A Novel Continuous Function for Approximation to the Factorial

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Abstract: - Various approximations to the factorial have been proposed in the literature to formulate continuous functions in which the argument is a non-negative variable. Most of these approximations are based on the classical De-Moivre-Stirling's or shortly Stirling's formula. Further approximations have been provided, either multiplying the Stirling's formula to a correction function, or introducing some structural modifications to the Stirling's formula. The characteristics of the various approximations can be pointed out by investigating how the correction function or the structurally modified formula can provide a better representation of the factorial for natural numbers with respect to the classical Stirling's formula. This paper starts with a tutorial illustration of the characteristics of various approximations to the factorial, and contains the proposal of a novel continuous function with relatively simple structure. The proposed function has very low relative approximation errors with respect to the factorial; furthermore, the relative approximation error is always positive. These characteristics enable the novel function to be used as an upper bound to the factorial. Application examples of the proposed formula in the pattern recognition domain are presented, in order to obtain factorial-free formulations for the calculation of orthogonal Fourier-Mellin moments and Pseudo-Zernike moments, with some notes on the possible computational complexity reduction obtainable by exploiting the proposed formulation with respect to the computation of the same moments using the factorials, in analogy to what has been done in the literature by using a different type of approximation.

Key-Words: - Factorial, Stirling's formula, Asymptotic convergence, Correction function, Relative approximation error, Orthogonal Fourier-Mellin moments, Pseudo-Zernike moments.

1 Introduction

The factorial $n!$ is defined for the discrete set of natural numbers $n \in \aleph$, with the basic properties that $n! = n\,(n-1)!$ and $0! = 1$. The factorial is used for determining different types of probabilities, such as binomial and hypergeometric ones, and multiple factorials appear in various applications such as pattern recognition and image processing. However, direct calculation of the factorial could be computationally quite complicate for large values of n, and in particular could cause overflows in the numerical representation of the outcomes. In addition, the discrete definition of the factorial could be a limiting aspect when there is a need for taking the derivatives with respect to n of terms containing the factorial. As such, approximate representations of the factorial through continuous functions have been defined in various ways. For instance, the factorial of a natural number $n \in \aleph$ can be written by using the Gamma function [1] as $n! = \Gamma(n+1)$ or, in the integral form,

$$n! = \int_0^\infty x^n e^{-x} dx \qquad (1)$$

However, the integral formulation makes Equation (1) difficult to handle. Simpler continuous explicit functions have then been proposed for an easier approximation to the factorial. A typical approximation is the well-known De Moivre-Stirling's formula, originated by the work of Abraham De Moivre [2] and James Stirling [3], often called in short *Stirling's formula*, as De Moivre obtained a formula with an undetermined constant, while Stirling found its expression with the constant set to $\sqrt{2\pi}$.

Using a natural number $n \in \aleph$, the Stirling's formula is typically expressed as

$$s(n) = n^{n+\frac{1}{2}}\, e^{-n} \sqrt{2\pi} \qquad (2)$$

or

$$s(n)=\left(\frac{n}{e}\right)^n \sqrt{2\pi n} \tag{3}$$

Another classical representation considers the logarithm of $n!$, formulating the expressions:

$$\ln(n!) \cong \ln(s(n)) = n \ln n - n + \ln\left(\sqrt{2\pi n}\right) \cong n \ln n - n \tag{4}$$

Proofs of the asymptotic convergence of the Stirling's formula to $n!$ for $n \to \infty$ and related discussions have been presented in several references [3]-[15].

The Stirling's formula is typically used for approximating the factorial $n!$ for very large values of n. However, its accuracy for relatively low values of n is limited, and can be further improved also for high values of n, as remarked below.

This paper deals with discussing the approximations to the factorial leading to lower approximation errors than the Stirling's formula for small and large values of n. Different approximations are discussed, comparing their behavior for high and relatively low values of n. An original contribution of this paper is the formulation of a continuous formula for approximation to the factorial with interesting characteristics of small approximation error and unilateral evolution of this error for all values of n. The proposed formula is included among the comparisons to show its effectiveness.

Section 2 of this paper recalls the formulations of a number of functions used to approximate the factorial. Section 3 introduces the novel continuous function and discusses its characteristics. Section 4 points out some fields of possible application of the approximation to the factorial in the pattern recognition field. Section 5 contains the concluding remarks.

2 Background on the approximations to the factorial

2.1 Formulation of different types of approximation

Various types of approximations to the factorial by using explicit continuous functions have been proposed [16]. A structural categorization of these approximations is indicated in the sequel.

A. One type of approximation can be written by taking the Stirling's formula (2) and multiplying it to a *correction function* $\zeta(n)$, such that

$$\psi(n)=\zeta(n)s(n) \tag{5}$$

In this case, a common formulation uses as correction function the first terms of the asymptotic series for the Gamma function [17]-[19]. For instance, using the first two terms of the series, the correction function is written in the form

$$\zeta_b(n)=\left(1+\frac{1}{b\,n}\right) \tag{6}$$

where the classical value $b = 12$ is assumed, to approximate with $\zeta_b(n)$ the way in which the ratio $n!/s(n)$ approaches unity for increasing values of n.

