhar [22], Aggarwal & Verma [23] and others have examined the effect of magnetic field for a layer of fluid. However in the literature, so far no research have been done to discuss the effect of a magnetic field on the thermal convection of a two-layer system consists of a horizontal fluid layer overlying a layer of porous medium saturated with the same fluid with uniform heating from below.

The implementation of Chebyshev tau method is highly useful in obtaining accurate eigenvalues for one and two layer problems. The application of a D^2 Chebyshev tau methods to a variety of subject areas is discussed by Abdullah & Lindsay [24-26], Abdullah [27], Al-Aidrous & Abdullah [28], Straughan [12-13], Carr, and Straughan [14], Carr [15] and others. Bukhari [9] used a first order Chebyshev tau method to obtain results for multi-layered continua and he showed that the first order Chebyshev tau method is more accurate than the D^2 Chebyshev tau method.

Convective heat transfer in porous media has been a subject of great interest for the last several decades. This interest was motivated by numerous thermal engineering applications in various disciplines. The enormous volume of work devoted to this field is well documented in the book of Nield and Bejan [29] and recent work in this field is considered by Cheng [30] and Riahi [31].

The problem of convection in a porous-fluid system has many industrial and geophysical applications. A thorough understanding of the physics of buoyancy-driven flow in fluids overlying porous media is essential when designing thermal installation systems, thermal energy storage systems and grain stores. There are numerous environmental and engineering circumstances in which fluid layers and fluid-saturated porous layers are heated together from below(e.g. horizontal layers of fibrous insulation, grain storage installations, post-accident cooling of nuclear reactors). In connection with the problem of post-accident cooling of nuclear reactors Somerton & Catton [4] and Rhee et al. [3] report the linear stability analysis of natural convection in a bed of heat generating particles cooled from above by a layer of fluid. The thermal circulation in lakes, shallow coastal waters and other reservoirs is influenced by the interaction between the body of water and the other water-saturated substrate.

The purposes of this paper are twofold: one is to study the effect of magnetic field on the thermal convection of a two-layer system consists of a horizontal fluid layer overlying a layer of porous medium saturated with the same fluid with uniform heating from below and the other is to derive a first order Chebyshev tau method to yield eigenvalues accurately for the two layer problem. This problem is similar to that discussed by Chen and Chen [6] but here the effect of magnetic field is included and the linear stability equations are solved numerically using first order Chebyshev tau method.

2 Mathematical Formulation

Consider two infinite horizontal layers L_1 and L_2 confined between two parallel horizontal boundaries such that the top of layer L_2 touches the bottom of layer L_1. The plane interface between the two layers is $x_3 = 0$, the upper boundary of L_1 is $x_3 = d_f$ and the lower boundary of L_2 is $x_3 = -d_m$ where the subscripts f and m denote fluid layer and porous medium layer respectively.

Suppose that the upper layer L_1 is filled by an incompressible thermally and electrically conducting viscous fluid and is governed by Navier-Stokes equations, whereas the lower layer L_2 is occupied by a porous medium permeated by the same fluid which is governed by Darcy's law. Both layers are subjected to a constant vertical magnetic field H. Gravity $\boldsymbol{g}$ acts in the negative direction of x_3 and the porous medium is heated at its lower boundary (see Fig. 1).

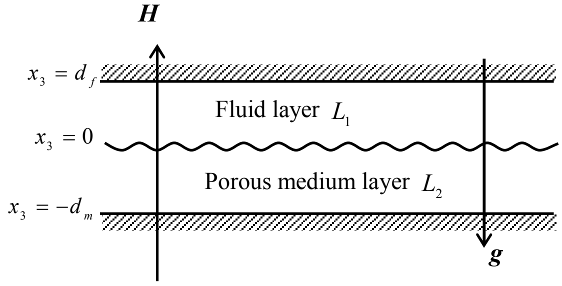

Fig. 1: Schematic diagram of the problem.

The Boussinesq approximation has been applied to both layers, as the density is constant everywhere except in the body force term. Let T be the Kelvin temperature of the fluid, T_0 be a constant reference Kelvin temperature, ρ_0 be the density of the fluid at T_0 and α be the coefficient of volume expansion of the fluid (constant), then the fluid density ρ_f is proportionate to T such that

$$\rho_f = \rho_0[1-\alpha(T-T_0)]. \tag{1}$$

The governing equations of the fluid layer are

$$\begin{aligned}\rho_0\left(\frac{\partial \boldsymbol{V}_f}{\partial t}+(\boldsymbol{V}_f\cdot\nabla)\boldsymbol{V}_f\right) &= -\nabla P_f+\mu\nabla^2\boldsymbol{V}_f+\rho_f\boldsymbol{g} \\ &\quad +\frac{\mu_f}{4\pi}(\boldsymbol{H}_f\cdot\nabla)\boldsymbol{H}_f, \\ (\rho_0 c_p)_f\left(\frac{\partial T_f}{\partial t}+\boldsymbol{V}_f\cdot\nabla T_f\right) &= k_f\nabla^2 T_f, \\ \frac{\partial \boldsymbol{H}_f}{\partial t} &= (\boldsymbol{H}_f\cdot\nabla)\,\boldsymbol{V}_f-(\boldsymbol{V}_f\cdot\nabla)\boldsymbol{H}_f+\eta_f\nabla^2 H_f,\end{aligned} \tag{2}$$

