Constitutive relations and conditions for reciprocity in bianisotropic media (Macroscopic approach)

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Abstract: - The paper deals with reciprocity relations in bianisotropic electromagnetic materials. Different forms of material constitutive relations are discussed and reciprocity conditions for different approaches to their derivation are considered.

Key-Words: - Reciprocity principle, bianisotropic electromagnetic media, constitutive relations, reciprocal materials

1 Introduction

For any electromagnetic materials, including the wide class of so called bianisotropic (BA) materials it is important to consider in what extent may be some general physical ideas helpfull in material research. In the phenomenological electromagnetics there are three general restrictions imposed on the material parameters. These are:

- restriction following from Onsager-Casimir principle expresses the symmetry of kinetic coefficients (expressed as so called Onsager-Casimir reciprocity relations),
- restrictions which follow from spatial symmetry of medium structure,
- restriction prescribed from the energy conservation principle.

Some problems are connected with the choise of constitutive relation for BA media form. Present author (in Sec. 2) prefer the so called E-B representation which is more suitable from physical point of view, namely for understanding the connection between microscopic and macroscopic treatment to material properties. The application od reciprocity principle known from electromagnetic field theory is based on the time-reversal symmetry of electromagnetic field equations. Then Onsager reciprocity relations and/or Onsager-Casimir reciprocity relations (Sec. 3) well-known from linear non-equilibrium thermodynamics, allowed express the macroscopic consequences of microscopic reversibility and may be used for derivation of material reciprocity conditions (in linear scope). Some misunderstandings are connected with the derivation of condition for material reciprocity.

Even nowdays one can see in the literature the different points of view (Sec.4.2) on this problem. For appreciate the medium to be reciprocal or not, the symmetry properties of field equations and of material relations are important. One must consider the symmetry with respect to time inversion (which decides on the validity of reciprocity principle) and the space-symmetry properties (parity) which dictate the structure of the material tensors.

2 Constitutive relations in electromagnetic field

2.1 Different forms of the constitutive relations

The relations between fields vectors in dielectric and/or magnetic media can be expressed as functional dependence in the form

$$\vec{D} = \vec{D}(\vec{E},\vec{H}),\ \vec{B} = \vec{B}(\vec{E},\vec{H}) \tag{1}$$

or

$$\vec{D} = \vec{D}(\vec{E},\vec{B}),\ \vec{H} = \vec{H}(\vec{E},\vec{B}). \tag{2}$$

In the first approach are $\vec{E}$ and $\vec{H}$ considered as field vectors (so called primary fields) while $\vec{D}$ and $\vec{B}$ as excitation vectors (so called induction fields). We will prefer here the approach (2) in which are $\vec{E}$ and $\vec{B}$ considered as field vectors (primary fields) while $\vec{D}$ and $\vec{H}$ as excitation vectors (induction fields), e.g. [1, 2]. Both approaches are used by many authors and for using each of them are in macroscopic treatment acceptable arguments. From macroscopic point of view are both forms applicable. The approach (2) is more convenient/worth from physical point of view (see e.g. [3]) namely for understanding the connection between microscopic and macroscopic treatment of material properties description. In this case the constitutive relations we can write in the form (in the frequency domain)

$$\vec{D} = \overline{\overline{\varepsilon}}\vec{E} + \overline{\overline{\alpha}}\vec{B}\,,\ \vec{H} = \overline{\overline{\beta}}\vec{E} + \overline{\overline{\mu^{-1}}}\vec{B}\,, \qquad \text{(3a,b)}$$

where the field $\vec{D}$ and $\vec{H}$ are expressed as linear functions of the fields $\vec{E}$ and $\vec{B}$. The constitutive relations are named E-B (or Boys-Post) representation. The tensors $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$ represent the medium permittivity and permeability and the tensors $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ describe the cross coupling between the electric and magnetic fields (so called magnetoelectric crosscoupling). All named tensors are functions of electromagnetic field frequency ω. The four 3 x 3 tensors of the second rank in the most general form contain 36 independent parameters. The relations (3) describe a broad class of media with spatial and frequency dispersion. If $\overline{\overline{\varepsilon}}$, $\overline{\overline{\alpha}}$, $\overline{\overline{\beta}}$ and $\overline{\overline{\mu}}$ are scalars, the medium is called biisotropic medium.