A correction function with more terms is indicated in [20], leading to an expansion of the type

$$\zeta_b(n)=1+\frac{1}{12n}+\frac{1}{288n^2}+ \\ -\frac{139}{51840n^3}-\frac{571}{2488320n^4}+O\left(\frac{1}{n^5}\right) \tag{7}$$

indicated by using the classical big-O representation accounting for the remaining terms. This representation, truncated at the fourth term, has been used by Szirtes in [21] to overcome the impossibility of direct computation of the factorial for large numbers with the computing facilities available at the time he wrote his paper.

Another type of approximation uses an exponential correction function [5], [9], [22], [23], such that:

$$\zeta_e(n)=e^{\tau(n)} \tag{8}$$

The exponential correction function has been exploited by Robbins [5] to introduce suitable expressions to be used as lower and upper bounds to the factorial, by respectively considering $\tau_l(n)=\frac{1}{12n+1}$ and $\tau_u(n)=\frac{1}{12n}$, such that:

$$s(n)\,\tau_l(n) < n! < s(n)\,\tau_u(n) \tag{9}$$

Variants to the lower boundary have been successively proposed by Nanjundiah [22]:

$$\tau_l'(n) = \frac{1}{12n} - \frac{1}{360n^3} \tag{10}$$

and by Maria [23]:

$$\tau_l'''(n) = \frac{1}{12n + \frac{3}{2(2n+1)}} \tag{11}$$

Furthermore, Whittaker and Watson [16], provided an asymptotic series that, despite not converging for any n, can be used in its first terms:

$$\tau_l''(n) = \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \frac{1}{1188n^9} + \dots \tag{12}$$

B. A further type of approximation is built by *modifying the structure* of the Stirling's formula. An example is

$$g(n) = n^n e^{-n} \sqrt{2\pi n + q} \tag{13}$$

corresponding for $q = \pi/3$ to the Gosper's approximation reported in [24].

Of course, all the representations recalled above can be used to write the factorial approximations in logarithmic terms.

2.2 Relative approximation error

In order to evaluate the accuracy of the approximation to the factorial by using a function $\psi(n)$, let us define the relative approximation error (*RAE*), expressed in percent as

$$RAE(n, \psi(n)) = 100 \, \frac{\psi(n) - n!}{n!} \tag{14}$$

With respect to the *RAE*, one of the properties of the asymptotic series (12) is that taking more terms of the series the absolute *RAE* becomes progressively smaller [16].

Fig. 1 shows the *RAE* values reached by using the Stirling's formula and other approximations introduced through Equations (5)-(6), Equation (8) with upper bound from [5] and lower bounds from [5], [22] and [23], and Equation (13) with $q = \pi/3$. The entries in the legenda of Fig. 1 show the functions in decreasing order of *RAE* (positive values first, up to the negative values). The list of approximations is explicitly reported below in the same order.

- Exponential correction (upper bound from Robbins [5]):

$$\psi(n) = e^{\frac{1}{12n}} \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \tag{15}$$

- Exponential correction from Whittaker and Watson [16], truncated at the fifth term:

$$\psi(n) = e^{\left(\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \frac{1}{1188n^9}\right)} \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \tag{16}$$

- Exponential correction (lower bound from Nanjundiah [22]):

$$\psi(n) = e^{\left(\frac{1}{12n} - \frac{1}{360n^3}\right)} \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \tag{17}$$

- Extended Stirling's formula from [20], neglecting the big-O terms:

$$\psi(n) = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4}\right) \tag{18}$$

- Exponential correction (lower bound from Maria [23]):

$$\psi(n) = e^{\frac{1}{12n + \frac{3}{2(2n+1)}}} \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \tag{19}$$

- Approximation Eq. (6) with $b = 12$:

$$\psi(n) = \left(1 + \frac{1}{12n}\right) \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \tag{20}$$

- Exponential correction (lower bound from Robbins [5]):

$$\psi(n) = e^{\frac{1}{12n+1}} \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \qquad (21)$$

- Gosper's approximation [24]):

$$\psi(n) = \left(\frac{n}{e}\right)^n \sqrt{2\pi\, n + \frac{\pi}{3}} \qquad (22)$$

The last entry is the Stirling's formula (2). The results indicated in Fig. 1 provide a clear confirmation that the approximated formulas exhibit much better characteristics and lower *RAE* than the Stirling's formula, especially for low values of *n*. However, in all these cases the maximum *RAE* occurs for $n = 1$ and is significantly higher than the errors reached for higher values of *n*.

Starting from these bases, the following section introduces and illustrates the characteristics of a novel continuous formula that improves the approximation accuracy in terms of limiting the maximum *RAE* to a very low value over the entire range of numbers $n \in \aleph$.

3 A novel continuous approximation to the factorial

The general formulation of the approximations adopting the correction function as in (3) is particularly interesting because of the simplicity of its representation. In particular, let us consider Equation (6) and write the corresponding approximation function as

$$\psi_b(n) = \zeta_b(n) s(n) \qquad (23)$$

A parametric study has been carried out by changing the (real) value of the parameter *b*, stepping beyond the classical value $b = 12$. The results have shown that the *RAE* is very sensitive to the value of *b*. Fig. 2 indicates some results. It can be seen that in part of the cases the higher absolute value of *RAE* occurs for $n = 1$, then the absolute *RAE* decreases by increasing *n*. However, this monotonically changing behavior occurring for both positive and negative values of *RAE* is not found for any value of *b* (otherwise there would be the best situation with *RAE* identically null for a given value of *b*). In some cases (as for $b = b' = 11.855$) the *RAE* has no monotonic behavior, but it has not always the same sign (with two initial negative values for $n = 1$ and $n = 2$, and successive positive values).