where $\boldsymbol{V}_f$, $\boldsymbol{H}_f$ and P_f are velocity, magnetic field and pressure in fluid layer and μ, μ_f, k_f, η_f, c_p are kinematic viscosity, magnetic permeability, thermal conductivity, electrical resistivity and specific heat at constant pressure in fluid layer. The governing equations of the porous medium layer L_2 are

$$\begin{aligned}\frac{\rho_0}{\phi}\frac{\partial \boldsymbol{V}_m}{\partial t} &= -\nabla P_m-\frac{\mu}{K}\boldsymbol{V}_m+\rho_f\boldsymbol{g}+\frac{\mu_m}{4\pi}(\boldsymbol{H}_m\cdot\nabla)\boldsymbol{H}_m, \\ (\rho_0 c_p)^*\frac{\partial T_m}{\partial t}+(\rho_0 c_p)_f\boldsymbol{V}_m\cdot\nabla T_m &= k^*\nabla^2 T_m, \\ \frac{\partial \boldsymbol{H}_m}{\partial t} &= (\boldsymbol{H}_m\cdot\nabla)\,\boldsymbol{V}_m-(\boldsymbol{V}_m\cdot\nabla)\boldsymbol{H}_m+\eta_m\nabla^2 H_m,\end{aligned} \tag{3}$$

where $\boldsymbol{V}_m$, $\boldsymbol{H}_m$ and P_m are velocity, magnetic field and pressure in porous medium layer and K, μ_m, η_m, ϕ are permeability of porous medium, magnetic permeability, electrical resistivity, and porosity in the porous medium layer and where

$$X^* = \phi\, X_f + (1-\phi)\, X_m$$

where in (3) X is replaced by k or $\rho_0 c_p$.

3 The Boundary Conditions

Suppose that $x_3 = d_f$ and $x_3 = -d_m$ are rigid and maintained at constant temperature T_u, and T_l respectively. In terms of w_f and w_m, the axial velocity components of the fluid in L_1 and L_2, these requirements lead to the following conditions

$$w_f(d_f)=0,\quad \frac{\partial w_f(d_f)}{\partial x_3}=0,\quad T_f(d_f)=T_u,\quad H_f(d_f)=0, \tag{4}$$

on the top boundary of L_1 and the conditions

$$w_m(-d_m)=0,\qquad T_m(-d_m)=T_l,\qquad H_m(-d_m)=0, \tag{5}$$

on the lower boundary of L_2 where H_f and H_m are the normal components of the magnetic field in L_1 and L_2 respectively and the subscripts u and l denote upper and lower boundaries respectively. The fluid/porous medium interface boundary conditions are based on the assumption that temperature, heat flux and normal fluid velocity are continuous. Thus at the interface plane $x_3 = 0$ we have

$$\begin{aligned} &w_f(0)=w_m(0), \qquad\qquad T_f(0)=T_m(0), \\ &k_f\frac{\partial T_f(0)}{\partial x_3}=k^*\frac{\partial T_m(0)}{\partial x_3},\quad -P_f(0)+2\mu\frac{\partial w_f(0)}{\partial x_3}=-P_m(0).\end{aligned} \tag{6}$$

These interfacial conditions mean respectively that the normal fluid velocity, temperature, heat flux, and normal component of the stress tensor are continuous. It is important to remark that when using the Darcy equation, the continuity of normal stress does not include the viscous contribution in the porous region. This leaves three final conditions to be specified on $x_3 = 0$, two of these are related to the magnetic field in L_1 and L_2 respectively which are

$$\frac{\partial H_f(0)}{\partial x_3}=0, \quad \frac{\partial H_m(0)}{\partial x_3}=0 \tag{7}$$

and the final condition is due to Beavers & Joseph [32] which has the form

$$\frac{\partial u_f}{\partial x_3}=\frac{\alpha_{BJ}}{\sqrt{K}}\left(u_f-u_m\right), \qquad \frac{\partial v_f}{\partial x_3}=\frac{\alpha_{BJ}}{\sqrt{K}}\left(v_f-v_m\right), \tag{8}$$

where u_f, v_f, u_m and v_m are the limiting tangential components of the fluid velocity as the interface is approached from the layers L_1 and L_2 respectively, and α_{BJ} is Beavers & Joseph constant.

The steady state solution, in whose stability we are interested, is one for which there is no fluid motion in either layer, the magnetic field is constant in the vertical direction in both layers and the temperatures at the upper and lower boundaries are held at fixed constant temperatures T_u and T_l respectively with $T_l > T_u$. Equations (2) and (3) have a steady state solution of the form

$$\begin{aligned}
&\boldsymbol{V_f}=\boldsymbol{0}, \quad \boldsymbol{V_m}=\boldsymbol{0}, \quad \boldsymbol{H_f}=(0,0,H_f), \quad \boldsymbol{H_m}=(0,0,H_m)\\
&T_f=T_0-(T_0-T_u)\frac{x_3}{d_f}, \qquad 0\le x_3\le d_f,\\
&T_m=T_0-(T_l-T_0)\frac{x_3}{d_m}, \qquad -d_m\le x_3\le 0,
\end{aligned} \tag{9}$$

where T_0 is the temperature at the interface, which is found by requiring continuity of temperature and heat flux as

$$T_0=\frac{k_f d_m T_u + k^* d_f T_l}{k_f d_m + k^* d_f}.$$

Since the pressures are not required explicitly in the ensuing instability analysis, we do not give them.