Note that constitutive relations (3) may be written in a cartesian coordinate system as

$$D_j = \sum_{k=1}^{3} \varepsilon_{jk} \circ E_k + \alpha_{jk} \circ B_k\,,$$
$$H_j = \sum_{k=1}^{3} \beta_{jk} \circ E_k + \mu^{-1}{}_{jk} \circ B_k\,. \qquad \text{(4a,b)}$$

Here operation $\circ$ indicates a temporal evolution operation in the time domain and means simple multiplication in the frequency domain.

There are several classes of materials that manifest magnetoelectric behavior and which are modelled by coupling between the electric and magnetic fields in the constitutive relations.

The media for them the constitutive relations (3) – (4) are valid are generally named bianisotropic media (BA media).

2.2 Conection between E-B and E-H approach

In modern E-B concept of microphysical basis of electromagnetism the macroscopic field is conceived as piecewise homogeneous entities not varying continuously in spacetime. In contrast, the E-H concept is essentially macroscopic. The present electromagnetism is a microscopic science, even though it is mostly used in its macroscopic form. In microscopic approach are the following two (primitive) fields: the electric field $\vec{E}$ and the magnetic field $\vec{B}$ and both vary extremely rapid as function of position $\vec{x}$ and time t. Their sources are the microscopic charge density $c(\vec{x},t)$ and the microscopic current density $\vec{j}(\vec{x},t)$. Macroscopic measuring devices average over relatively large spatial and temporal intervals. By means of macroscopic charge density $\rho(\vec{x},t)$ and current density $\vec{J}(\vec{x},t)$ is possible formulate macroscopic Maxwell equations (in which are the free charge and free current densities considered as sources) [4, 5].

For induced fields $\vec{D}$ and $\vec{H}$ the following well-known relations are valid

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}\,,\ \vec{H} = \mu_0^{-1}\vec{B} - \vec{M}\,, \qquad (5)$$

where $\vec{P}$ and $\vec{M}$ are electric polarization and magnetization.

3 Reciprocity relations

3.1 Onsager reciprocity relations (ORR) and Onsager - Casimir reciprocity relations (OCRR)

In two seminal papers published in 1931 [6] L. Onsager derived, with the assumption of microscopic reversibility, (i.e. reversibility at the microscopic scale) a set of reciprocity relations applicable to proper and coupled linear phenomena which are valid at macroscopic length scale, in variety of situations.

In 1945 H.B. Casimir [7] improved the foundations of the ORR and formulated generalized reciprocity relations (OCRR). ORR and OCRR were initially considered as applicable to purely instantaneous phenomena or, at least, in processes when time lag may be neglected. Later were reciprocity relations (RR) widened in their scope as a result of the fluctuation-dissipation theorem by H.B. Callen and R.F. Green (1952) and then for time-harmonic phenomena by H.B.Callen, M.L. Barash and J.L. Jackson (1952) [8, 9].

3.2 Onsager´s approach

Onsager derived the relations between so-called thermodynamic fluxes J_i and thermodynamic forces X_i. He obtained the relations (see e.g. [6])

$$J_i = \sum_j L_{ij} X_j \qquad (i, j = 1, 2, \dots, f) \tag{6}$$

with

$$L_{ij} = L_{ji}\,. \tag{7}$$

The phenomenological coefficients L_{ij} are constant, i.e. independent on X_j . Coefficients of type L_{ij} $(i = j)$ are called proper and these of type L_{ij} $(i \neq j)$ are called mutual or cross coefficients.

Let we note that ORR cannot be obtained from purely macroscopic arguments alone, but only with the help of supplementary hypotheses, such as the time reversal symmetry of equilibrium correlation functions and linear regression of fluctuations. In macroscopic approach ORR must be postulated.

3.3 Casimir´s approach

We can here remember that ORR were derived for α-type parameters which are the even functions of particles velocities. Casimir gave a modification of ORR for so-called β-type parameters, which are the odd functions of particle velocities.

The general form of OCRR include both mentioned situations, e.g. [7, 10]. In the scalar form is

$$L_{ij} = \varepsilon_i\, \varepsilon_j\, L_{ji} \qquad (i, j = 1, 2, \dots, f)\,, \tag{8}$$

where *f* is the number of independent scalar fluxes and (in this time) forces which are present in this linear constitutive equations. Here $\varepsilon_i = \varepsilon_j = 1$ for the case that coefficients L_{ij} deal with crosseffects which can be described by only α-type or only by β-type parameters.

