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# Analysis of Nonlinear Isentropic Sound Wave-Propagation in a Cylindrical Tube Filled with Fluid-Saturated Porous Media

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*Abstract:* - A theory of sound waves propagation in porous media that includes the nonlinear effects of Forchheimer type with nonzero radial velocity effects is laid out utilizing variational solutions technique. It is shown that the main parameters governing the propagation of sound waves are shear wave number, reduced frequency number, porosity, Darcy number, and Forchheimer number. The manner in which the flow influences the attenuation and the phase velocities of forward and backward propagating isentropic acoustic waves is deduced. It is found that increasing Darcy number and Forchheimer number increased wave's attenuation and phase velocity for both forward and backward sound waves, whereas increasing the porosity decreased attenuation and phase velocity for both waves. The effect of increasing the reduced frequency is found to increase attenuation of the forward waves and decrease attenuation and phase velocity of the backward sound waves. Moreover, the effect of the steady flow is found to decrease the attenuation and phase velocities for forward sound waves and enhance them for the backward sound waves.

*Key-Words:* - sound waves, porous medium, fluid flow, noise isolator.

## 1 Introduction

The acoustic wave propagation problem finds its application in many engineering applications such as geophysical exploration, seismology, earthquake engineering, and rock dynamics, etc. For instance, if the acoustic improvements are restricted to interior spaces, porous material such as mineral wools or open pore foams are utilized to fill double wall cavities, and floors and ceilings as noise isolators, which helps control reverberation time, and avoid undesired reflections. If acoustic improvements are restricted to outdoor problems, granular materials such as porous concrete or similar materials are utilized as acoustic noise barriers against traffic noise, as such materials behave better with bad weather and other atmospheric phenomena.

In porous materials the absorption process of the acoustic wave takes place through viscosity and thermal losses of the acoustic energy inside the micro tubes forming the material. The problem of propagation of sound waves in fluids contained in a plain medium is a classical one, to which famous names are tied to, like Helmholtz [1], Kirchhoff [2] and Rayleigh [3]. By extending the problem of a single tube model to a bulk porous media, the basic equations for the wave propagation in anisotropic porous media were first formulated by Biot [4] [5]. A variational treatment of the problem of sound transmission in narrow tubes is described by Cummings [6] as an alternative to the more popular analytical procedure, which is limited to mathematically tractable geometries. A first approximation to the effects of mean flow on sound propagation through cylindrical capillary tubes is achieved by Peat [7]. Furthermore, sound transmission in narrow pipes with superimposed uniform mean flow and acoustic modeling of automobile catalytic converters is achieved by Dokumaci [8]. Jeong and Ih [9] studied numerically the propagation of sound waves through capillary tubes with mean flow, whereas an approximate dispersion equation for sound waves in a narrow pipe with ambient gradients is formulated by Dokumaci [10]. Recently, a thermoacoustic theory for a bulk random medium is developed from that for a single pore, based on parallel, capillary-tube-based theories by Roh et al [11]. In their work, basic equations such as the Navier-Stokes equation, the equation of state, and the equation of heat transfer for a bulk porous medium are formulated for the derivation of the thermoacoustic wave equation as analogously as possible to those for a single pore.

The problem of sound waves propagation in a stationary or flowing fluid in a porous medium is not addressed yet. An attempt is made in this article to develop a simplified nonlinear theory that predicts the propagation characteristics of a stationary or flowing fluid in saturated porous media. This theory is an extension of the classical plain medium theory, utilizing a modification to Darcy's law due to the Forchheimer effects and assuming a nonzero radial velocity effects. Analytical expressions for the propagation constant are obtained from variational solutions. Comparison with previous work in the limit of plain medium shows an excellent agreement.