4 The Perturbation Equations

Let the equilibrium solution be perturbed by the following linear perturbation quantities

$$\boldsymbol{V}_f=\boldsymbol{0}+\boldsymbol{v}_f, \ P_f=P_f(x_3)+p_f, \ T_f=T_0-(T_0-T_u)\frac{x_3}{d_f}+\theta_f,$$

$$\boldsymbol{H}_f=H_f\boldsymbol{e}_3+\boldsymbol{h}_f, \ \boldsymbol{V}_m=\boldsymbol{0}+\boldsymbol{v}_m, \ P_m=P_m(x_3)+p_m,$$

$$T_m=T_0-(T_l-T_0)\frac{x_3}{d_m}+\theta_m, \ \boldsymbol{H}_m=H_m\boldsymbol{e}_3+\boldsymbol{h}_m,$$

and derive linearized equations for these quantities in the usual manner. To obtain the non-dimensional equations for the fluid layer L_1 we choose length to be d_f, time to be $d^2{}_f/\lambda_f$, velocity to be λ_f/d_f, pressure to be $\mu\lambda_f/d^2{}_f$, temperature to be $|T_0-T_u|$ and magnetic field to be $H_f\lambda_f/\eta_f$. For the porous layer L_2 we choose length to be d_m, time to be $d^2{}_m/\lambda_m$, velocity to be λ_m/d_m, pressure to be $\mu\lambda_m/K$, temperature to be $|T_l-T_0|$ and magnetic field to be $H_m\lambda_m/\eta_m$ where $\lambda_f=k_f/(\rho_0 c_p)_f$ is the thermal diffusivity of the fluid phase. Thus the linearized version of equations (2) and (3) can be written in the non-dimensional form

$$\begin{aligned}
&\frac{\partial \boldsymbol{v}_f}{\partial t_f}=P_{r_f}\left(-\nabla p_f+\nabla^2\boldsymbol{v}_f+Ra_f\theta_f\boldsymbol{e}_3+Q_f\frac{\partial \boldsymbol{h}_f}{\partial x_3}\right),\\
&\frac{\partial \theta_f}{\partial t_f}-\beta\, w_f=\nabla^2\theta_f,\\
&P_{m_f}^{-1}\frac{\partial \boldsymbol{h}_f}{\partial t_f}=\frac{\partial \boldsymbol{v}_f}{\partial x_3}+\nabla^2\boldsymbol{h}_f,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\frac{Da}{\phi}\frac{\partial \boldsymbol{v}_m}{\partial t_m}=P_{r_m}\left(-\nabla p_m-\boldsymbol{v}_m+Ra_m\theta_m\boldsymbol{e}_3+Da\,Q_m\frac{\partial \boldsymbol{h}_m}{\partial x_3}\right),\\
&G_m\frac{\partial \theta_m}{\partial t_m}-\beta\, w_m=\nabla^2\theta_m,\\
&P_{m_m}^{-1}\frac{\partial \boldsymbol{h}_m}{\partial t_m}=\frac{\partial \boldsymbol{v}_m}{\partial x_3}+\nabla^2\boldsymbol{h}_m,
\end{aligned} \tag{11}$$

where

$$\beta=\frac{(T_l-T_0)}{|T_l-T_0|}=\frac{(T_0-T_u)}{|T_0-T_u|}=\begin{cases}-1, & \text{when heating from above,}\\ 1, & \text{when heating from below.}\end{cases}$$

Notice that P_{r_f}, P_{m_f}, Q_f and Ra_f are non-dimensional numbers which denote respectively, the viscous Prandtl number, magnetic Prandtl number, Chandrasekhar number and Rayleigh number of the fluid layer and P_{r_m}, P_{m_m}, Q_m and Ra_m are the corresponding non-dimensional numbers in the porous medium layer. These non-dimensional numbers are given by

$$P_{r_f}=\frac{\nu}{\lambda_f}, \ P_{m_f}=\frac{\eta}{\lambda_f}, \ Q_f=\frac{\mu_f H_f^2 d_f^2}{4\pi\rho_0\nu\eta}, \ Ra_f=\frac{g\alpha\, d_f^3|T_0-T_u|}{\nu\,\lambda_f},$$

$$P_{r_m}=\frac{\nu}{\lambda_m}, \ P_{m_m}=\frac{\eta_m}{\lambda_m}, \ Q_m=\frac{\mu_m H_m^2 d_m^2}{4\pi\rho_0\nu\eta_m}, \ Ra_m=\frac{g\alpha\, K d_m|T_l-T_0|}{\nu\,\lambda_m},$$

and $Da = \dfrac{K}{d_m{}^2}$ is the Darcy number, $G_m = (\rho_0 c_p)^* / (\rho c_p)_f$ and $\lambda_m = k^* / (\rho_0 c_p)_f$ is the thermal diffusivity of the porous medium phase. We shall discuss the problem in the case of heating the porous medium layer from below, so we shall assume that $\beta = 1$. We now apply the normal mode expansion of the form

$$\psi(\boldsymbol{x}, t) = \psi(x_3)\exp\left[i(rx_1 + qx_2) + \sigma\ t\right], \qquad (12)$$

for the functions $w_f, \theta_f, h_f, w_m, \theta_m$ and h_m. Thus the fluid and porous layers equations become

$$\begin{aligned} \frac{\sigma_f}{P_{r_f}}\left(D_f^2 - a_f^2\right) w_f - \sigma_f Q P_{m_f}^{-1} D_f h_f &= \left(D_f^2 - a_f^2\right)^2 w_f \\ &\quad - Q D_f^2 w_f - a_f^2 Ra_f \theta_f, \\ \sigma_f \theta_f &= w_f + \left(D_f^2 - a_f^2\right)\theta_f, \\ \sigma_f P_{m_f}^{-1} h_f &= D_f w_f + + \left(D_f^2 - a_f^2\right) h_f, \end{aligned} \qquad (13)$$