In mixed (combined) case $\varepsilon_i = 1$ and $\varepsilon_j = -1$. It means that

$$\varepsilon_i \varepsilon_j = 1 \text{ (Onsager)}, \quad \varepsilon_i \varepsilon_j = -1 \text{ (Casimir)} \tag{9}$$

and we can speak about symmetric ORR when (7) is valid and about antisymmetric OCRR when

$$L_{ij} = -L_{ji} \tag{10}$$

holds. When among f independent parameters the number of α-parameters is 1,2,……m and the number of β-parameters is m+1, m+2, …f, so OCRR have the form

$$\left.\begin{array}{ll} L_{ij} = L_{ji}\ (i,j = 1, 2, \dots, m) & \text{(Onsager)} \\ L_{i\gamma} = -L_{\gamma i} & \\ (i = 1, 2, \dots, m\ ;\ \gamma = m+1, \dots, f) & \text{(Casimir)} \\ L_{\gamma\lambda} = L_{\lambda\gamma}\ (\gamma,\lambda = m+1, m+2, \dots, f) & \text{(Onsager)} \end{array}\right\} \tag{11}$$

From (9) it follows that even (only even) or odd (only odd) variables are coupled by a symmetric matrix and even and odd variables (mixed) are coupled by an antisymmetric matrix.

If an external magnetic field is acting, it must be reverse not only all the velocities ($\vec{v}_i \to -\vec{v}_i$) but also the magnetic field orientation ($\vec{B} \to -\vec{B}$). Only then the particles retrace their former path – this is a consequence of the expression of the Lorentz force. The reciprocity relations (8) have now to be replaced by

$$L_{ij}(\vec{B}) = \varepsilon_i\, \varepsilon_j\, (-\vec{B})\,. \tag{12}$$

The same reasoning can be applied for processes taking place in non-inertial frames rotating with an angular velocity $\vec{\omega}$. This is the consequence of the form of the Coriolis force. Here not only the velocities of particles but also orientation of angular velocity ($\vec{\omega} \to -\vec{\omega}$) of system must be reversed.

4 Reciprocity in electromagnetic field

4.1 Electromagnetic reciprocity – general

The application of reciprocity principle in electromagnetic (EM) field is based on the time-reversal symmetry of electromagnetic field equations. ORR and/or OCRR allow express the macroscopic consequences of microscopic reversibility in linear electromagnetic materials (see e.g. [10, 11]). Reciprocity relations may be e.g. applied to the constitutive relations (3) of linear homogeneous BA materials, in which $\vec{E}$ and $\vec{B}$ are considered as the primary fields while $\vec{D}$ and $\vec{H}$ are the induction fields presented a response in/of material. In the same time must be respect that vectors $\vec{D}$ and $\vec{E}$ are even and $\vec{H}$ and $\vec{B}$ are odd with respect to time reversal.

The symmetry properties of field equations and of material relations are very important in electromagnetic media knowledge. In particular, symmetries with respect to time inversion define whether a particular material is reciprocal or not. Space symmetry dictates the structure of the material tensors.

There are many attempts to generalize the reciprocity principle (RP) for BA media. Gyrotropic (i.e. gyromagnetic and/or gyroelectric) media with non-symmetrical constitutive tensors caused by an applied dc magnetic field ($\overrightarrow{\mathrm{B}_0}$) have been called nonreciprocal because the usual reciprocity theorem (RT) as Lorentz, Rayleigh-Carson or Feld-Tai reciprocity does not apply to them (more about see e.g. [12]). The quantity „reaction“ introduced by Rumsey [13] made possible to derive the reciprocity theorem and so called „modified RT“ [14].