On the basis of the above concepts, it is possible to find a particular situation in which the *RAE* is null for $n = 1$. By elaborating the calculations using Equation (6), the condition $RAE = 0$ is satisfied for the value

$$b = b^* = \frac{\sqrt{2\pi}}{e - \sqrt{2\pi}} = 11.843 \qquad (24)$$

In these conditions, the maximum *RAE* evaluated on $n \in \aleph$ is 0.0102% for $n = 5$. This case corresponds to the limit case for which all *RAE* values are nonnegative. For $b > b^*$ the *RAE* becomes negative for $n = 1$. From Equation (6), the condition $b = b^*$ corresponds to

$$\zeta_{b^*}(n) = \left(1 + \frac{e - \sqrt{2\pi}}{n\sqrt{2\pi}}\right) \qquad (25)$$

Results of a more detailed analysis are shown in Fig. 3, where the maximum *RAE* is reported for different values of *n* in function of the parameter *b*. Fig. 3 shows that only the cases $n = 1$, $n = 4$ and $n = 5$ are involved in defining the maximum *RAE* as the parameter *b* changes. In particular, the maximum RAE occurs for $n = 1$ when the parameter *b* varies from about 11.823 to about 11.855.

Let us then compute the minimum value of the maximum *RAE*:

$$\varepsilon_{\min}^{\%} = \min_n \left\{ \max_b \left\{ \varepsilon^{\%}\left(n, \psi_b(n)\right) \right\} \right\} \qquad (26)$$

For $n \in \aleph, \varepsilon_{\min}^{\%} = 0.0085\%$ occurs for $n = 5$ and $b = b' = 11.855$. However, as already pointed out (Fig. 2), in this case the *RAE* values are negative for $n < 3$ and positive for $n \geq 3$.

Let us focus on the case with parameter $b = b^*$, for which the *RAE* is always nonnegative and the maximum *RAE* is relatively close to its minimum value. By using Equation (2) and Equation (25) and substituting the terms into Equation (23), it is possible to represent the approximation to the factorial in a simple form, for $n \in \aleph$:

$$\nu(n) = \left[e + (n-1)\sqrt{2\pi}\right] n^{n-\frac{1}{2}}\, e^{-n} \qquad (27)$$

The formulation of Equation (27) merges the simplicity of representation with a very low value of the maximum *RAE* and with *RAE* values always non-negative for $n \in \aleph$. The latter property allows for considering Equation (27) as an *upper bound* to the factorial for any $n \in \aleph$.

The asymptotic convergence of $\nu(n)$ to $n!$ for $n \to \infty$ is guaranteed by the fact that $\zeta_b(n)$ tends to unity for $n \to \infty$ and by the existing proof of convergence of the Stirling's formula to $n!$ for $n \to \infty$, such that

$$\lim_{n\to\infty}\left(\frac{\nu(n)}{n!}\right)=\lim_{n\to\infty}\left(\frac{s(n)}{n!}\right)=1 \qquad (28)$$

In spite of the asymptotic convergence of the approximated functions to the factorial, stated from the limit (28), it is well-known that the difference $n!$-$s(n)$ does not converge to zero. In effect, in absolute terms this difference $n!$-$s(n)$ increases by increasing n, however with an increase much lower than the increase of $n!$ or $s(n)$. The same asymptotic property holds for the proposed formulation $\nu(n)$ in Equation (27). Fig. 4 shows the trend of increase of the logarithmic difference defined as

$$\delta(n)=\ln\left|n!-\psi(n)\right| \qquad (29)$$

where also the basic expression $s(n)$ of the Stirling's formula (2) is considered among the functions $\psi(n)$. According to the definition (29), no distinction is made in Fig. 4 among the formulations providing upper bounds, lower bounds, or no explicit bound to the factorial. From Fig. 4, the formulation $s(n)$ is the one providing the least accurate approximation to the factorial with respect to all the other variants considered, including the proposed formulation $\nu(n)$.

Equation (27) is then proposed here as a novel simple approximation to the factorial. The logarithmic form of the proposed equation is

$$\ln(\nu(n))=\ln\left(e+(n-1)\sqrt{2\pi}\right)+\left(n-\frac{1}{2}\right)\ln n-n \qquad (30)$$

The possible applications of this novel formulation are the same as those of the original Stirling's formula and of its existing approximations. Some fields of application are recalled in the following section.