$$\begin{aligned} -\frac{Da}{P_{r_m}\phi}\sigma_m\left(D_m^2 - a_m^2\right) w_m + \sigma_m Da\, Q_m P_{m_m}^{-1} D_m h_m &= \\ \left(D_m^2 - a_m^2\right) w_m + a_m^2 Ra_m \theta_m + Da Q_m D_m^2 w_m, \\ G_m \sigma_m \theta_m &= w_m + \left(D_m^2 - a_m^2\right)\theta_m, \\ \sigma_m P_{m_m}^{-1} h_m &= D_m w_m + \left(D_m^2 - a_m^2\right) h_m, \end{aligned} \qquad (14)$$

where $a_f = \sqrt{r_f^2 + q_f^2}$ and $a_m = \sqrt{r_m^2 + q_m^2}$ are non-dimensionalised wave numbers in the fluid layer and porous medium layer respectively, σ_f and σ_m are the growth rates and

$$a_f = \hat{d} a_m, \quad \sigma_f = \frac{\hat{d}^2}{\hat{k}}\sigma_m, \quad D_f = \frac{\partial}{\partial x_3} \quad x_3 \in (0,1),$$

$$D_m = \frac{\partial}{\partial x_3} \quad x_3 \in (-1,0).$$

The boundary conditions in the final form are

$$w_f = 0, \quad D_f w_f = 0, \quad \theta_f = 0, \quad h_f = 0, \quad \text{on } x_3 = 1, \qquad (15)$$

$$\left.\begin{aligned} &\varepsilon_T w_f = w_m, \quad \theta_f = \varepsilon_T \theta_m, \\ &D_f \theta_f = D_m \theta_m, \\ &\frac{\varepsilon_T Da}{\hat{d}^3}\left(D_f^2 - 3a_f^2 - \frac{\sigma_f}{P_{r_f}}\right) D_f w_f = \\ &-\left(\frac{Da\sigma_m}{P_{r_m}\phi} + 1\right) D_m w_m, \\ &D_f h_f = 0, \qquad D_m h_m = 0, \\ &\frac{\varepsilon_T}{\hat{d}}\left(D_f w_f - \frac{\sqrt{Da}}{\hat{d}\alpha_{BJ}} D_f^2 w_f\right) = D_m w_m, \end{aligned}\right\} \text{on } x_3 = 0, \qquad (16)$$

$$w_m = 0, \quad \theta_m = 0, \quad h_m = 0, \qquad \text{on } x_3 = -1, \qquad (17)$$

where $\hat{d}, \hat{k}$ and ε_T are given by

$$\hat{d} = d_f / d_m, \quad \hat{k} = k_f / k^*, \quad \varepsilon_T = \frac{T_l - T_0}{T_0 - T_u} = \frac{\hat{k}}{\hat{d}}.$$

We notice that

$$\begin{aligned} &P_{r_f} = \frac{1}{\hat{k}} P_{r_m}, \qquad P_{m_f} = \frac{1}{\hat{k}} P_{m_m}, \\ &Q = \hat{d}^2 Q_m, \quad Ra_f = \frac{\hat{d}^4}{Da\,\hat{k}^2} Ra_m. \end{aligned} \qquad (18)$$

The complete eigenvalue system to be solved comprises (13)-(17). This system consists of an eighth order ODE in the fluid layer and a sixth order ODE in the porous medium layer, with 14 boundary conditions. The method of solution used to solve this system is the first order Chebyshev tau method. The implementation of this method is important since it is a highly accurate method and we are able to calculate as many eigenvalues as we need which is useful in cases when the eigenvalues of interest are changing in parameter space. We can also calculate the eigenfunctions easily.

We first map $x_3 \in (0,1)$ and $x_3 \in (-1,0)$ into the Chebyshev domain $(-1,1)$ by the transformations $z = 2x_3 - 1$ and $z = 2x_3 + 1$ respectively, then we suppose that

$$y_r(z) = \sum_{k=0}^{M-1} \alpha_{kr} T_k(z), \qquad 1 \le r \le 14, \qquad z \in (-1,1),$$

to be the Chebyshev expansion of the variables y_r which are defined by

$y_1 = w_f$, $y_2 = Dw_f$, $y_3 = D^2 w_f$, $y_4 = D^3 w_f$,

$y_5 = \theta_f$, $y_6 = D\theta_f$, $y_7 = h_f$, $y_8 = Dh_f$,

$y_9 = w_m$, $y_{10} = Dw_m$, $y_{11} = \theta_m$, $y_{12} = D\theta_m$,

$y_{13} = h_m$, $y_{14} = Dh_m$.

then equations (13) and (14) can be written as a system of 14 first order ordinary differential equations of the form $\frac{d\boldsymbol{Y}}{dz} = \boldsymbol{EY} + \sigma \boldsymbol{FY}$, to yield an eigenvalue problem of the form $\boldsymbol{Ax} = \sigma \boldsymbol{Bx}$. This generalized matrix eigenvalue problem is solved using the QZ algorithm which we employed via the NAG routine F02BJF.

