Dynamic Behaviour of Plates Subjected to a Flowing Fluid

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Abstract: - Elastic structures subjected to a flowing fluid undergo a considerable change in their dynamic behaviour and can lose their stability. In this article we describe the development of a fluid-solid finite element to model plates subjected to flowing fluid under various boundary conditions. The mathematical model for the structure is developed using a combination of a hybrid finite element method and Sanders' shell theory. The membrane displacement field is approximated by bilinear polynomials and the transversal displacement by an exponential function. Fluid pressure is expressed by inertial, Coriolis and centrifugal fluid forces, written respectively as function of acceleration, velocity and transversal displacement. Bernoulli's equation for the fluid-solid interface and a partial differential equation of potential flow are applied to calculate the fluid pressure. The impermeability condition ensures contact between the system of plates and the fluid. Mass and rigidity matrices for each element are calculated by exact integration. Calculated results are in reasonable agreement with other analytical theories.

Key-Words: - Vibration, Finite element, Plates, Potential flow, Fluid structure interaction, Critical velocity.

1 Introduction

Systems of plates subjected to fluid flow are often found in contemporary industries such as nuclear reactors and aerospace. Generally these industries require high rates of fluid flow and low plate thicknesses. Under these conditions, if the length of the plates is excessive the structure becomes very susceptible to failure.

Earlier works in this field were carried out on engineering test reactor (ETR) systems consisting of many thin plates stacked in parallel with narrow channels between the plates to let coolant flow through. Miller [13] was the first to present a theoretical analysis predicting the critical flow velocity for divergence. His analysis applies a method of 'neutral equilibrium' whereby pressure and plate restoring forces are balanced, leading to a derivation of critical velocity of flow for various types of support. It is important to underline that the motion of a plate excited by fluid flow displaces the nearby fluid, and then the fluid reactive motion may further deform the plate. Excessive fluid reactive motion at a certain flow velocity over the surface of the plate is referred to as the divergent velocity, which may also be considered a critical flow velocity [8]. Rosenberg and Youngdahl [14] have formulated a dynamic model describing the motion of a fuel plate in a parallel plate assembly. They found that good agreement exists between the results of the dynamic model and that of the neutral equilibrium used by Miller [13]. Three parallel plate assemblies were tested by Groninger and Kane [2] to investigate the flow-induced deflections of the individual plates. The model showed that adjacent plates always move in opposite directions at high flow rates, causing alternate opening and closing of the channel. They detected a violent dynamic instability at 1.9 times Miller's collapse velocity.

The assumptions of Miller [13], and Rosenberg and Youngdahl [14] were the same. They linearized the pressure drop expression using only a first-order approximation. Wambsganss [19] retained the second-order terms in an attempt to assess their influence on stability. The second-order terms generate an additional stability criterion in the form of an upper bound on the amplitude of quasi-static deflections for stable oscillations. He derived a new expression for critical velocity. Smissaert [17 and performed analytical and experimental 18] investigations on an MTR-type flat-plate fuel element. The experimental results [17] show that for low velocities the plates will deform as a result of static pressure differences in the channels between these plates. At high fluid velocities a highamplitude flutter vibration is observed. This flutter does not appear below a minimum average water velocity referred to as the flutter velocity, which is approximately equal to two times the Miller velocity of the assembly. In the analytical study, Smissaert [18] indicated that a plate assembly is characterized by two velocities; Miller's velocity and flutter velocity. One explanation of the dynamic instability is that the exciting frequency of the fully-developed turbulent flow approaches the in-fluid natural frequency of the plate. Theoretically, under this condition the amplitude of the plate vibration becomes large. Weaver and Unny [20] studied the dynamic behaviour of a single flat plate, one side of which is exposed to high flow velocity of a heavy fluid such as water. They examined the variation of natural frequencies according to the rate of flow. They concluded that for a given mass rapport, the neutral zone of stability is followed by a zone of static instability. After this stage the plate quickly returns to neutral stability, which continues until the occurrence of dynamic instability. Kornecki et al. [9] considered a flat panel of infinite width and finite length embedded in an infinite rigid plane with uniform incompressible potential flow over its upper surface. The studied plates were constrained (clamped or simply supported) along their leading and trailing edges. The case of a panel clamped at its leading edge and free at its trailing edge was investigated both theoretically and experimentally. The obtained results demonstrate that a panel fixed at its leading and trailing edges loses its stability by divergence (static instability), while the cantilevered panel loses its stability by flutter. Other investigators have studied the fluid flow effect on dynamic behaviour of rectangular plates; i.e. Ishii [6], Dowell [1] and Holmes [5]. More recently, Kim and Davis [8] developed an analytical model of a system of thin rectangular flat plates. Their model was used to investigate static and dynamic instabilities of the system. Guo and Paidoussis [3] studied theoretically the stability of rectangular plates with free side-edges in inviscid channel flow. They treated the plate as one dimensional and the channel flow as two-dimensional. The Galerkin method was employed to solve the plate equation, while the Fourier transform technique was employed to obtain the perturbation pressure from the potential flow equations. They investigated every possible combination of classical supports at the leading and trailing edges of the plates. They concluded that divergence and coupled mode flutter may occur for plates with any type of end supports, while single mode flutter only arises for nonsymmetrically supported plates. Guo and Paidoussis [4] have also conducted a theoretical study of the hydro-elastic instabilities of rectangular parallelplate assemblies. They considered the plates as twodimensional, with a finite length, and the flow field was assumed to be inviscid and three-dimensional. Two types of instability were found; single-mode flutter and coupled-mode flutter. They also concluded that the frequency at a given flow velocity decreases as the aspect ratio increases and the channel height to plate-width ratio decreases.

The purpose of this paper is to develop a solidfluid finite element to study the dynamic response of a rectangular plate subjected to potential flow. This new finite element permits us to obtain the low as well as the high frequencies of fluid-structure systems with precision for any combination of boundary conditions without changing the displacement field. This finite element is applied to simulate a number of plates and set of parallel plates subjected to flowing fluid. The mathematical model for the structure is developed using a combination of the finite element method and Sanders' shell theory. The velocity potential and Bernoulli's equation are adopted to express the fluid pressure acting on the structure.

2 Solid finite element

The geometry of the mean surface of the rectangular plate and the co-ordinate systems used for this analysis are shown in Fig. 1.b. A typical four-node element and nodal degrees of freedom are shown in Fig. 1.a. Each node has six degrees of freedom consisting of in-plane and out-of-plane displacement components and their spatial derivatives.

2.1 Equilibrium equations and displacement functions

To develop the equilibrium equations for rectangular plates, the Sanders' equations for cylindrical shells are used assuming the radius to be infinite, $\theta = y$ and $rd\theta = dy$. Both membrane and bending effects are taken into account in this theory. It is worthy to note that Sanders' shell theory is based on Love's first approximation theory but leads to zero strains for the case of rigid body motion. The developed displacement functions therefore satisfy the convergence criteria for the proposed finite element.