## 2 Problem Formulation

Consider a rigid tube filled with a saturated porous material, the fluid is assumed to be stationary or moving inside the tube. The x-coordinate is measured along the tube and the r-coordinate is measured normal to the axial direction. Under the boundary layer approximations the basic equations which govern acoustic wave propagation in a rigid tube filled with a porous media are the continuity and momentum equations, as given below:

$$\varepsilon\frac{\partial\rho^*}{\partial t^*}+u^*\frac{\partial\rho^*}{\partial x^*}+v^*\frac{\partial\rho^*}{\partial r^*}+\rho^*\left(\frac{\partial v^*}{\partial r^*}+\frac{v^*}{r^*}+\frac{\partial u^*}{\partial x^*}\right)=0 \quad (1)$$

$$\rho^*\left[\varepsilon^{-1}\frac{\partial u^*}{\partial t^*}+\varepsilon^{-2}\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial r^*}\right)\right]=-\frac{\partial p^*}{\partial x^*} \quad (2)$$

$$-\frac{\mu}{K}u^*-\frac{C_F\rho^* u^{*2}}{K^{1/2}}+\mu\varepsilon^{-1}\left(\frac{\partial^2 u^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial u^*}{\partial r^*}\right)$$

$$-\frac{\partial p^*}{\partial r^*}=0 \quad (3)$$

Where $u^*, v^*$ are the velocity components in the axial and normal directions; respectively. $\rho^*$ and $p^*$ are the fluid density and pressure, $\mu$ is the absolute viscosity, K and $\varepsilon$ are the permeability and porosity of the porous medium, respectively. Since one is dealing not only with capillary tubes, the radial velocity effect might be expected to be significant. This effect of the radial velocity couples the continuity equation (3) and the momentum equation (4). It is assumed that the flow through the capillary duct is a superposition of a fully developed laminar, incompressible, axial steady flow and a small harmonic acoustic disturbance of frequency $\omega$. The steady flow is taken to have constant density $\bar{\rho}$ and a speed of sound $\bar{a}$ such that the fluid variables can be expanded in the form:

$$\rho^*=\bar{\rho}\left(1+\alpha\rho(\eta)e^{\Gamma\xi}e^{i\omega t^*}\right) \quad (4)$$

$$u^*=\bar{a}\left(M_0(\eta)+\alpha u(\eta)e^{\Gamma\xi}e^{i\omega t^*}\right) \quad (5)$$

$$v^*=\bar{a}\alpha v(\eta)e^{\Gamma\xi}e^{i\omega t^*} \quad (6)$$

$$p^*=(\bar{\rho}\bar{a}^2/\gamma)\left(p_0(\xi)+\alpha p(\eta)e^{\Gamma\xi}e^{i\omega t^*}\right) \quad (7)$$

Where $\alpha << 1$ and $\gamma$ is the ratio of specific heats. It is seen that the steady flow variables $p_0$ and Mach number $M_0$ together with acoustic variables $\rho, u, v$ and $p$ are dimensionless. Now, introduce the following variables in the transformations:

$$\xi=\omega x^*/\bar{a}\,;\ \eta=r^*/R \quad (8)$$

Where $R$ is the radius of the capillary duct. Assume that the axial acoustic wave motion has a complex propagation constant $\Gamma$, which can be expanded in the form:

$$\Gamma=\Gamma'+i\Gamma'' \quad (9)$$

Where $\Gamma'$ represents the wave attenuation per unit distance and $\Gamma''$ represents the phase shift over the same distance. The assumed forms of the variables, equations (4-7) are substituted into the governing equations (1), (2) and (3), while terms of similar order of $\alpha$ are equated. It is found that for zeroth order, the steady flow solution, equations of continuity and radial momentum are identically satisfied, thus the axial momentum equation (3) takes the following form:

$$\frac{s^2}{\gamma}\frac{dp_0}{d\xi}=\frac{1}{\varphi\eta}\frac{d}{d\eta}\left(\eta\frac{dM_0}{d\eta}\right)-Da^2M_0-\frac{C_F s^2}{k}M_0^2 \quad (10)$$

Here $s=R\sqrt{\bar{\rho}\omega/\mu}$ is the shear wave number, $k=\omega R/\bar{a}$ is the reduced frequency parameter, $Da=R^2/K$ is the Darcy number and $C_F$ is the Forchheimer number. This is the classical equation of Hagen-Poiseuilli flow, the solution of which, with no-slip boundary conditions, gives a parabolic velocity profile:

$$M_0=\frac{s^2}{\gamma}\frac{dp_0}{d\xi}\left(\frac{1-\eta^2}{4}\right)=2\bar{M}(1-\eta^2) \quad (11)$$