4 Applications

Approximations to the factorial are particularly useful in all the applications in which the factorial plays a basic role on the computational side. Some examples can be found with respect to q-parametric operators [25]-[27]. Other examples refer to cases in which it may be convenient to obtain factorial-free formulations to avoid the evaluation of one or more factorials. In this paper, some applications are shown concerning the exploitation of moments for pattern recognition purposes. The use of moments in this respect can be addressed by following the approach introduced by Hu [28], leading to the use of orthogonal moments [29] of different categories. Some widely used categories of orthogonal moments are Fourier-Mellin, Zernike (based on the principles introduced in [30]) and pseudo-Zernike [31]. The orthogonal moments can be exploited for instance in image processing for reconstructing the characteristics of a given object by using a finite number of moments. However, multiple factorials appear in the definition of the orthogonal moments, leading to heavy computational burden for the numerical procedures to calculate the high-order moments. As such, specific formulations to compute the orthogonal moments have been developed by avoiding the direct use of factorials, as summarized in [32]. An approach to obtain factorial-free formulations of the approximate orthogonal moments using the extended Stirling's formula has been presented recently [32]-[34]. Following the lines of this approach, the proposed formula (27) for approximation to the factorial is used here to write dedicated and factorial-free versions of some components used to determine the approximate orthogonal Fourier-Mellin and pseudo-Zernike moments. With these formulations, it is possible to reduce the computational complexity evaluated in terms of numbers of multiplications required to compute the orthogonal moments.

4.1 Orthogonal Fourier-Mellin moments

The orthogonal moments have been introduced in [35] by using a set of complex polynomials whose kernel is given by a set of orthogonal radial polynomials expressed in polar coordinates as

$$\begin{aligned} Q_p(r) &= \sum_{k=0}^{p}(-1)^{p+k}\frac{(p+k+1)!}{(p-k)!k!(k+1)!}r^k = \\ &= \sum_{k=0}^{p}(-1)^{p+k}\,T_p(k)r^k \end{aligned} \qquad (31)$$

where r is the radius, $p \in \aleph$ and q is a positive or negative integer such that $0 \le |q| \le p$, for $p = 0, 1, 2, \ldots, \infty$.

Following the developments reported in [33], based on the general property $n! = n\cdot(n-1)!$ for $n \in \aleph$, the term $T_p(k)$ containing the factorials can be rewritten as

$$T_p(k)=\frac{(p-k+1)(p+k+1)!(k+1)}{(p-k+1)!(k+1)!\,(k+1)!} \qquad (32)$$

Taking the logarithm of (32) yields

$$\begin{aligned}&\ln(T_p(k))=\ln(p-k+1)+\ln(k+1)+\\&+\ln((p+k+1)!)-\ln((p-k+1)!)-2\ln((k+1)!)\end{aligned} \qquad (33)$$

By replacing $\ln(n!)$ with $\ln(\nu(n))$ for $n \in \aleph$ in all the terms containing the factorials, the following expressions are found:

$$\begin{aligned}&\ln(T_p(k))=\\&\ln(p-k+1)+\ln(k+1)+\ln\left(e+(p+k)\sqrt{2\pi}\right)+\\&+\left(p+k+\frac{1}{2}\right)\ln(p+k+1)-(p+k+1)+\\&-\ln\left(e+(p-k)\sqrt{2\pi}\right)-\left(p-k+\frac{1}{2}\right)\ln(p-k+1)+\\&+(p-k+1)-2\ln\left(e-k\sqrt{2\pi}\right)-2\left(k+\frac{1}{2}\right)\ln(k+1)+\\&+2(k+1)=\\&=\left(\frac{1}{2}-p+k\right)\ln(p-k+1)+\left(\frac{1}{2}+p+k\right)\ln(p+k+1)+\\&+2k\ln(k+1)-\ln\left(e+(p-k)\sqrt{2\pi}\right)+\ln\left(e+(p+k)\sqrt{2\pi}\right)-\\&-2\ln\left(e-k\sqrt{2\pi}\right)+2\end{aligned} \qquad (34)$$

Hence, by reporting the expressions of the right-hand side to a single logarithm and comparing the arguments at the left-hand and right-hand sides, the final factorial-free expression of the term $T_p(k)$ is

$$\begin{aligned}T_p(k)=&(p-k+1)^{\frac{1}{2}-p+k}(p+k+1)^{\frac{1}{2}+p+k}\cdot\\&\cdot\frac{(k+1)^{2k}e^2}{\left(e-k\sqrt{2\pi}\right)}\frac{\left(e+(p+k)\sqrt{2\pi}\right)}{\left(e+(p-k)\sqrt{2\pi}\right)}\end{aligned} \qquad (35)$$

The expression (35) is relatively simple to be computed with respect to (32). In order to quantify the related advantages, computational complexity issues [36]-[38] can be addressed as indicated in [33]. To simplify the determinations, we assume here that the operations of addition and subtraction are ignored under the hypothesis that their computational burden is much lower than the one of multiplications for the processor used. The few ratio operations are treated here as multiplications for the sake of simplicity, assuming that priority is given to the products of all the terms separately at the upper (lower) side, leaving the ratio as the last operation. On the above concepts, computational complexity is calculated approximately by enumerating the number of equivalent multiplication operations. For this purpose, the hypotheses used here are the same ones indicated in [33]. In particular, square roots are assumed to be calculated with the Newton-Raphson method, with each square root equivalent to 24 multiplications. Exponentials are assumed to be computed with the FASTEXP method [20], according to which the complexity of each exponential calculation is given by the base-2 logarithm of the exponent.