The equilibrium equations of a rectangular plate according to Sanders' theory can be written as a function of displacement components with respect to the reference surface:

$$P_{22}\frac{\partial^2 V}{\partial y^2} + P_{21}\frac{\partial^2 U}{\partial x \partial y} + P_{33}\left(\frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 V}{\partial x^2}\right) = 0 \quad (1.a)$$

$$P_{II}\frac{\partial^2 U}{\partial x^2} + P_{I2}\frac{\partial^2 V}{\partial x \partial y} + P_{33}\left(\frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 U}{\partial y^2}\right) = 0 \quad (1.b)$$

$$P_{44}\frac{\partial^4 W}{\partial x^4} + (P_{45} + P_{54} + 2P_{66})\frac{\partial^4 W}{\partial x^2 \partial y^2} + P_{55}\frac{\partial^4 W}{\partial y^4} = 0 \quad (1.c)$$







Note that both circumferential and longitudinal hybrid elements used in the dynamic analysis of vertical Lakis and Paidoussis [22] and horizontal Lakis and Selmane [23] open cylindrical shells were developed based on exact solution of the equilibrium equations. This approach resulted in a very precise element which leads to fast convergence and less numerical difficulties from the computational point of view. This encouraged us to develop a new finite element using the same approach for dynamic analysis of rectangular plates. Generally, exact solution of the equilibrium equations for the case of rectangular plates is difficult. To overcome this we present the in-plane membrane displacement components in terms of bilinear polynomials and the out-of-plane bending displacement component by an exponential function. Hence, the displacement field may be defined as follows:

$$U(x, y, t) = C_1 + C_2 \frac{x}{A} + C_3 \frac{y}{B} + C_4 \frac{xy}{AB}$$
(2.a)

$$V(x, y, t) = C_5 + C_6 \frac{x}{A} + C_7 \frac{y}{B} + C_8 \frac{xy}{AB}$$
 (2.b)

$$W(x, y, t) = \sum_{j=9}^{24} C_j e^{i\pi \left(\frac{x}{A} + \frac{y}{B}\right)} e^{i\omega t}$$
(2.c)

where U and V represent the in-plane displacement components of the middle surface in X and Y directions, respectively, W is the transversal displacement of the middle surface, A and B are the plate dimensions in X and Y directions, " ω " is the natural frequency of the plate (rad/sec), "i" is a complex number and C_i are unknown constants.

Equation (1.c) can be developed in Taylor's series as follows:

$$W(x, y, t) = C_{9} + C_{10} \frac{x}{A} + C_{11} \frac{y}{B} + C_{12} \frac{x^{2}}{2A^{2}} + C_{13} \frac{xy}{AB} + C_{14} \frac{y^{2}}{2B^{2}} + C_{15} \frac{x^{3}}{6A^{3}} + C_{16} \frac{x^{2}y}{2A^{2}B} + C_{17} \frac{xy^{2}}{2AB^{2}} + C_{18} \frac{y^{3}}{6B^{3}} + (3)$$

$$C_{19} \frac{x^{3}y}{6A^{3}B} + C_{20} \frac{x^{2}y^{2}}{4A^{2}B^{2}} + C_{21} \frac{xy^{3}}{6AB^{3}} + C_{22} \frac{x^{3}y^{2}}{12A^{3}B^{2}} + C_{23} \frac{x^{2}y^{3}}{12A^{2}B^{3}} + C_{24} \frac{x^{3}y^{3}}{36A^{3}B^{3}}$$

We can write the displacements U, V and W in matrix form:

$$\begin{cases} U \\ V \\ W \end{cases} = [R] \{C\}$$
 (4)

where [*R*] is a matrix of order (3x24) in which the components are the *x* and *y* terms of Equations (2.a, 2.b and 3) without the unknown constants (see Appendix) and $\{C\}$ is the vector for the unknown constants.

The components of this last vector can be determined using twenty-four degrees of freedom for a plate element as shown in Fig. 1. The displacement vector of each element is given as:

$$\{\delta\} = \left\{\!\left\{\delta_i\right\}^T, \left\{\delta_j\right\}^T, \left\{\delta_k\right\}^T, \left\{\delta_l\right\}^T, \left\{\delta_l\right\}^T\right\}^T$$
(5)

Each node, i.e. "node i", possesses a nodal displacement vector composed of the following terms:

$$\left\{\delta_{i}\right\} = \left\{U_{i}, V_{i}, W_{i}, \partial W_{i} / \partial x, \partial W_{i} / \partial y, \partial^{2} W_{i} / (\partial x \partial y)\right\}^{T} (6)$$

where U_i and V_i are nodal in-plane displacement components and W_i represent the nodal displacement components normal to the middle surface as shown in Fig. 1.a.

By introducing Equations (2.a, 2.b and 3) into relation (5), the elementary displacement vector can be defined as:

$$\left\{\delta\right\} = \left[A\right]\left\{C\right\} \tag{7}$$

The vector $\{C\}$ in Equation (7) will be then replaced by the generalized displacement vector of a quadrilateral finite element. The displacement field may be described by the following relation:

$$\begin{cases} U \\ V \\ W \end{cases} = [R][A]^{-1}\{\delta\} = [N]\{\delta\}$$
(8)

where matrix [N] of order (3×24) is the displacement shape function of the finite element and the terms of matrix $[A]^{-1}$ are given in the Appendix.

2.2 Kinematics Relations

Strain-displacement relations for the rectangular plates are given as Sanders [15]:

$$\begin{cases} \varepsilon_{x} \\ \varepsilon_{y} \\ 2\varepsilon_{xy} \\ \kappa_{x} \\ \kappa_{x} \\ \kappa_{xy} \end{cases} = \begin{cases} \frac{\partial U/\partial x}{\partial V/\partial y} \\ \frac{\partial V/\partial y}{\partial V/\partial x + \partial U/\partial y} \\ -\frac{\partial^{2}W}{\partial x^{2}} \\ -\frac{\partial^{2}W}{\partial y^{2}} \\ -2\partial^{2}W/(\partial x\partial y) \end{cases}$$
(9)

Substituting the displacement components defined in Equation (8) into the strain-displacement relationship (9), one obtains an expression for the strain vector as a function of nodal displacements.

$$\{\varepsilon\} = [Q][A]^{-1}\{\delta\} = [B]\{\delta\}$$
(10)

where matrix [Q] of order (6 ×24) is given in the Appendix.