Where $\bar{M}$ is the mean Mach number of the steady flow. The linearized acoustic equations follow from equating terms of first order of $\alpha$ in the governing equations, are:

$$k\left[\frac{i\rho}{\varepsilon}+\Gamma u+2\bar{M}\Gamma(1-\eta^2)\rho\right]+\frac{dv}{d\eta}+\frac{v}{\eta}=0 \quad (12)$$

$$\left[\frac{iu}{\varepsilon}+\frac{2\bar{M}\Gamma}{\varepsilon^2}(1-\eta^2 u]+\frac{4\bar{M}}{k\varepsilon^2}\eta v=(-\Gamma/\gamma)p+ \quad (13)$$

$$(1/s^2\varepsilon)\left[\frac{d^2u}{d\eta^2}+(1/\eta)du/d\eta\right]-(Da^2/s^2)u-(2C_F Da/k)M_0 u$$

The case of $\varepsilon=1$ or $Da=0$ corresponds to the plain medium without the presence of the solid matrix, and any values of $0<\varepsilon<1$ or $Da>0$ represent a porous medium with different pore spaces. For the case of $\varepsilon=1$ and $Da=0$, the governing equations (12) and (13) reduces to the case of a pure plain medium obtained by Peat [5]. In the limit of zero steady flow, $\bar{M}=0$, these equations are found to be reduced to those for the reduced frequency solution of Tijdeman [12].

It will be assumed that the tubes are rigid, which implies the no-slip boundary condition of the fluid velocity at the wall of the tube, i.e.,

$$u = 0 \text{ at } \eta = 1 \tag{14}$$

The solution of equations (12)-(13) is greatly simplified if one assumes that the acoustic disturbances occur isentropically, i.e.,

$$p = \rho\gamma \tag{15}$$

## 3 Variational Solution

As the pressure, $p$ , is constant over the radial cross-section, the problem reduced to only solving the continuity and momentum equations for the velocity components and pressure. Variational solutions will be obtained based on the following trial parabolic form of acoustic velocity variation,

$$u = C(1-\eta^2) \quad \text{where } c \text{ is constant} \tag{16}$$

Consider the continuity equation (12), with the assumption of isentropic disturbances and the given form of the trial solution of the axial velocity, equations (15) and (16), and integrating the resulted expression with the boundary condition of $v = 0$ when $\eta = 0$ yields:

$$-\frac{\eta v}{k} = \frac{\eta^2}{2}\left[\frac{i}{\varepsilon} + 2\bar{M}\Gamma\left(1-\frac{\eta^2}{2}\right)\right]\frac{p}{\gamma} + \Gamma\frac{\eta^2}{2}\left(1-\frac{\eta^2}{2}\right)\frac{u}{1-\eta^2} \tag{17}$$

This expression can now be substituted into the full momentum equation (13) to give:

$$\frac{iu}{\varepsilon} + \frac{2\bar{M}\Gamma}{\varepsilon^2}(1-\eta^2)u - \frac{4\bar{M}}{\varepsilon^2}\frac{\eta^2}{2}\left[\frac{i}{\varepsilon} + \frac{2\bar{M}\Gamma}{\varepsilon^2}\left(1-\frac{\eta^2}{2}\right)\right]\frac{p}{\gamma} + \left(1-\frac{\eta^2}{2}\right)\Big]\frac{u}{(1-\eta^2)\Big]} = (-\Gamma/\gamma)p + \frac{1}{s^2\eta\varepsilon}\left[\frac{d}{d\eta^2}\left(\eta\frac{du}{d\eta}\right)\right] - (Da^2/s^2)u - \frac{C_s^* Da\bar{M}}{k} \tag{18}$$

where $C_s^* = 2C_F$ . Equation (18) corresponds to the minimum of the functional:

$$G = \int_0^1 \begin{bmatrix} (\eta/s^2\varepsilon)\left(\frac{du}{d\eta}\right)^2 + \frac{iu^2\eta}{\varepsilon} + \frac{2\bar{M}\Gamma}{\varepsilon^2}(1-\eta^2)\eta u^2 + \\ \frac{4i\bar{M}}{\varepsilon^3}\frac{p}{\gamma}\eta^3 u + \frac{8\bar{M}\Gamma}{\varepsilon^2}\frac{p}{\gamma}\eta^3\left(1-\frac{\eta^2}{2}\right)u + \\ \frac{\bar{M}\Gamma}{\varepsilon^2}\eta^3(2-\eta^2)\frac{u^2}{1-\eta^2} + \frac{2\Gamma p}{\gamma}\eta u + \\ \frac{Da^2}{s^2}\eta u^2 + \frac{C_s^* Dau^2\eta(2\bar{M}(1-\eta^2)}{k} \end{bmatrix} d\eta \tag{19}$$