5 Results and Discussion

The numerical results obtained show that the onset of instability is in the form of stationary convection even in the presence of magnetic field. This explained the assumption of Chen and Chen [6] from the beginning that the principle of exchange of instabilities holds.

Numerical results and stability curves are obtained for the present problem with thermal conductivity ratio $\hat{k} = 0.7$, $Da = 4\times10^{-6}$, $\alpha_{BJ} = 0.1$, $\phi = 1$, $P_{m_m} = 1$, $P_{r_m} = 1$, $G_m = 1$, and for variety of reciprocal depth ratio $\hat{d}$ ranging from 0.001 to 1 and for various values of, Q_m, ranging from 0 to 10^5.

Fig.2 shows the relation between Q_m and the critical Rayleigh number in the porous medium layer, Ra_m, for different values of $\hat{d}$ (see Table 1.a and Table 1.b). It is clear from the figure that Ra_m increases continuously as Q_m increases which means that the magnetic field has a stabilizing effect in this system. Moreover, Ra_m decreases continuously as the thickness of the fluid layer increases and this result is in agreement with the result obtained by Sun [1] in the absence of magnetic field.

The marginal stability curves at a number of depth ratios are shown in Fig. 3 and Fig. 4 for $Q_m = 0$ and $Q_m = 50000$ respectively. The results in Fig.3, $Q_m = 0$, are consistent with those of Chen and Chen [6] in which he showed that the marginal stability curves are bimodal, exhibiting two relative minima when $\hat{d}$ is less than a certain critical value. In the presence of magnetic field, as in the present problem, the marginal stability curves are also bimodal but for large depth ratios. Fig.4 shows that when $\hat{d} \geq 0.33$ and $Q_m = 50000$ the marginal stability curves are bimodal. In fact Q_m must exceeds a certain critical value in order to have two minimum values for Ra_m. As an example, this value has been calculated numerically when $\hat{d} = 0.33$ and we found that it should be greater than 7000.

Moreover in Fig.3, $Q_m = 0$, for $\hat{d} \leq 0.1$ the long wave branch is the most unstable whereas for $\hat{d} > 0.1$ the short wave branch is the most unstable. In the presence of magnetic field as in Fig.4, $Q_m = 50000$ for $\hat{d} \leq 0.33$ the short wave branch is the most unstable.

Figure (5) shows the relation between $\hat{d}$ and critical wave number of porous medium layer a_m for different values of Q_m. It is clear from this figure that as $\hat{d}$ increases there is a rapid change of the critical wave number a_m.

Figure (6) shows the relation between $\hat{d}$ and critical Ra_m for different values of Q_m. It is clear from this figure that there is a precipitation drop of the values of the critical Ra_m as $\hat{d}$ increases. This precipitation decreases as Q_m increases. The numerical results related to this figure is listed in table (2).

The critical Rayleigh number in the fluid layer, Ra_f, can be calculated from equation (18) and the relations between $\hat{d}$ and the critical thermal Rayleigh numbers in the porous and fluid layers when $Q_m = 0, 50000$ are displayed in Fig. 7. As $\hat{d}$ increases Ra_f increases and it quickly exceeds the critical value and convection ensues. In fact when $\hat{d}$ is large the problem reduces to a horizontal layer of fluid heated from below and the critical Rayleigh number of Ra_f approaches the value 1100 which is the value of the critical Rayleigh number with one rigid and one free boundary in the classical Benard problem. Here the porous interface behaves more like a free boundary than a rigid one. Similarly in the presence of magnetic field, when $\hat{d}$ is large the problem reduces to a horizontal layer of fluid heated from below and subject to a vertical magnetic field and the critical Rayleigh number Ra_f approaches the value of the critical Rayleigh number with one rigid and one free boundary the classical magnetic Benard problem according to the value of Q_m. This figure shows also the precipitous drop of the critical Rayleigh number Ra_m with increasing $\hat{d}$. The numerical results related to this figure is listed in table (3).

The precipitous drop of the critical Rayleigh number Ra_m with increasing $\hat{d}$ and the rapid change of the critical wave number a_m are shown in Fig. 8 when $Q_m = 0{,}50000$.

The marginal stability curves for different values of α_{BJ} when $\hat{d} = 0.04, 0.13, 0.5$ are shown in Fig.9-11, respectively when $Q_m = 0$. It appears from the figures that Ra_m is sensitive to α_{BJ} for small values of $\hat{d}$ ($\hat{d} \leq 0.13$) when a_m exceeds a certain value, and this value of a_m is decreased when α_{BJ} is increased. Generally Ra_m increases by increasing α_{BJ}. Moreover if $\hat{d} > 0.13$, Ra_m is no longer sensitive to α_{BJ}.

In the presence of magnetic field, the marginal stability curves for different values of α_{BJ} when $\hat{d} = 0.04, 0.13, 0.5$ are shown in Fig.12-14 respectively when $Q_m = 50000$. It appears from the figures that Ra_m is sensitive to α_{BJ}, and that the effect of α_{BJ} is different when magnetic field is applied. Here Ra_m increases as α_{BJ} increases for small values of a_m but when a_m exceeds a certain value Ra_m decreases as α_{BJ} increases.

The problem is also solved when the upper surface is free. The conditions on the upper layer is replaced by the following conditions

$$w_f = 0, \quad D_f^2 w_f = 0, \quad \theta_f = 0, \quad D_f h_f = 0, \qquad \text{on } x_3 = 1 .$$

Numerical results are obtained for the same values of the parameters suggested in the case when the upper boundary is rigid. The effects of the parameters in this case are similar to the case when the upper layer is rigid but the numerical results are quantitatively less.