4.2 Macroscopic conditions for medium reciprocity

Macroscopic conditions of reciprocity for linear homogeneous BA medium one can obtain as direct consequence of Maxwell equations and of known constitutive relation (CR) e.g. in the form (3). If we know (or define) what electrodynamic system is reciprocal (with help of mentioned notion „reaction“), so from Maxwell equations with using of considered CR we obtain the reciprocity conditions for the medium parameters. Following authors [12, 15, 16] we can obtain, with using of OCRR (11), the conditions for medium reciprocity in the form

$$\left.\begin{aligned} &\varepsilon_{jk}(\vec{B}_0) = \varepsilon_{kj}(-\vec{B}_0) \\ \text{or} \quad &\\ &\overline{\overline{\varepsilon}}(\vec{B}_0) = \overline{\overline{\varepsilon^T}}(-\vec{B}_0), \end{aligned}\right\} \quad (13a)$$

$$\left.\begin{aligned} &\mu_{jk}^{-1}(\vec{B}_0) = \mu_{kj}^{-1}(-\vec{B}_0), \\ \text{or} \quad &\\ &\overline{\overline{\mu^{-1}}}(\vec{B}_0) = \overline{\overline{\mu^{-1T}}}(-\vec{B}_0), \end{aligned}\right\} \quad (13b)$$

$$\left.\begin{aligned} &\alpha_{jk}(\vec{B}_0) = -\beta_{kj}(-\vec{B}_0), \\ \text{or} \quad &\\ &\overline{\overline{\alpha}}(\vec{B}_0) = \overline{\overline{-\beta^T}}(-\vec{B}_0). \end{aligned}\right\} \quad (13c)$$

Here $\overrightarrow{\mathrm{B}_0}$ is the time odd parameter, namely an external magnetic (magnetostatic) field and symbol ^{T} note the transpose matrix.

We can see that the tensors $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ (magnetoelectric coupling) are mutual anti-symmetric.

Form (13) it follows, that isotropic media are reciprocal and anisotropic media with symmetrical permitivity and permeability tensors are also reciprocal. Bianisotropic media that satisfy (13a) and (13b) and in the same time condition $\overline{\overline{\alpha}} = \overline{\overline{\beta^+}}$ are reciprocal if $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ are purely imaginary matrices.

Interesting is the approach of Lakhtakia and Depine [17] which consider the (linear) CR for electric polarization $\vec{P}$ and magnetization $\vec{M}$ as a linear functions $\vec{E}$ and $\vec{B}$. Using the OCRR (11) and (12), they obtained the relations

$$\overline{\overline{\varepsilon}}(\vec{B}_0) = \overline{\overline{\varepsilon^T}}(-\vec{B}_0), \quad (14a)$$

$$\overline{\overline{\mu^{-1}}}(\vec{B}_0) = \overline{\overline{\mu^{-1T}}}(-\vec{B}_0), \quad (14b)$$

$$\overline{\overline{\alpha}}(\vec{B}_0) = \overline{\overline{\beta^T}}(-\vec{B}_0), \quad (14c)$$

Still open is the question how to clarify the different results (13c) and (14c). Note that Onsager in his papers considered the motion od microscopic particles e.g. in heat conduction, diffusion flows and related transport processes. In this sense are the Onsager fluxes directly concerned with particular motion. In dielectric and magnetic materials corresponds to these processes the transport of electric charges in processes of electric polarization and magnetization. Thus, in order to correctly exploit the OCR in electromagnetics, one must consider $\vec{E}$ and $\vec{B}$ as primitive fields and isolate those parts of $\vec{D}$ and $\vec{H}$, which indicate the presence of a material, because the mentioned microscopic processes cannot occur in vacuum. The quantities represent corresponding parts of $\vec{D}$ and $\vec{H}$ are $\vec{P}$ and $\vec{M}$, i.e. electric polarization and magnetization (see eq. (5)). In this sense the condition (14c) corresponds with Onsager symmetry relations (7).

But the expression (derivation) of reciprocity conditions and of correct application of OCR is, unfortunately, influenced by some misunderstandings. The ambiguity is grounded in conflict between an assumption of material response without any delay whatsoever and the Onsager relations in conception of Callen et al. The former approach corresponds to a noncausal assumption in electromagnetic theory, in other words corresponds to a false premise that materials respond purely instanteously.

5 Conclusion

The reciprocity principle represents the general physical law which can be formulated both from macroscopic and from microscopic point of view. The application of reciprocity principle in electromagnetics is based on the reciprocity relations known in nonequilibrium linear thermodynamics and the problem of linear response of system may be in both situations formulated. In this paper was analyzed the notion of reciprocal medium and the possibilities for derivation of material reciprocity criteria. The different approaches to choice of used constitutive relations are comment and consequences of their nonunific results are analyzed.

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