The assumed form of trial solution for $u$, equation (16), is substituted into this expression and the minimum is found by setting:

$$\partial G/\partial C = 0 \tag{20}$$

This results in an expression for the constant C; as in equation (20),

$$C = -\frac{p}{\gamma}\left(\frac{\Gamma}{2} + \frac{i\bar{M}}{3\varepsilon^3} + \frac{\bar{M}^2\Gamma}{2\varepsilon^2}\right) \Bigg/ \left(\frac{2}{s^2\varepsilon} + \frac{i}{3\varepsilon} + \frac{3\bar{M}\Gamma}{4\varepsilon^2} + \frac{Da^2}{6s^2} + \frac{C_s^* Da}{4k}\right) \tag{21}$$

Substitution the same trial solution of $u$ into equation (17) and utilizing the boundary condition of $v = 0$ at $\eta = 1$ leads to a second expression of the propagation constant,

$$C = -\frac{2p}{\Gamma\gamma}\left(\frac{i}{\varepsilon} + \bar{M}\Gamma\right) \tag{22}$$

Equations (21) and (22) enable C to be eliminated which results in an expression for the propagation constant:

$$\left(1-\frac{2\bar{M}^2}{\varepsilon^2}\right)\Gamma^2 - \left(\frac{8}{s^2\varepsilon} + \frac{4i}{3\varepsilon} + \frac{3i}{\varepsilon^3} - \frac{2i}{3\varepsilon^3} + \frac{4Da^2}{3s^2} + \frac{C_s^* Da}{k}\right)\bar{M}\Gamma + \left(\frac{4}{3\varepsilon^2} - \frac{8i}{s^2\varepsilon^2} - \frac{4Da^2 i}{3s^2\varepsilon} - \frac{C_s^* Dai}{k\varepsilon}\right) = 0 \tag{23}$$

Solving for the propagation constant yields the following formula:

$$\Gamma = \frac{\left(\frac{8}{s^2\varepsilon} + \frac{3i}{\varepsilon^3} + \frac{4i}{3\varepsilon} + \frac{4Da^2}{3s^2} - \frac{2i}{3\varepsilon^2} + \frac{C_s^* Da}{k}\right)\bar{M}}{2\left(1-\frac{2\bar{M}^2}{\varepsilon^2}\right)} \pm \frac{\left\{\left(\frac{8}{s^2\varepsilon} + \frac{3i}{\varepsilon^3} + \frac{4i}{3\varepsilon} + \frac{4Da^2}{3s^2} - \frac{2i}{3\varepsilon^2} + \frac{C_s^* Da}{k}\right)^2 \bar{M}^2 - 4\left(1-\frac{2\bar{M}^2}{\varepsilon^2}\right)\left(\frac{4}{3\varepsilon^2} - \frac{8i}{s^2\varepsilon} - \frac{4Da^2 i}{3s^2\varepsilon} - \frac{C_s^* Dai}{k\varepsilon}\right)\right\}^{1/2}}{2\left(1-\frac{2\bar{M}^2}{\varepsilon^2}\right)} \tag{24}$$

Note that when $\varepsilon = 1$ or $Da = 0$ the propagation constant, equation (23) is reduced to those obtained by Peat [5] for the case of a pure plain medium. It is important also to note that the $\bar{M}$ number will reflect the effect of steady flow on the acoustic problem under consideration; the case of $\bar{M} = 0$ corresponds to the absence of mean flow velocity and to the acoustic problem in a stationary porous media.

## 4 Results and Discussion

Comparison of variational solution with exact solution as given by Peat [5] in the limits of plain medium for $\varepsilon = 1$ and $Da = 0$ is shown in Table 1. The table shows a good agreement between present results and Peat's results for both attenuation and phase shift.