2.3 Constitutive Equations

The stress-strain relationship of an isotropic rectangular plate is defined as follows:

$$\{\sigma\} = [P]\{\varepsilon\}$$
(11)

where [P] is the elasticity matrix for an isotropic plate and no bending-membrane coupling is present (see Appendix). Substituting Equation (10) into

Equation (11) results in the following expression for the stress tensor:

$$\{\sigma\} = [P][B]\{\delta\}$$
(12)

The mass and stiffness matrices for one finite element can be expressed as:

$$[k_s]^e = \iint_{S} [B]^T [P] [B] dS$$
(13.a)

$$[m_s]^e = \rho_s h \iint_{S} [N]^T [N] dS \qquad (13.b)$$

where *S* is the element surface area, *h* is the plate thickness, ρ_s is the material density and [*P*], [*N*] and [*B*] are defined in Equations (11, 8 and 10), substituting them into Equations (13.a and 13.b) we obtain:

$$\begin{bmatrix} k_s \end{bmatrix}^e = \begin{bmatrix} [A]^{-1} \end{bmatrix}^T \left(\int_0^{y_e x_e} [Q]^T [P] [Q] dx dy \right) \begin{bmatrix} A \end{bmatrix}^{-1}$$
(14.a)
$$\begin{bmatrix} m_s \end{bmatrix}^e = \rho_s h \begin{bmatrix} A \end{bmatrix}^{-1} \end{bmatrix}^T \left(\int_0^{y_e x_e} [R]^T [R] dx dy \right) \begin{bmatrix} A \end{bmatrix}^{-1}$$
(14.b)

where x_e and y_e are dimensions of an element according to the X and Y coordinates, respectively. These integrals are calculated analytically using Maple mathematical software.

3 Fluid-solid interaction

The fluid pressure acting upon the structure is generally expressed as a function of out-of-plane displacement and its derivatives i.e. velocity and acceleration. These three terms are respectively known as the centrifugal, Coriolis and inertial forces [16]. The fluid matrices will be combined with solid matrices as follows:

$$\begin{bmatrix} M_s \end{bmatrix} - \begin{bmatrix} M_f \end{bmatrix} \{ \ddot{\delta}_T \} + \begin{bmatrix} C_s \end{bmatrix} - \begin{bmatrix} C_f \end{bmatrix} \{ \dot{\delta}_T \} + \\ \begin{bmatrix} K_s \end{bmatrix} - \begin{bmatrix} K_f \end{bmatrix} \{ \delta_T \} = \{ 0 \}$$

$$(15)$$

where $[M_s]$, $[C_s]$ and $[K_s]$ are the global matrices of mass, damping and rigidity of the elastic plate, $[M_f]$, $[C_f]$ and $[K_f]$ represent the inertial, Coriolis and centrifugal forces of potential flow and $\{\delta_T\}$ is the global displacement vector. The elementary matrices of solid are calculated in Equation (14).

3.1 Fluid-solid finite element

The fluid-solid model is developed based on the following hypotheses: (i) the fluid flow is potential; (ii) vibration is linear (small deformations); (iii) the fluid mean velocity distribution (U_x) is constant

across a plate section and (iv) the fluid is incompressible.

Taking these assumptions into consideration, the velocity potential must satisfy the Laplace equation. This relation is expressed in the Cartesian system by:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$
(16)

where ϕ is the potential function. The Bernoulli equation is given by:

$$\left. \frac{\partial \phi}{\partial t} + \frac{1}{2} V_r^2 + \frac{P}{\rho_f} \right|_{z=0} = 0$$
 (17)

where *P* is the fluid pressure, V_r is the fluid velocity and ρ_f is the fluid density.

The components of fluid velocity along X, Y, Z directions, respectively are defined by:

$$V_x = U_x + \frac{\partial \phi}{\partial x}$$
 $V_y = \frac{\partial \phi}{\partial y}$ $V_z = \frac{\partial \phi}{\partial z}$ (18)

where U_x is the mean velocity of fluid in the xdirection. Fig. 2 depicts a fluid-solid finite element subjected to flowing fluid on its upper surface.

Introducing Equation (18) into (17) and neglecting the non-linear terms we can write the dynamic pressure at the solid-fluid interface as follows (see Fig. 2):

$$P\Big|_{z=0} = -\rho_f \left(\frac{\partial \phi}{\partial t} + U_x \frac{\partial \phi}{\partial x}\right)\Big|_{z=0}$$
(19)



Fig. 2: Fluid-solid finite element

The impermeability condition ensures contact between the shell and the fluid. This should be:

$$\left. \frac{\partial \phi}{\partial z} \right|_{z=0} = \left(\frac{\partial W}{\partial t} + U_x \frac{\partial W}{\partial x} \right)$$
(20)

The following separate variable relation is assumed for the potential velocity function:

$$\phi(x, y, z, t) = F(z)S(x, y, t)$$
(21)

where F(z) and S(x, y, t) are two separate functions to be defined.

The following expression may be defined by introducing Equation (21) into (20)

$$S(x, y, t) = \frac{1}{dF(0)/dz} \left(\frac{\partial W}{\partial t} + U_x \frac{\partial W}{\partial x}\right) \quad (22)$$

For x and y in the finite element domain (see Fig. 2) the potential and pressure at the interface are coupled by the transverse movement of the plate W(x,y,t) and its derivatives. Equation (22) describes the function S(x,y,t) in terms of this transverse movement of the plate which itself varies as a function of plate geometry and time. Therefore, the movement of the fluid at any point on the interface (including the boundaries x and y) is intimately linked to the movement of the edges of the structure. Then, substituting Equation (22) into (21), results in the following expression for the potential function:

$$\phi(x, y, z, t) = \frac{F(z)}{dF(0)/dz} \left(\frac{\partial W}{\partial t} + U_x \frac{\partial W}{\partial x}\right) \quad (23)$$

The only unknown function in Equation (22) is F(z). By introducing Equation (23) into Equation (16), we obtain the following differential equation of second order:

$$\frac{d^2 F(z)}{dz^2} - \mu^2 F(z) = 0$$
(24)

where:

$$\mu = \pi \sqrt{l/A^2 + l/B^2}$$

The solution of this last equation is:

$$F(z) = A_1 e^{\mu z} + A_2 e^{-\mu z}$$
(25)

where A_1 and A_2 are unknown constants. By substituting Equation (25) into Equation (23) the potential function becomes:

$$\phi(x, y, z, t) = \frac{\left(A_{I}e^{\mu z} + A_{2}e^{-\mu z}\right)}{dF(0)/dZ} \left(\frac{\partial W}{\partial t} + U_{x}\frac{\partial W}{\partial x}\right) \quad (26)$$

 A_1 and A_2 are two constants to be determined using the fluid boundary conditions. We note that at the solid-fluid interface (Z=0) we have the impermeability condition (20) which is common for all cases. At the second limit of fluid (Z=h₁), we have a boundary condition corresponding to either a rigid wall, an elastic plate or an infinite fluid level.

For each case we have a distinct solution. Below we will calculate the pressure of fluid acting on only one side $(Z=h_1)$ of the plate. If the fluid pressure acts on two sides, the total dynamic pressure will be a combination of the pressures corresponding to the fluid boundary conditions at both top and bottom $(Z=h_2)$ surfaces of the plate.