For the purpose of computational complexity assessment, the expression (35) is written in the form

$$\begin{aligned}T_p(k)=&((p-k+1)(p+k+1))^{\frac{1}{2}}\frac{(p+k+1)^{p+k}}{(p-k+1)^{p-k}}\cdot\\&\cdot\frac{(k+1)^{2k}e^2}{\left(e-k\sqrt{2\pi}\right)}\frac{\left(e+(p+k)\sqrt{2\pi}\right)}{\left(e+(p-k)\sqrt{2\pi}\right)}\end{aligned} \qquad (36)$$

Thus, in the worst case the exponentials depending on p on the upper side of the ratio have complexity $\log_2(2p)$ each (for $k = p$) and the exponential at the lower side has complexity $\log_2(2p)$ for $k = 0$. The square root corresponds to 24 multiplications. The other multiplications refer to the 3 products with the terms containing $\sqrt{2\pi}$ (considered as a given constant), 5 other products on the upper side (including e^2), 2 other products on the lower side and eventually 1 ratio. The overall complexity is of about $2\log_2(2p)+\log_2(p)+35$. As indicated in [33], the computational complexity of Equation (32) containing the factorials is $5p$-1, and the one obtained by using the approximation provided in Equation (20) is $9\log_2(2p+4)+18$. The variations of the computational complexity for increasing values of the moment order p in the cases indicated in [33] and using the proposed approximation are summarized in Fig. 5. It emerges that the expression (35) has a computational complexity higher than the one of Equation (32) for relatively low values of p, but its complexity increases according to a logarithm-based law and not linearly with respect to p as it occurs by using Equation (32).

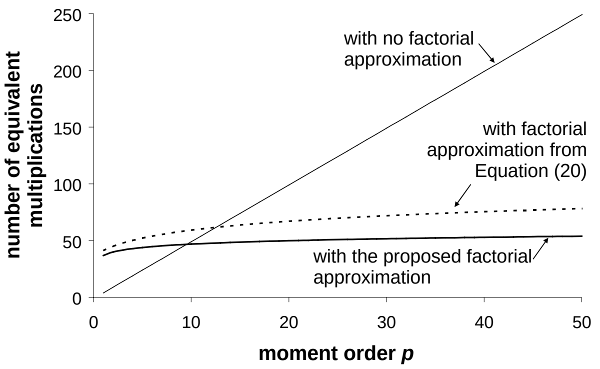

Fig. 5. Computational complexity comparisons for the orthogonal Fourier-Mellin moments.

As it already happened for the approximated formulation used in [33], for computational complexity reduction purposes the application of the proposed approximated formulation is not convenient for small values of the moment order, reaches the break-even point for a moment order of about 9-10 for the proposed formulation and about 12-13 for the representation shown in [33], and becomes more and more convenient for increasing values of the moment order p.

4.2 Pseudo-Zernike moments

The approximate pseudo-Zernike polynomials are defined in [34] as the pseudo-Zernike polynomials in which the factorial representation is replaced by the Stirling approximation (6).

The pseudo-Zernike moments are defined by using in their kernel an orthogonal set of pseudo-Zernike radial polynomials expressed in polar coordinates as [32]

$$S_{pq}(r)=\sum_{k=0}^{p-|q|}(-1)^k\frac{(2p+1-k)!}{k!(p+|q|+1-k)!\,(p-|q|-k)!}r^{p-k}=$$
$$=\sum_{k=0}^{p-|q|}(-1)^k\,T_{pq}(k)r^{p-k} \tag{37}$$

where r is the radius, $p \in \aleph$ and q is a positive or negative integer such that $0 \le |q| \le p$, for $p = 0, 1, 2, \ldots, \infty$.

After the developments reported in [32], based on the general property $n! = n\cdot(n-1)!$ for $n \in \aleph$, the term $T_{pq}(k)$ containing the factorials can be rewritten as

$$T_{pq}(k)=\frac{(k+1)(p-|q|-k+1)(2p-k+1)!}{(p+|q|-k+1)!(p-|q|-k+1)!\,(k+1)!} \tag{38}$$

Taking the logarithm of (38) yields

$$\begin{aligned}\ln(T_{pq}(k)) &= \ln(k+1)+\ln(p-|q|-k+1)+\\ &+\ln((2p-k+1)!)-\ln((p+|q|-k+1)!)+\\ &-\ln((p-|q|-k+1)!)-\ln((k+1)!)\end{aligned} \tag{39}$$

By replacing $\ln(n!)$ with $\ln(\nu(n))$ for $n \in \aleph$ in all the terms containing the factorials, the following expressions are found:

$$\begin{aligned}&\ln(T_{pq}(k))=\\ &=\left(\frac{1}{2}-k\right)\ln(k+1)+\left(\frac{1}{2}-p+|q|+k\right)\ln(p-|q|-k+1)+\\ &-\left(\frac{1}{2}+p+|q|+k\right)\ln(p+|q|+k+1)+\\ &+\left(2p-k+\frac{1}{2}\right)\ln(2p-k+1)+\ln\left(e+(2p-k)\sqrt{2\pi}\right)+\\ &-\ln\left(e+(p+|q|-k)\sqrt{2\pi}\right)-\ln\left(e+(p-|q|-k)\sqrt{2\pi}\right)+\\ &-\ln\left(e+k\sqrt{2\pi}\right)+2\end{aligned} \tag{40}$$