6 Conclusion

In this paper we have implemented a detailed linear stability analysis in a superposed fluid and porous layers in the presence of a vertical magnetic field. The effect of magnetic field has been investigated in detail. A wide range of parameters have been used to allow for the possibility of overstability and the numerical results showed that instability occurs only via stationary convection.

We have shown that even in the presence of magnetic field the onset of convection had a bimodal nature in which convection may be dominated by the porous medium or the fluid layer depending on the value of the depth ratio $\hat{d}$.

The first order Chebyshev tau method is used to obtain the numerical solutions of the problem. This method is important for this class of problems since it is highly accurate. Several numerical results are obtained relating the critical Rayleigh numbers of fluid and porous layers, the wave number, the depth ratio, the magnetic parameter and the Beavers and Joseph constant.

Table 1.a: The relation between Q_m and critical Ra_m.

Q_m	critical Ra_m at $\hat{d}=$				
	0.001	0.01	0.04	0.1	0.11
0	39.418	36.754	24.822	19.121	14.743
100	39.426	36.760	24.831	19.145	15.105
500	39.458	36.786	24.866	19.233	16.516
1000	39.497	36.818	24.911	19.329	18.218
5000	39.812	37.072	25.285	19.858	19.246
10000	40.203	37.389	25.758	20.306	19.692
50000	43.273	39.896	28.946	22.640	21.946
100000	46.976	42.998	32.206	24.972	24.187

Table 1.b: The relation between Q_m and critical Ra_m.

Q_m	critical Ra_m at $\hat{d}=$				
	0.12	0.13	0.33	0.5	1
0	10.572	7.782	0.214	0.043	0.003
100	10.876	8.041	0.254	0.059	0.007
500	12.058	9.045	0.394	0.115	0.018
1000	13.477	10.244	0.551	0.177	0.032
5000	18.664	18.108	1.608	0.599	0.126
10000	19.117	18.572	2.790	1.081	0.237
50000	21.307	20.713	11.266	4.620	1.079
100000	23.470	22.802	21.193	8.828	2.098

Table 2. The relation between $\hat{d}$ and critical Ra_m.

$\hat{d}$	critical Ra_m at $Q_m =$				
	0	500	5000	50000	100000
0.01	36.754	36.786	37.072	39.896	42.998
0.04	24.822	24.866	25.285	28.946	32.206
0.1	19.121	19.233	19.858	22.640	24.972
0.11	14.743	16.516	19.246	21.946	24.187
0.12	10.572	12.058	18.664	21.307	23.470
0.13	7.782	9.045	18.108	20.713	22.802
0.33	0.214	0.394	1.608	11.266	21.193
0.5	0.043	0.115	0.599	4.620	8.828
1	0.003	0.018	0.126	1.079	2.098

Table 3. The relation between $\hat{d}$ and critical Ra_f .

$\hat{d}$	critical Ra_f at Q_m =				
	0	500	5000	50000	100000
0.01	0.188	0.188	0.189	0.204	0.219
0.04	32.420	32.478	33.026	37.808	42.065
0.1	975.578	981.279	1013.175	1155.086	1274.101
0.11	1101.306	1233.752	1437.635	1639.360	1806.773
0.12	1118.478	1275.725	1974.625	2254.153	2483.001
0.13	1133.998	1318.002	2638.636	3018.270	3322.717
0.33	1296.833	2384.471	9731.718	68166.882	128229.380
0.5	1355.766	3678.292	19113.365	147322.370	281500.200
1	1430.448	9416.549	64342.363	550264.866	1070585.13

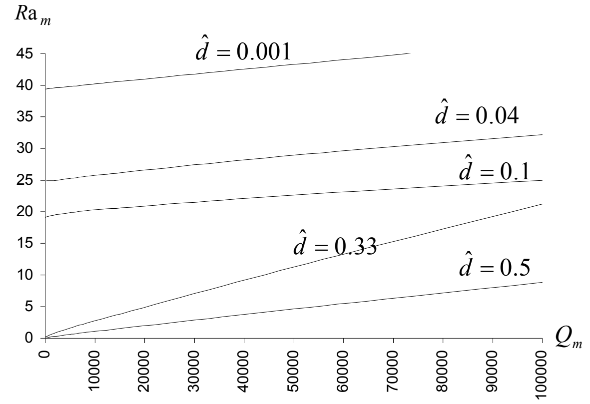

Fig.2: The relation between Q_m and Ra_m.

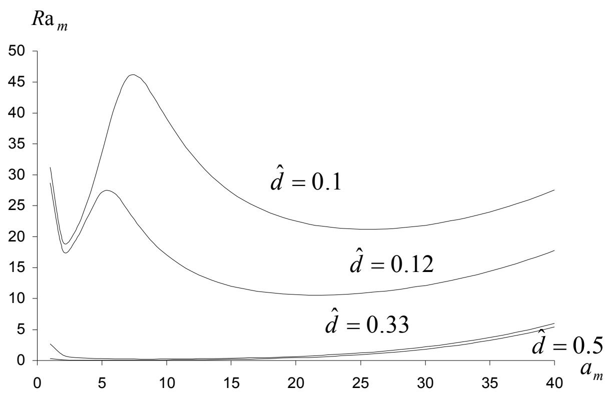

Fig.3: The relation between a_m and Ra_m ($Q_m = 0$).