The presented approach may be adapted to model curved structures subjected to flowing fluid

such as turbine blades. To accomplish this the local matrices must be transformed to the global system before assembling into the global matrices [21].

3.1.1 Fluid-solid finite element subject to flowing fluid with infinite level of fluid

When the flowing fluid height on and/or under the plate $(h_1 \text{ and/or } h_2)$ is infinite (see Fig. 3), we assume that very far from the plate the potential is null. This boundary condition is written as follows:



Fig. 3: Fluid-solid finite element subjected to flowing fluid with infinite height

In order to avoid an infinite potential, the constant A_1 of Equation (26) must be null. Equation (20) permits us to calculate the second constant A_2 . The potential expression becomes:

$$\phi(x,y,z,t) = -\frac{F'(0)e^{-\mu z}}{\mu} \left(\frac{\partial W}{\partial t} + U_x \frac{\partial W}{\partial x}\right)$$
(28)

The introduction of Equation (28) into relation (19), results in the following expression for the pressure function

$$P = \frac{\rho_f}{\mu} \left[\frac{\partial^2 W}{\partial t^2} + 2U_x \frac{\partial^2 W}{\partial x \partial t} + U_x^2 \frac{\partial^2 W}{\partial x^2} \right]$$
(29)

or:

$$P = Z_{f1} \left[\frac{\partial^2 W}{\partial t^2} + 2U_x \frac{\partial^2 W}{\partial x \partial t} + U_x^2 \frac{\partial^2 W}{\partial x^2} \right]$$
(30)

3.1.2 Fluid-solid finite element subject to flowing fluid bounded by rigid wall

As shown in Fig. 4, fluid flows between a rigid wall and an elastic plate. This provides another boundary condition at $Z=h_1$ when the impermeability condition is taken into account. This boundary condition is adopted by Lamb [11], McLachlan [12] and is expressed by:

$$\left. \frac{\partial \phi}{\partial z} \right|_{z=h_{I}} = 0 \tag{31}$$



Fig. 4: Fluid solid finite element in contact with flowing fluid bounded by a rigid wall

Using Equations (20) and (31) we can calculate the constants A_1 and A_2 corresponding to this last boundary condition. Substituting these constants into (26) we obtain:

$$\phi(x,y,z,t) = \frac{F'(0)}{\mu} \frac{(e^{-2\mu h_t} e^{\mu z} + e^{-\mu z})}{(e^{-2\mu h_t} - 1)} \times \left(\frac{\partial W}{\partial t} + U_x \frac{\partial W}{\partial x}\right)$$
(32)

Replacing Equation (32) into (19), the corresponding dynamic pressure becomes:

$$P = -\frac{\rho_f(e^{-2\mu h_i} + 1)}{\mu(e^{-2\mu h_i} - 1)} \left[\frac{\partial^2 W}{\partial t^2} + 2U_x \frac{\partial^2 W}{\partial x \partial t} + U_x^2 \frac{\partial^2 W}{\partial x^2} \right] (33)$$

or:

$$P = Z_{f^2} \left[\frac{\partial^2 W}{\partial t^2} + 2U_x \frac{\partial^2 W}{\partial x \partial t} + U_x^2 \frac{\partial^2 W}{\partial x^2} \right]$$
(34)

3.1.3 Fluid-solid finite element subject to flowing fluid bounded by elastic plate

When fluid flows through two parallel elastic plates (see Fig. 5) two transverse vibration modes, inphase and out-of-phase, should be considered. The impermeability condition at the solid-fluid interface remains the same for both modes while the boundary condition at Z= h_1 changes according to the mode of vibration (in-phase or out-of-phase).

3.1.3.1 In-phase mode

In the case of the in-phase mode the boundary condition at fluid limits $Z=h_1$ is expressed as follows [10]:

$$\left. \frac{\partial \phi}{\partial z} \right|_{z=h_1} = \left(\frac{\partial W}{\partial t} + U_x \frac{\partial W}{\partial x} \right)$$
(35)



Fig. 5: Fluid-solid finite element in contact with flowing fluid bounded by an elastic plate

Similarly, A_1 and A_2 can be calculated by introducing Equation (26) into relations (20) and (35). The substitution of these constants in Equation (26) enables us to develop the following expression for the potential:

$$\phi(x, y, z, t) = \frac{1}{\mu} \left[\frac{(1 - e^{-\mu \theta_1})e^{\mu z} + (1 - e^{\mu \theta_1})e^{-\mu z}}{e^{\mu \theta_1} - e^{\mu \theta_1}} \right] \times \frac{\partial W}{\partial t} + U_x \frac{\partial W}{\partial x}$$
(36)

By replacing Equation (36) into (19) we obtain the following expression for pressure:

$$P = -\frac{\rho_f}{\mu} \left(\frac{2 - e^{-\mu h_f} - e^{\mu h_f}}{e^{\mu h_f} + e^{-\mu h_f}} \right) \times \left[\frac{\partial^2 W}{\partial t^2} + 2U_x \frac{\partial^2 W}{\partial x \partial t} + U_x^2 \frac{\partial^2 W}{\partial x^2} \right]$$
(37)

or:

$$P = Z_{f^3} \left[\frac{\partial^2 W}{\partial t^2} + 2U_x \frac{\partial^2 W}{\partial x \partial t} + U_x^2 \frac{\partial^2 W}{\partial x^2} \right] \quad (38)$$

3.1.3.2 Out-of-phase mode

The boundary condition for the out-of-phase mode is expressed as follows [10]:

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\frac{h_i}{2}} = 0 \tag{39}$$

Here again, A_1 and A_2 can be calculated using Equations (26, 39 and 20). The potential function may be written as follows:

$$\phi(x,y,z,t) = \frac{F'(0)}{\mu} \frac{(e^{-\mu h_l} e^{\mu z} + e^{-\mu z})}{(e^{-\mu h_l} - 1)} \left(\frac{\partial W}{\partial t} + U_x \frac{\partial W}{\partial x}\right) (40)$$

Using Equations (40) and (19) we obtain the following dynamic pressure:

$$P = -\frac{\rho_f \left(e^{-\mu h_1} + 1\right)}{\mu \left(e^{-\mu h_1} - 1\right)} \left[\frac{\partial^2 W}{\partial t^2} + 2U_x \frac{\partial^2 W}{\partial x \partial t} + U_x^2 \frac{\partial^2 W}{\partial x^2}\right] (41)$$

or:

$$P = Z_{f4} \left[\frac{\partial^2 W}{\partial t^2} + 2U_x \frac{\partial^2 W}{\partial x \partial t} + U_x^2 \frac{\partial^2 W}{\partial x^2} \right]$$
(42)

3.2 Calculation of Fluid-Induced Force

The elementary vector of fluid-induced force is expressed by:

$$\left\{F\right\}^{e} = \int_{A} \left[N\right]^{T} \left\{P_{\nu}\right\} dS$$
(43)

where [N] is the shape function matrix of the finite element defined in Equation (8), $\{P_{\nu}\}$ is a tensor expressing the pressure applied by the fluid on the plate and S is the elementary fluid-structure interface area.