Again, by reporting the expressions on the right-hand side to a single logarithm and comparing the arguments on the left-hand and right-hand sides, the final factorial-free expression of the term $T_{pq}(k)$ is

$$\begin{aligned}T_{pq}(k)=&\frac{(p-|q|-k+1)^{\frac{1}{2}-p+|q|+k}}{(p+|q|-k+1)^{\frac{1}{2}+p+|q|-k}}(2p-k+1)^{\frac{1}{2}+2p-k}\cdot\\ &\frac{\left(e+(2p-k)\sqrt{2\pi}\right)}{\left(e+(p-|q|-k)\sqrt{2\pi}\right)\left(e+(p+|q|-k)\sqrt{2\pi}\right)}\cdot\frac{(k+1)^{\frac{1}{2}-k}e^2}{\left(e+k\sqrt{2\pi}\right)}\end{aligned} \tag{41}$$

For computational complexity assessment purposes, Equation (41) is rewritten as

$$\begin{aligned}T_{pq}(k)=&\left(\frac{(p-|q|-k+1)(2p-k+1)(k+1)}{(p+|q|-k+1)}\right)^{\frac{1}{2}}\cdot\\ &\frac{(2p-k+1)^{2p-k}}{(p-|q|-k+1)^{p-|q|-k}(p+|q|-k+1)^{p+|q|-k}(k+1)^k}\cdot\\ &\frac{\left(e+(2p-k)\sqrt{2\pi}\right)}{\left(e+(p-|q|-k)\sqrt{2\pi}\right)\left(e+(p+|q|-k)\sqrt{2\pi}\right)}\cdot\frac{e^2}{\left(e+k\sqrt{2\pi}\right)}\end{aligned} \tag{42}$$

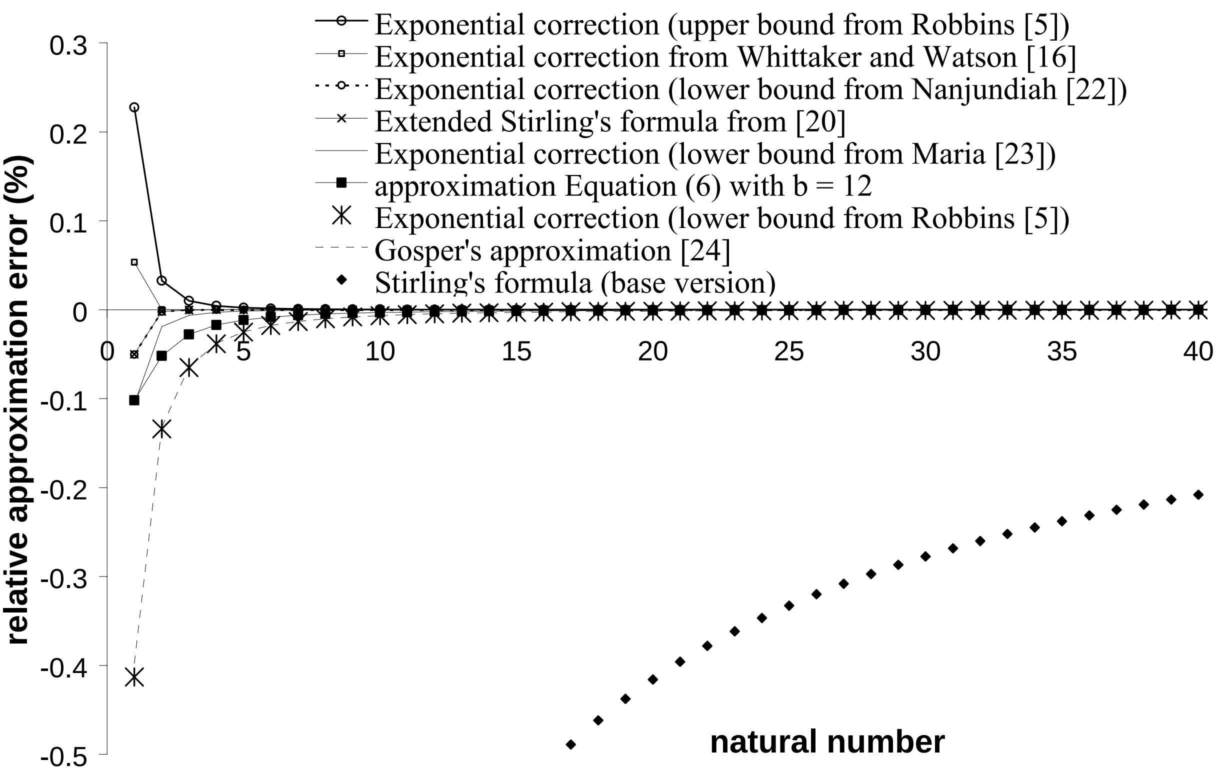

Fig. 1. Relative approximation errors for the Stirling's formula and other approximations to the factorial.