Figure 1 is a plot of the modulus of wave attenuation per unit distance, $\Gamma'$, and phase shift, $|\Gamma''|$, for varying shear wave number, Mach number, and for $Da = 10, C_s^* = 0.1, \varepsilon = 0.8$ and $k = 0.15\pi$ . It is clear that as the Mach number is increased the attenuation is decreased and the phase velocities are increased for the forward waves, whereas both the attenuation and phase velocities for the backward sound waves increased; this result is due to the collision effects of the forward sound waves and favorable vertical velocity effects which results in more damping of the backward sound waves. For shear wavenumber less than one and the limit of zero steady flow, the attenuation of the forward and backward waves decreased rapidly and reached a constant plateau.

Figure 2 shows the effect of increasing Darcy numbers $Da = 0, 0.1, 1, 5, 10$ for $\overline{M} = 0.1, C_s^* = 0.1$, it is clear that as the Darcy number is increased the attenuation and phase velocities for both the forward and backward sound waves decreased; this is due to favorable effects of the solid matrix in damping sound waves.

Figure 3 shows the effect of porosity on attenuation and phase velocities for selected values of $Da = 10, C_s^* = 0.1$, $\overline{M} = 0.1$ and $k = 0.15\pi$, it is shown that increasing the porosity decreases the attenuation and phase velocities for both the forward and backward waves; this is due to the small effect of the solid matrix as moving toward the plain media limit. Figure 4 shows the effect of Forchheimer term $C_s^* = 0.1, 1, 5, 10$ on attenuation and phase velocities for selected values of Darcy number and reduced frequency, it is found that as the Forchheimer term is increased the attenuation and phase velocities are increased for the forward and backward sound waves; this is due to favorable damping effects of the fluid inside the large pores of the solid matrix. Finally, Figure 5 shows the effect of increasing reduced frequency parameter on the attenuation and phase velocities for Darcy number of 10, porosity of 0.8, and Mach number of 0.1 , it is found that as the reduced frequency is increased, the attenuation is increased and the phase velocities are decreased for the forward sound waves, whereas both the attenuation and phase velocities are decreased for the backward sound waves; this is could be attributed to the higher frequency of the impacted sound waves on the solid matrix. It is important to note that the same behavior is noticed for sound waves propagated in a plain medium.

# 4 Conclusion

A simplified nonlinear theory which predicts the propagation characteristics of sound wave in a stationary or flowing fluid in saturated porous media has been developed. Based on the results presented in this article:

1- It is demonstrated that the main parameters governing the propagation of sound waves are shear wavenumber, reduced frequency number, porosity, Darcy number, and Forchheimer number.

2- It is found that the effect of increasing Darcy number or Forchheimer number is to increase the attenuation and phase velocities for both forward and backward sound waves; this is due to favorable role of solid matrix in damping sound waves.

3- It is also found that the effect of increasing porosity or reduced frequency parameter is to decrease attenuation and phase velocities for both forward and backward sound waves; this is due to the absence of the role of porous matrix and high incident sound waves strength, respectively.

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*Nomenclature*

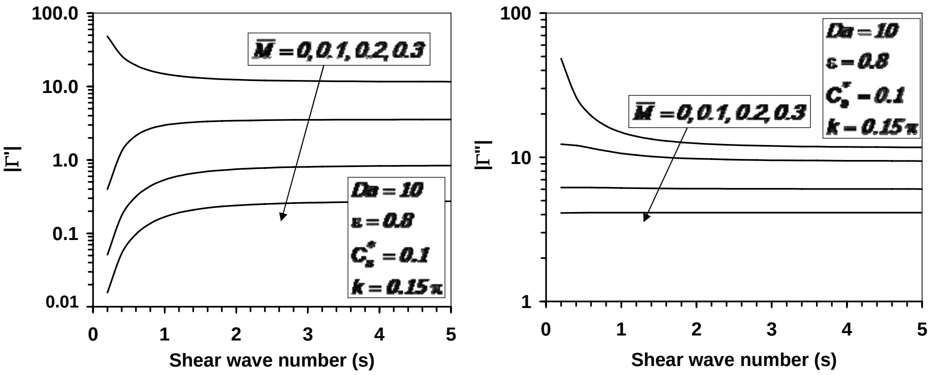
| | |
|---|---|
| $\bar{a}$ | Mean speed of sound of steady flow |
| $C$ | Constant defined in equation (16) |
| $Da$ | Darcy number, $R^2/K$ |
| $i$ | Imaginary number, $\sqrt{-1}$ |
| $M_0$ | Steady flow Mach number |
| $\overline{M}$ | Mean Mach number of steady flow, |
| $p$ | Acoustic pressure |
| $p_0$ | Steady flow pressure |
| $s$ | Shear wave number |
| $u^*, v^*$ | The velocity components in *x*-and *y*-directions |
| $u, v$ | Acoustic velocity components in *x*- and *y*-directions |
| $x^*, r^*$ | Axial and normal coordinates |