By placing the matrix [N] of Equation (8) into Equation (43), the element load vector becomes:

$$\{F\}^{e} = \iint_{A} \left[\left[A \right]^{-I} \right]^{T} \left[R \right]^{T} \{P_{v}\} dS$$
(44)

The dynamic pressures of Equations (29, 33, 37 and 41) may be rewritten as:

$$P = Z_{fi} \left[\frac{\partial^2 W}{\partial t^2} + 2U_x \frac{\partial^2 W}{\partial x \partial t} + U_x^2 \frac{\partial^2 W}{\partial x^2} \right]$$
(45)

where Z_{fi} (*i*=1, 4), depends on the boundary conditions (see Equations 30, 34, 38 and 42) and *P* is the only non-zero component in the pressure tensor $\{P_{\nu}\}$. Substituting Equation (2.c) into (45) the pressure expression becomes:

$$P = Z_{fi} \left[\frac{\partial^2 W}{\partial t^2} + 2U_x \frac{i\pi}{A} \frac{\partial W}{\partial t} + U_x^2 \frac{i^2 \pi^2}{A^2} W \right] \quad (46)$$

The transversal displacement can be separated from Equation (8) as follows:

$$\begin{cases} 0\\0\\W \end{cases} = \left[R_f \right] \left[A \right]^{-1} \left\{ \delta \right\}$$
(47)

where $[R_f]$ is a (3x24) matrix given in the Appendix. Substituting Equation (47) into (46), we obtain the following expression for pressure:

$$\{P_{\nu}\} = Z_{fi} \left[R_{f}\right] \left[A\right]^{-1} \left\{\left\{\ddot{\delta}\right\} + 2U_{x}\frac{i\pi}{A}\left\{\dot{\delta}\right\} + U_{x}^{2}\frac{i^{2}\pi^{2}}{A^{2}}\left\{\delta\right\}\right\} (48)$$

By combining Equations (44) and (48) the element load vector is given by the following relation:

$$\{F\}^{e} = Z_{fi} \begin{pmatrix} \int_{A} [[A]^{-1}]^{T} [R]^{T} [R_{f}] [A]^{-1} \{\ddot{\delta}\} dS + \\ 2U_{x} \frac{i\pi}{A} \int_{A} [[A]^{-1}]^{T} [R]^{T} [R_{f}] [A]^{-1} \{\dot{\delta}\} dS \\ + U_{x}^{2} \frac{i^{2}\pi^{2}}{A^{2}} \int_{A} [[A]^{-1}]^{T} [R]^{T} [R_{f}] [A]^{-1} \{\delta\} dS \end{pmatrix}$$
(49)

Note that the force induced by a flowing fluid is a function of acceleration, velocity and displacement of the solid finite element. From Equation (49) we can separate the added matrices induced by flowing fluid, respectively describing inertial, Coriolis and centrifugal effects as follows:

$$\left[m_{f}\right]^{e} = Z_{fi} \int_{A} \left[\left[A\right]^{-1}\right]^{T} \left[R\right]^{T} \left[R_{f}\right] \left[A\right]^{-1} dA \qquad (50.a)$$

$$\left[c_{f}\right]^{e} = 2U_{x}Z_{fi}\frac{i\pi}{A}\int_{A}\left[\left[A\right]^{-1}\right]^{T}\left[R\right]^{T}\left[R_{f}\right]\left[A\right]^{-1}dA$$
(50.b)

$$\left[k_{f}\right]^{e} = U_{x}^{2} Z_{fi} \left(\frac{i\pi}{A}\right)^{2} \int_{A} \left[\left[A\right]^{-I}\right]^{T} \left[R\right]^{T} \left[R_{f}\right] \left[A\right]^{-I} dA (50.c)$$

Dynamic equilibrium requires a combination of three elementary matrices the last with corresponding matrices given in Equation (14).

4 Eigenvalue problem

A rectangular plate is subdivided into a series of quadrilateral finite elements such that each of them is a smaller rectangular plate (see Fig. 1.b). The positions of the nodal points of the elements are chosen in such a way that the local and global coordinates are parallel. An in-house computer code has been developed to establish the structural matrices of each element based on the equations developed using this theoretical approach. The global matrices mentioned in Equation (15) are obtained by superimposing the matrices for each individual element. After applying the boundary conditions these matrices are reduced to square matrices of order 6*N-NC, where N is the number of nodes in the structure and NC is the number of constraints applied. The eigenvalue problem is solved by means of the equation reduction technique. Equation (15) may be rewritten as follows:

$$\begin{bmatrix} \begin{bmatrix} 0 \\ M \end{bmatrix} \begin{bmatrix} \left\{ \ddot{\delta}_{T} \right\} \\ \left\{ \dot{\delta}_{T} \right\} \end{bmatrix} + \begin{bmatrix} -\begin{bmatrix} M \\ 0 \end{bmatrix} \begin{bmatrix} \left\{ \dot{\delta}_{T} \right\} \\ \left\{ \delta_{T} \right\} \end{bmatrix} = \{0\} \quad (51)$$

where:

$$\begin{bmatrix} M \end{bmatrix} = \begin{bmatrix} M_s \end{bmatrix} - \begin{bmatrix} M_f \end{bmatrix}, \quad \begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} C_f \end{bmatrix}, \quad \begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} K_s \end{bmatrix} - \begin{bmatrix} K_f \end{bmatrix}, \\ \{\delta_T\} \text{ is the global displacement vector and structural damping is neglected. The eigenvalue$$

problem is given by:

$$\left[[DD] - \Lambda [I] \right] = 0 \tag{52}$$

$$\begin{bmatrix} DD \end{bmatrix} = \begin{bmatrix} 0 & [I] \\ [K]^{-1}[M] & [K]^{-1}[C] \end{bmatrix}, \ \Lambda = 1/i\omega^2 \text{ and } [I] \text{ is}$$

the identity matrix.

whore .

5 Results and discussions

The precision of the finite element method depends on the number of elements used to discretize the physical problem. The first set of calculations is therefore to determine the requisite number of elements for a precise determination of the natural frequencies.

The variation of the first five frequencies versus the number of finite elements of rectangular plate simply supported on its four sides is plotted in Fig. 6 and shows the minimum required number of elements to assure fast convergence in determining both low and high frequencies. The values of the material and geometrical properties used in the calculations are: Young's modulus E=196GPa, material density ρ =7860 kg/m³. Poissson's ratio v=0.3, thickness h=2.54 mm, A=609.6 mm and B=304.8mm.