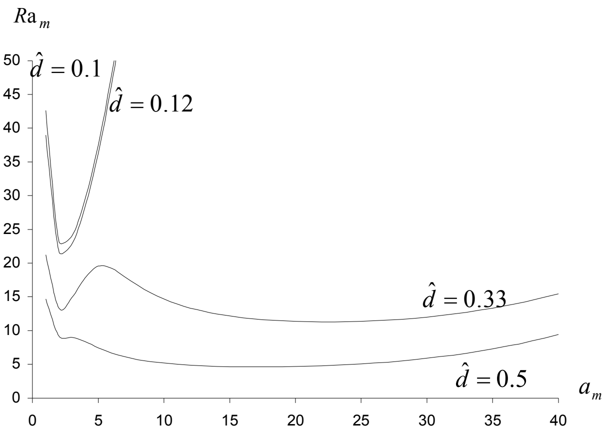

Fig.4: The relation between a_m and Ra_m ($Q_m = 50000$).

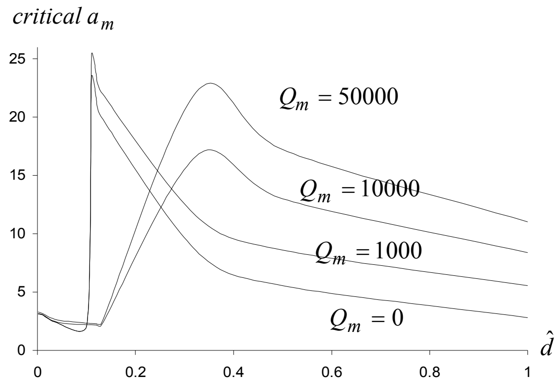

Fig. 5 : The relation between $\hat{d}$ and critical a_m

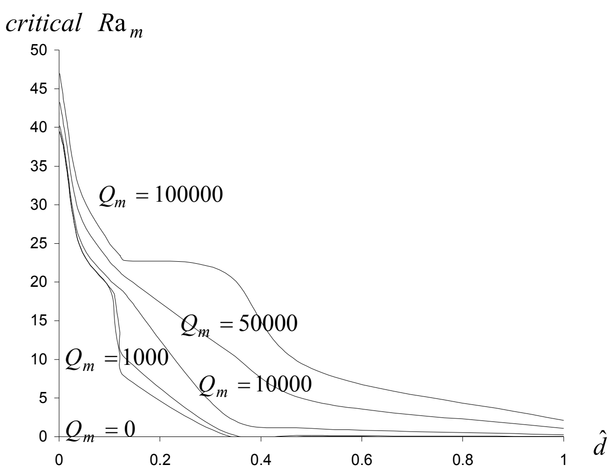

Fig. 6 : The relation between $\hat{d}$ and critical Ra_m

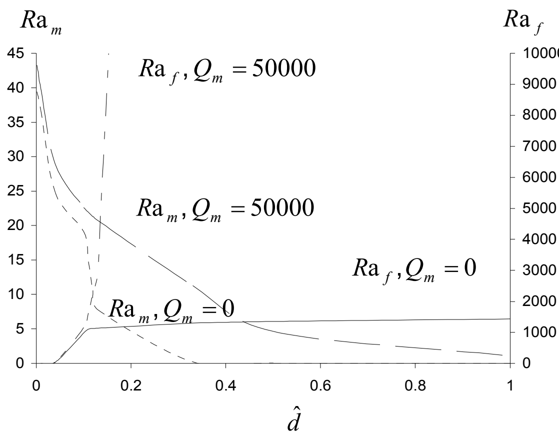

Fig.7: The relation between $\hat{d}$, Ra_m and Ra_f.

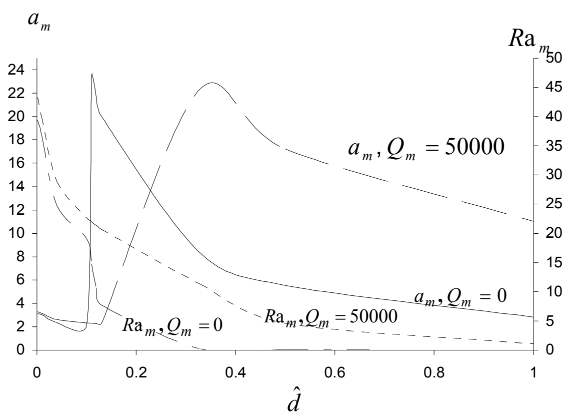

Fig.8: Variation of Ra_m and a_m with $\hat{d}$.

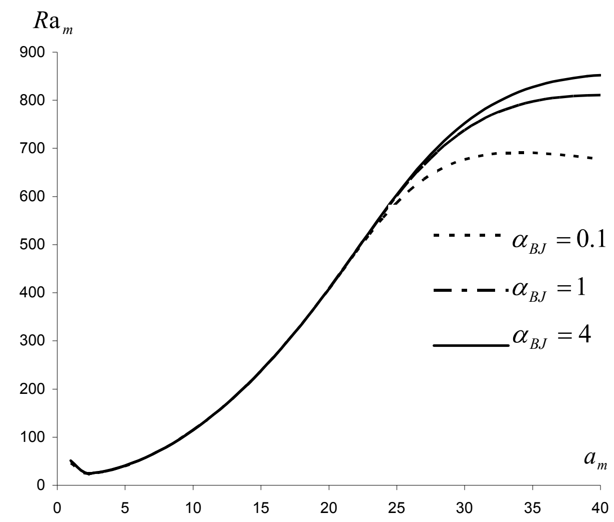

Fig.9: The relation between wave number a_m and Rayleigh number in the porous layer Ra_m for different values of α_{BJ} when $\hat{d} = 0.04$ and $Q_m = 0$.