*Greek symbols:*

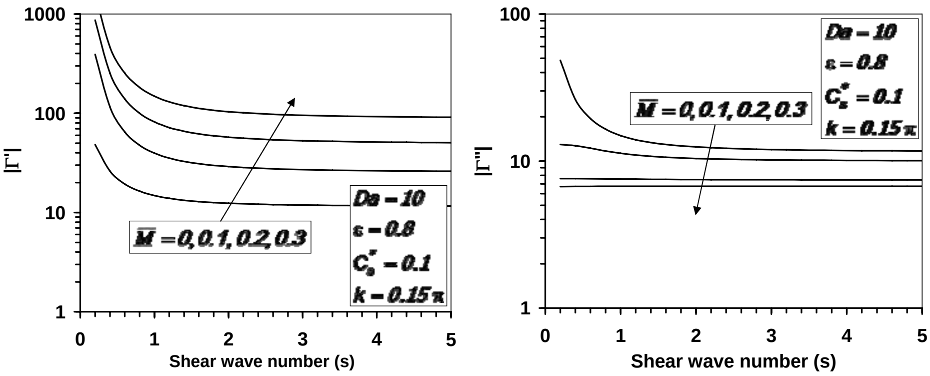
| | |
|---|---|
| $\alpha$ | Perturbation parameter |
| $\rho$ | Acoustic density |
| $\bar{\rho}$ | Mean steady flow density |
| $\rho^*$ | Fluid density |
| $\xi$ | Dimensionless axial coordinate |
| $\eta$ | Dimensionless normal coordinate |
| $\gamma$ | Ratio of specific heats |
| $\omega$ | Harmonic disturbance frequency |
| $\varepsilon$ | Porosity |
| $\mu$ | Dynamic viscosity |
| $\Gamma$ | Propagation constant |
| $\Gamma'$ | Attenuation |
| $\Gamma''$ | Phase shift angle |

Table 1. Comparison of the results of attenuation $|\Gamma'|$ and phase shift $|\Gamma''|$ for the case of $\bar{M} = 0$

| Shear wavenumber, s | $\lvert\Gamma'\rvert$ | | $\lvert\Gamma''\rvert$ | |
|---|---|---|---|---|
| | Present | Peat [5] | Present | Peat [5] |
| 0.2 | 9.967 | 9.975 | 10.033 | 10.025 |
| 0.4 | 4.934 | 4.950 | 5.067 | 5.050 |
| 0.6 | 3.235 | 3.259 | 3.435 | 3.409 |
| 0.8 | 2.370 | 2.402 | 2.637 | 2.602 |
| 1.0 | 1.841 | 1.879 | 2.173 | 2.129 |
| 2.0 | 0.732 | 0.786 | 1.367 | 1.272 |
| 3.0 | 0.367 | 0.411 | 1.212 | 1.081 |
| 4.0 | 0.213 | 0.243 | 1.174 | 1.029 |
| 5.0 | 0.138 | 0.158 | 1.163 | 1.012 |
| 6.0 | 0.096 | 0.110 | 1.159 | 1.006 |
| 7.0 | 0.071 | 0.081 | 1.157 | 1.003 |
| 8.0 | 0.054 | 0.062 | 1.156 | 1.002 |




(a) Attenuation and Phase Shift – Forward wave




(b) Attenuation and Phase Shift – Backward wave

Fig. 1 Effect of Mach number on attenuation and phase shift