Eight elements are sufficient to calculate the two first modes, whereas for other modes convergence requires at least twenty five elements. This number of elements required by the present method is much lower than that of other existing approaches. In all of the following examples 64 elements are used, which assures that the results will be independent of mesh size.

In order to show that the developed model provides accurate results, calculations were performed on the same plate used in the convergence test. The first six natural frequencies are listed in Table 1 along with analytical results and ANSYS output data. It can be seen that the present method gives fairly good results compared to the exact solution and the commercial finite element code

An extensive study has been conducted to test the solid finite element in vacuum in reference [7]. Free vibrations of rectangular plates were obtained for a variety of boundary conditions and plate dimension ratios (A/B). The computed natural

frequencies were compared to those obtained by other theories [24] and from experiments. The results were in very good agreement.





Table 1 Natural frequency (Hz) of a plate simply supported at its four sides

Mode	Present	ANSYS	Analytical
	model	(shell 63)	solution Leissa
1	82.93	80.99	83.5
2	133.44	129.33	133.61
3	213.72	209.86	217.12
4	275.27	275.02	283.9
5	315.52	322.51	334

In the following examples, we present some calculations to test the solid-fluid model in the case of plates subjected to flowing fluid. To put the results in the non-dimensional form, the following parameters are defined:

$$\psi = \frac{\rho_f B}{\rho_s h} \tag{53}$$

$$\overline{\omega} = B^2 \sqrt{\frac{\rho_s h}{K}} \omega \tag{54}$$

$$\overline{U} = B \sqrt{\frac{\rho_s h}{K}} U_x \tag{55}$$

where ψ is the mass ratio, $\overline{\omega}$ is the dimensionless frequency, and \overline{U} is the dimensionless velocity.

The first example is a thin plate clamped on two opposite edges (see Fig. 7) subjected to flowing fluid on its upper and lower surfaces. The fluid level $(h_1 \text{ and/or } h_2)$ is assumed to be infinite. The corresponding dynamic pressure would be twice the

pressure calculated in Equation (29). The geometric ratios and dimensionless parameters for the structure are:

$$\psi = 0.93, h_1/A \rightarrow \infty, h_2/A \rightarrow \infty, A/B = 1$$

Numerical results were used to plot the curves shown in Fig. 8. We note that the plate becomes increasingly vulnerable to static instability as the rate of flow increases. Beyond the critical velocity we expect the occurrence of a large deflection of the plate [8].



Fig. 7: Plate clamped on two opposite edges subjected to flowing fluid $(h_1/A \rightarrow \infty \text{ and } h_2/A \rightarrow \infty)$



Fig. 8: Variation of frequency ϖ versus fluid velocity \overline{U} for plate clamped on two opposite edges subjected to flowing fluid.

In order to investigate the effect of boundary conditions on the critical velocity value, the same plate considered in the first example is studied again, but this time with the two opposite edges simply supported (see Fig. 9) instead of the two clamped edges. The variation of dimensionless frequencies for the first three modes versus dimensionless velocity of the fluid is shown in Fig. 10. It can be seen that the critical velocities for the first three modes are lower than those of the clamped plate. It can be concluded that clamped plates are more stable than simply supported plates which is in good agreement with the observations of Kim and Davis [8].



Fig. 9: Plate simply supported on two opposite edges subjected to flowing fluid





Fig. 10: Variation of frequency ϖ versus fluid velocity \overline{U} for a plate simply supported on two opposite edges subjected to flowing fluid.



Fig. 11: Cantilevered plate subjected to flowing fluid ($h_1/A \rightarrow \infty$ and $h_2/A \rightarrow \infty$)

We have calculated the critical velocities corresponding to the first three modes in the case of a cantilevered plate (see Fig. 11) which has the same material and geometrical parameters as those of the two previous examples. The case of a cantilevered plate subjected to flowing fluid is often encountered in practice. In Table 2 we have compared the critical velocities of a cantilevered plate with those of plates that are simply supported and clamped on two opposite edges (see Figs. 8 and 10). We conclude that the cantilevered plate is more vulnerable to static instability

Table 2: Dimensionless critical velocity (\overline{U}) of plates with various boundary conditions

	Dimensionless critical			
Boundary	velocity $\overline{U} = B\sqrt{\rho_s h/K} U_x$			
conditions	Mode 1	Mode 2	Mode 3	
Clamped on				
two opposite	9.58	11.36	18.83	
edges (Fig. 6)				
Simply				
supported on	4.22	6.98	15.83	
two opposite				
edges (Fig. 8)				
Cantilevered	1.494	3.73	9.25	
(Fig.10)				

Parallel-plate assemblies are often used in power nuclear reactors. Many thin plates are stacked in parallel and between the plates are narrow channels to let coolant flow through (see Fig. 12). All the plates have the same size and they are uniformly distributed. When channel height is relatively low, kinetic energy travels through the fluid from one plate to another. Vibration of the plates modifies the distribution of pressure and velocity along the channel. Therefore the fluid in the channels interacts simultaneously with both upper and lower plates. As mentioned previously, the plates vibrate according to two modes; in-phase and out-of-phase. The dynamic pressure for each case is distinct. It has been proven that the dynamic behaviour of parallel-plate assemblies clamped at two lateral walls can be sufficiently predicted using only one plate which vibrates in opposite directions relatively to its adjacent plates [4]. The model of Groninger and Kan [2] showed that the adjacent plates always move in opposite directions at high flow rates, causing alternate opening and closing of the channel. This condition provides the lower critical velocity [4]. Miller [13] derived relations expressing the critical velocity of an engineering test reactor system. For the case of a flat plate clamped on two opposite edges (see Fig. 12.a), the developed formula is:

$$U_{Miller} = \sqrt{\frac{15Eh_{l}h^{3}}{\rho_{f}(1-\nu^{2})B^{4}}}$$
(56)

where *E* is Young modulus, ρ_f is the fluid density, *h_i* is the channel height, ν Poisson's coefficient, *h* is the plate thickness and *B* is the plate width. Using our numerical model, we have calculated the out-of-phase vibrations of an internal plate for the parallel-plate assembly shown in Fig. 12.a.





(b)

Fig. 12: Engineering test reactor (ETR) system subjected to flowing fluid (a) parallel plates clamped to the lateral walls (b) parallel plates simply supported at the lateral walls

The corresponding dynamic pressure would be twice the pressure calculated in Equation (41). Fig. 13 shows the variation of the dimensionless critical velocities of the first out-of-phase mode as a function of channel height to plate length ratio computed by the present method and by Miller's analytical formula. By examining Fig. 13 it is clear that the critical velocity for a given plate can be increased by increasing the channel height. In the same figure we can see that at low fluid height good agreement is found between the numerical and analytical results, however for high fluid levels we observe a considerable discrepancy. This can be explained by the fact that Miller's formula is derived specifically for parallel plate systems with very low ratios (h_1/A) . On the other hand, it is important to note that beyond a certain fluid height, increasing ' h_1 ' or ' h_2 ' doesn't have any influence on the dynamic behaviour of a plate subjected to a flowing fluid. This will be confirmed in the following examples.