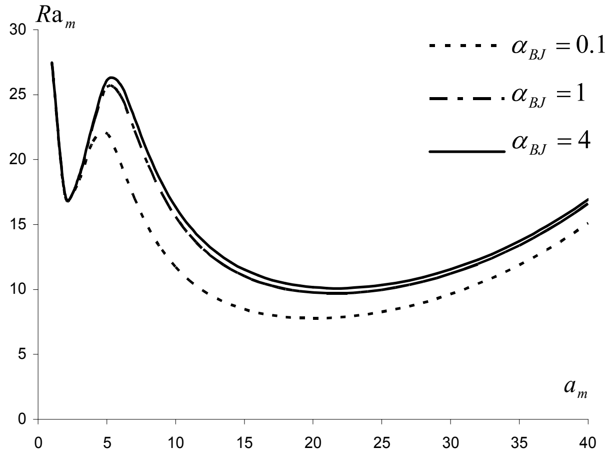

Fig.10: The relation between wave number a_m and Rayleigh number in the porous layer Ra_m for different values of α_{BJ} when $\hat{d} = 0.13$ and $Q_m = 0$.

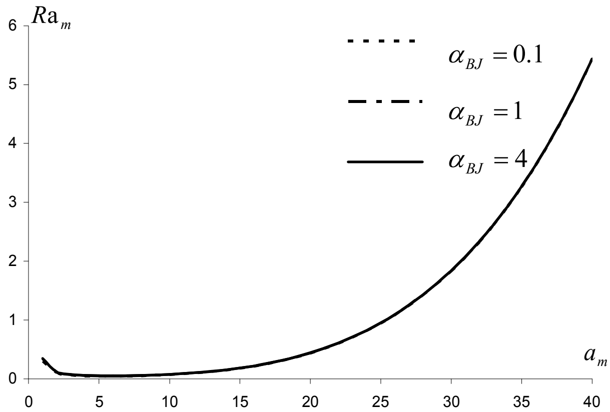

Fig.11: The relation between wave number a_m and Rayleigh number in the porous layer Ra_m for different values of α_{BJ} when $\hat{d} = 0.5$ and $Q_m = 0$.

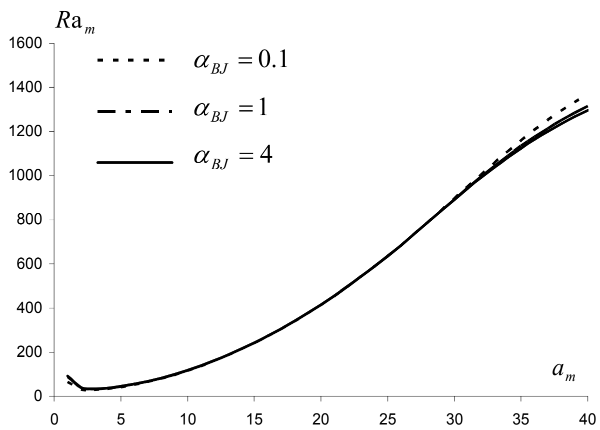

Fig.12: The relation between wave number a_m and Rayleigh number in the porous layer Ra_m for different values of α_{BJ} when $\hat{d} = 0.04$ and $Q_m = 50000$.

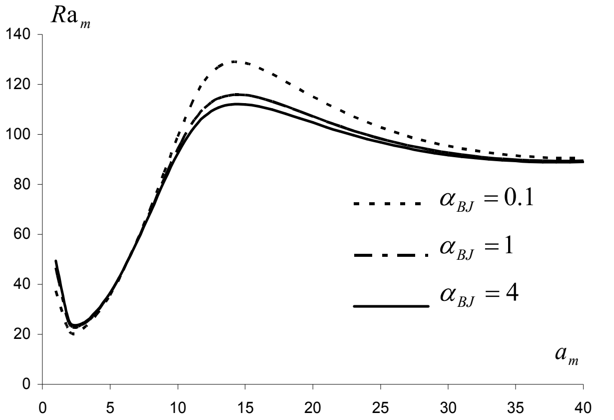

Fig.13: The relation between wave number a_m and Rayleigh number in the porous layer Ra_m for different values of α_{BJ} when $\hat{d} = 0.13$ and $Q_m = 50000$.

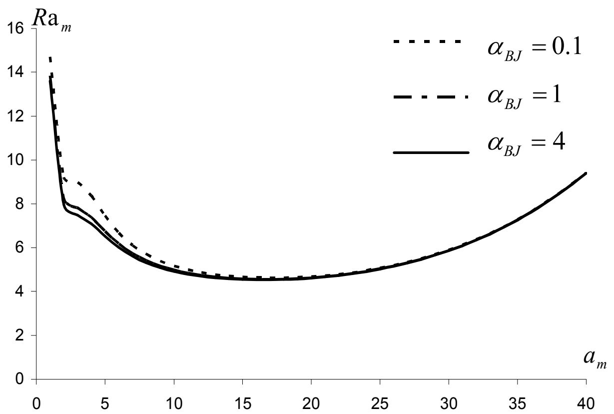

Fig.14: The relation between wave number a_m and Rayleigh number in the porous layer Ra_m for different values of α_{BJ} when $\hat{d} = 0.5$ and $Q_m = 50000$.

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