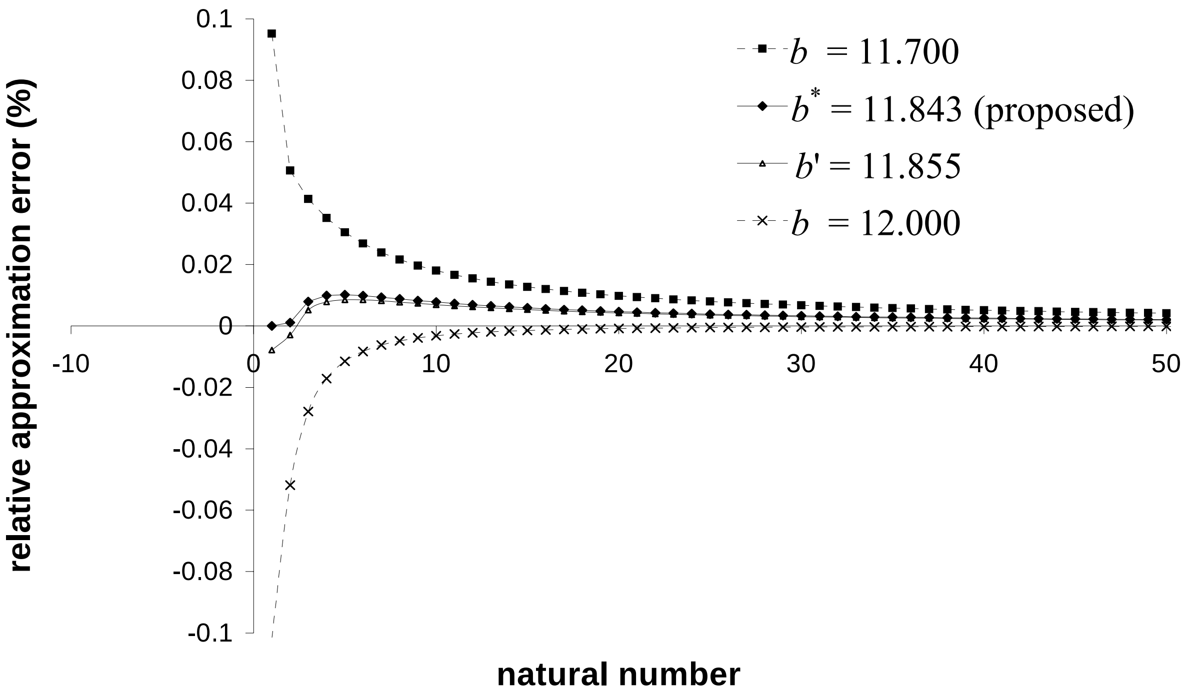

Fig. 2. Relative approximation errors for different values of the parameter b.

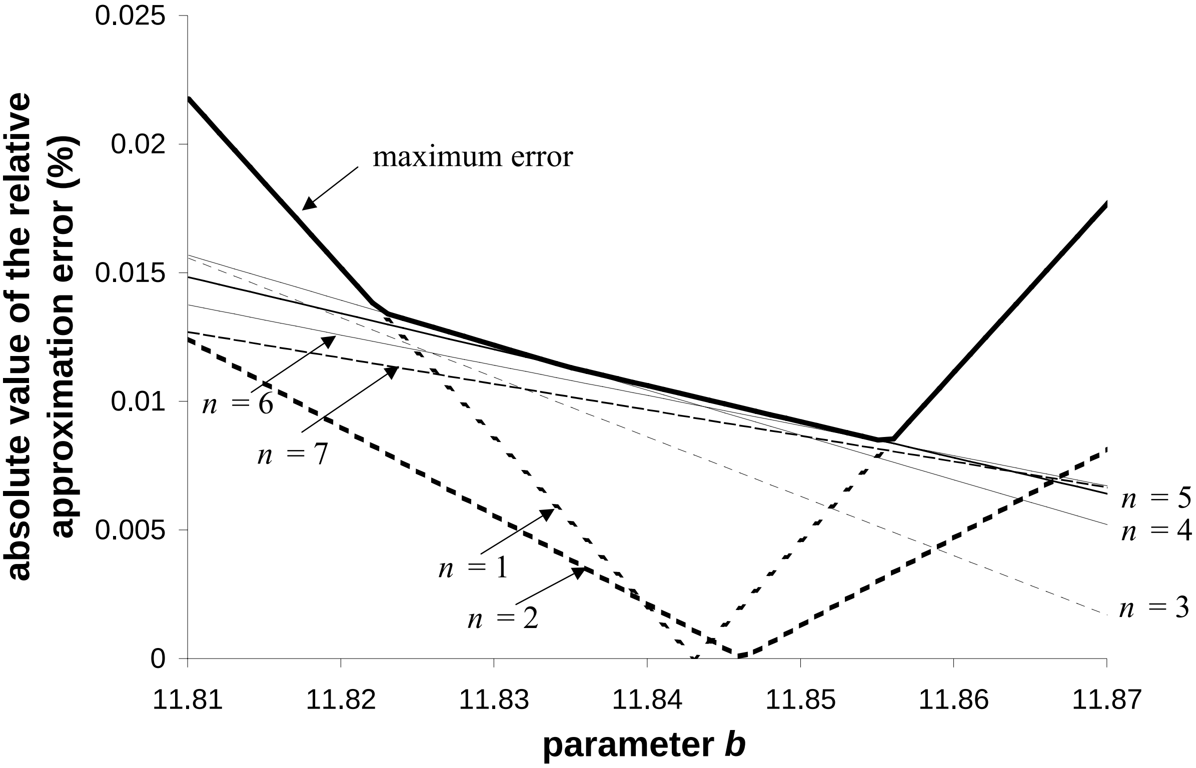

Fig. 3. Absolute values of the relative approximation error.