Fig. 13: Dimensionless critical velocity (\overline{U}) of an internal plate in an ETR system clamped on two opposite sides (see Fig. 12.a) versus channel height to plate length ratio (h₁/A), $\psi = 0.93$, A/B = 1.



Fig. 14: Dimensionless critical velocity (\overline{U}) of an internal plate in an ETR system simply supported on two opposite sides (see Fig. 12.b) versus channel height to plate length ratio (h₁/A),

 $\psi = 0.93, A/B = 1.$

Miller [13] has also derived a relation expressing the critical velocity of an engineering test reactor system when the parallel plates are simply supported on two opposite edges (see Fig. 12.b), the developed formula is:

$$U_{Miller} = \sqrt{\frac{5Eh^{3}h_{I}}{2\rho_{f}B^{4}(1-\nu^{2})}}$$
(57)

where *E* is Young modulus, ρ_f is the fluid density, h_i is the fluid level on the plate, ν Poisson's coefficient and B is the width of plate. We have used the solid-fluid finite element developed in this work to calculate the critical velocity corresponding to the first mode when adjacent simply supported plates move in opposite directions (out-of-phase mode). The results are obtained for different channel heights. Fig. 14 presents the dimensionless critical velocities computed for internal simply supported plates in an engineering test reactor. The agreement between our results and those calculated by Miller's formula is very good, especially for low channel heights.

The dynamic behaviour of structures may be influenced by changing the fluid level and/or fluid boundary conditions. Variation of the critical velocities as a function of fluid height and fluid boundary conditions are verified in the following examples. We will determine the limit values of fluid level (h_{1lim}) beyond which the increase of fluid level doesn't have any influence on the dynamic behaviour of structure. We initially considered the case of a plate clamped on two lateral sides placed in a channel of rigid walls (see Fig. 15). The plate is subjected to flowing fluid on both upper and lower surfaces. The corresponding pressure is twice the pressure calculated in Equation (33). We have gradually increased the fluid height and calculated for each (h_1) value the corresponding critical velocity for the first two modes. Fig. 16 shows that initially the critical velocity is increased by increasing the ratio (h_1/A) . However, there is a limit value for this ratio beyond which an increase in the fluid level doesn't change the critical velocity. For the case of a plate subjected to flowing fluid bounded by two rigid walls the limit value of this ratio is 0.5.



Fig. 15: Plate subjected to flowing fluid bounded by two rigid walls

We have also studied the effect of channel height on the critical velocity corresponding to the

first two modes of an internal plate in a parallelplate assembly (see Fig.11.a.). The dynamic pressure applied on each side is given by Equation (41). As shown in Fig. 17, the critical velocity first increases as channel height increases. When the ratio h_1/A reaches a value of 1, the dimensionless critical velocity remains constant even if we increase the channel height.



Fig. 16: Critical velocity of a single plate clamped on two opposite edges subjected to flowing fluid bounded by two rigid walls (see Fig. 15) versus fluid height to plate length ratio (h_I/A) ,

 $\psi = 0.93, A/B = 1$



Fig. 17: Critical velocity of an internal plate in parallel-plate assembly clamped on two opposite edges subjected to flowing fluid bounded by two elastic plates which vibrate in out-of-phase mode versus channel height to plate length ratio (h_1/A), $\psi = 0.93$, A/B = 1.

6 Conclusions

A solid-fluid finite element is developed for dynamic analysis of plates subjected to the dynamic

pressure induced by potential flow. The structural mathematical model is developed based on a combination of the finite element method and Sanders' shell theory. The in-plane and out-of-plane displacement components are modelled using bilinear polynomials and exponential functions, respectively. The general equations of the displacement functions are derived from the equilibrium equations of a rectangular plate.

The mass, damping and stiffness matrices corresponding to a solid and fluid are determined by exact analytical integration for each element. The fluid pressure is derived from a potential; it is a function of acceleration, velocity and the transverse displacement of the plate, respectively known as inertial, Coriolis and centrifugal effects.

Several plates with various boundary conditions were studied. The frequencies of vibration were calculated for each mean velocity of flow and for each fluid height, until the critical velocity was reached. Establishment of the critical velocity is very important for the design of plate systems subjected to fluid flow. We note that the boundary conditions and the fluid level on the plate strongly influence the dynamic behaviour of the plate.

The developed element can be used for analysis of rectangular plates with any boundary conditions contrary to previous analytical methods which were developed for particular cases.

The critical velocities calculated using our element agree well with those obtained using the analytical formulas derived by Miller, especially for the low fluid heights.

The limit value of fluid height was calculated for a plate subjected to flowing fluid bounded by two rigid walls and for an internal plate in an engineering test reactor system. The fluid-solid finite element developed in this work can be adapted to study the dynamic behaviour of structures with more complex forms subjected to flowing fluid forces.

Moreover, it is worthy to note that the present method can be used for rectangular plates, either uniform or non-uniform (thickness or other geometric discontinuities) subjected to any boundary conditions.

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Appendix

A.1 Non-zero elements of matrix [R] (3x24)

R(1,1) = 1	R(1,2) = x/A
R(1,3) = y/B	R(1,4) = xy/AB
R(2,5) = 1	R(2,6) = x/A
R(2,7) = y/B	R(2,8) = xy/AB

R(3,9) = 1	R(3,10) = x/A
R(3,11) = y/B	$R(3,12) = x^2/2A^2$
R(3,13) = xy/AB	$R(3,14) = y^2/2B^2$
$R(3,15) = x^3/6A^3$	$R(3,16) = x^2 y / 2AB$
$R(3,17) = xy^2/2AB$	$R(3,18) = y^3/2AB^2$
$R(3,19) = yx^3/6A^3B$	$R(3,20) = x^2 y^2 / 4A^2 B^2$
$R(3,21) = xy^3/6AB^3$	$R(3,22) = x^3 y^2 / 12 A^3 B^2$
$R(3,23) = x^2 y^3 / 12 A^2 B^3$	$R(3,24) = x^3 y^3 / 36 A^3 B^3$

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A.2 Matrix [R_f] (3x24)

 $R_{f}(3, j) = R(3, j)$ for j=1 to 24 $R_{f}(i, j) = 0$ for i=1 to 2 and j=1 to 24

A.3 Non-zero elements of Matrix [Q]

Q(1,2) = 1/AQ(1,2) = y/ABQ(2,7) = 1/BQ(2,8) = x/ABQ(3,3) = 1/BQ(3,4) = x/ABQ(3,8) = y/ABQ(3,6) = 1/A $Q(4,12) = -1/A^2$ $Q(4,15) = x/A^3$ $Q(4,16) = -y/A^2B$ $Q(4,19) = -xy/A^{3}B$ $Q(4,20) = -y^2/2A^2B^2$ $Q(4,22) = -xy^2/2A^3B^2$ $Q(4,23) = -y^3/6A^2B^3 Q(4,24) = -xy^3/6A^3B^3$ $Q(5,14) = -1/B^2$ $Q(5,17) = -x/AB^2$ $Q(5,18) = -y/B^3$ $Q(5,20) = -x^2/2A^2B^2$ $Q(5,22) = -x^3/2A^3B^2$ $Q(5,21) = -xy/AB^{3}$ $Q(5,23) = -x^2 y/6A^2B^3 Q(5,24) = -x^3 y/6A^3B^3$ Q(6,13) = -2/AB $Q(6,16) = -2x/A^2B$ $Q(6,17) = -2y/A^2B$ $Q(6,19) = -x^2/A^3B$ $Q(6,20) = -2xy/A^2B^2 Q(6,21) = -y^2/AB^3$ $Q(6,22) = -x^2 y / A^3 B^2 Q(6,23) = -x y^2 / A^2 B^3$ $O(6,24) = -x^2 y^2 / 2A^3 B^3$

A.5 Elasticity matrix P

The elasticity matrix [P] is of order (6x6). In the case of isotropic material the non vanishing terms of the elasticity matrix are:

 $P_{11} = P_{22} = D, P_{44} = P_{55} = K, P_{12} = P_{21} = vD,$ $P_{45} = P_{54} = vKP_{33} = (1 - v)D/2, P_{66} = (1 - v)K/2$ where $K = Eh^3/12(1 - v^2)$ and $D = Eh/(1 - v^2)$,

A.6 Non-zero elements of matrix [A]⁻¹

 $A^{-1}(1,1) = 1, A^{-1}(2,1) = A^{-1}(2,7) = -A/x$ $A^{-1}(3,1) = -A^{-1}(3,19) = -B/y_{a}$ $A^{-1}(4,1) = -A^{-1}(4,7) = A^{-1}(4,13) = -A^{-1}(4,19) = -AB/x_{0}y_{0}$ $A^{-1}(5,2) = 1$, $A^{-1}(6,2) = -A^{-1}(6,8) = -A/x_a$ $A^{-1}(7,2) = -B/y_{e}, A^{-1}(7,20) = -AB/x_{e}y_{e}$ $A^{-1}(8,2) = -A^{-1}(8,8) = A^{-1}(8,14) = A^{-1}(8,14) =$ $-A^{-1}(8,20) = AB/x$ $A^{-1}(9,3) = 1$, $A^{-1}(10,4) = A$, $A^{-1}(11,5) = B$ $A^{-1}(12,3) = -A^{-1}(12,9) = -6A^2/x_e^2$ $A^{-1}(12,4) = 2A^{-1}(12,10) = -4A^2/x_a$ $A^{-1}(13,6) = A.B$ $A^{-1}(14,3) = -A^{-1}(14,21) = -12A^3/x_a^2$ $A^{-1}(14,5) = A^{-1}(14,3) = 2A^{-1}(14,23) = 2B/y_a$ $A^{-1}(15,3) = -A^{-1}(15,9) = 12 A^3 / x_e^3$ $A^{-1}(15,4) = A^{-1}(15,10) = 6A^3/x_e^2$ $A^{-1}(16.5) = -A^{-1}(16.11) = -6 A^2 B/x_a^2$ $A^{-1}(16,6) = A^{-1}(16,12) = -2 A^2 B/x_e$ $A^{-1}(17,4) = -A^{-1}(17,22) = -6 AB^2 / v_{\pm}^2$ $A^{-1}(17,6) = A^{-1}(17,24) = -4AB^2/y_{a}$ $A^{-1}(18,3) = -A^{-1}(18,21) = 12 B^{3}/y_{e}^{3}$ $A^{-1}(18.5) = A^{-1}(18.23) = 6 B^3 / v_{\pm}^2$ $A^{-1}(19,5) = A^{-1}(19,11) = 12 A^{3} B / x_{e}^{3}$ $A^{-1}(19,6) = A^{-1}(19,12) = 6 B A^3 / x_a^2$ $A^{-1}(20,4) = 2A^{-1}(20,10) = -A^{-1}(20,16) =$ $-A^{-1}(20,22) = 24 B^2 A^2 / x_a y_a^2$ $A^{-1}(20,5) = -2A^{-1}(20,9)/3 = -A^{-1}(20,11) =$ $-2A^{-1}(20,17) = 2A^{-1}(20,23) = 24A^{2}B^{2}/x_{2}^{2}y_{3}$

$$\begin{split} &A^{-1}(20,6) = 2A^{-1}(20,12) = 4A^{-1}(20,18) = \\ &2/3A(20,24) = 16A^2B^2/x_e y_e \\ &A^{-1}(21,4) = -A^{-1}(21,22) = 12AB^3/y_e^3 \\ &A^{-1}(21,6) = A^{-1}(21,24) = 6AB^3/y_e^2 \\ &A^{-1}(22,3) = -A^{-1}(22,9) = A^{-1}(22,15) = \\ &-A^{-1}(22,21) = 72AB^2/x_e^3 y_e^2 \\ &A^{-1}(22,24) = A^{-1}(22,10) = -A^{-1}(22,16) = \\ &-A^{-1}(22,22) = -36A^3B^3/x_e^2 y_e^2 \\ &A^{-1}(22,5) = -A^{-1}(22,11) = -2A^{-1}(22,17) = \\ &2A^{-1}(22,23) = -48A^3B^2/x_e^3 y_e \\ &A^{-1}(22,24) = -24A^3B^2/x_e^2 y_e \\ &A^{-1}(23,3) = -A^{-1}(23,9) = A^{-1}(23,15) = \\ &A^{-1}(23,21) = 72A^2B^3/x_e^2 y_e^2 \\ &A^{-1}(23,22) = -48A^2B^3/x_e y_e^3 \\ &A^{-1}(23,22) = -48A^2B^3/x_e y_e^3 \\ &A^{-1}(23,5) = -A^{-1}(23,11) = -A^{-1}(23,17) = \\ &A^{-1}(23,23) = -36A^2B^3/x_e^2 y_e^2 \\ &A^{-1}(23,24) = -24A^2B^3/x_e y_e^3 \\ &A^{-1}(23,24) = -24A^2B^3/x_e y_e^3 \\ &A^{-1}(24,3) = -A^{-1}(24,19) = A^{-1}(24,15) = \\ &-A^{-1}(24,21) = 144A^3B^3/x_e^3 y_e^3 \\ &A^{-1}(24,22) = 72A^3B^3/x_e^2 y_e^2 \\ &A^{-1}(24,23) = -72A^3B^3/x_e^2 y_e^3 \\ &A^{-1}(24,24) = 36A^3B^3/x_e^2 y_e^2 \\ &A^{-1}(24,24) = 36A^3B^3/x_e^2 y_e^2 \end{split}$$