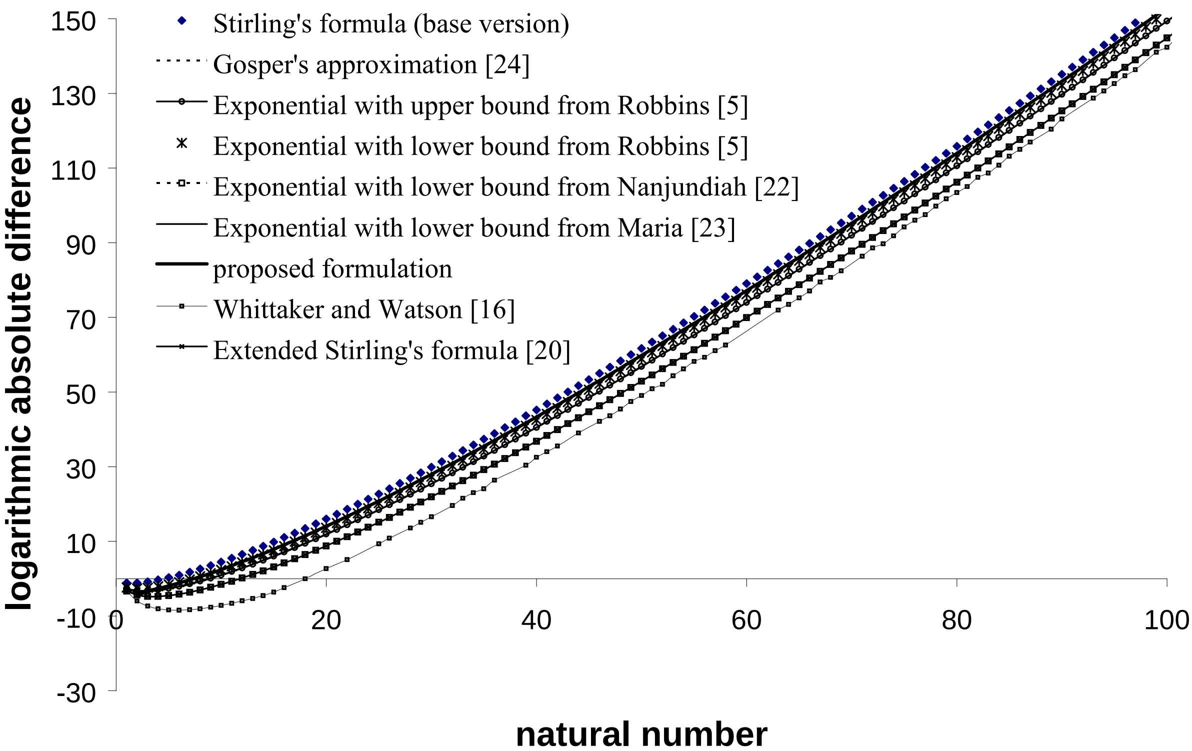

Fig. 4. Logarithmic absolute difference of the approximated formulations with respect to the factorial.

By using the same hypotheses and assumptions for the computation of square roots and exponentials indicated in Section 4.2, the overall complexity becomes of about $2\log_2(2p)+2\log_2(p)+40$ in the worst case. This result is compared in Fig. 6 with the computational complexity of Equation (38) containing the factorials, indicated as $4p+1$ in [32], and with the one obtained by using the approximation provided in Equation (20), determined as $3\log_2(2p+4)+35$ in [32]. Again, the approximated versions are not convenient for small values of the moment order, reach the break-even point for a moment order of about 12-13 for the representation shown in [32] and of about 14-15 for the proposed formulation.

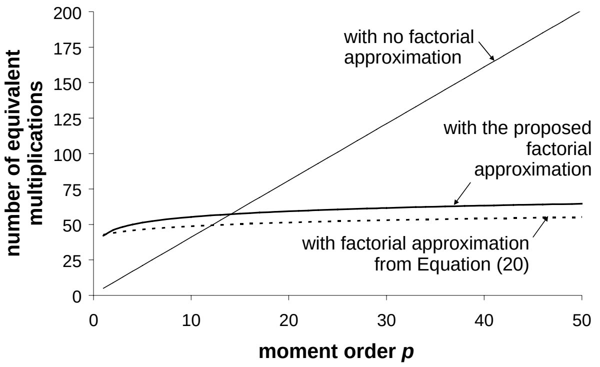

Fig. 6. Computational complexity comparisons for the pseudo-Zernike moments.

5 Conclusions

Various continuous approximations to the factorial have been proposed in the literature. This paper has presented a novel and simple formula leading to an effective approximation to the factorial by using an explicit continuous function. This formula shares the asymptotic properties of the classical Stirling's formula and exhibits excellent performance for approximating the factorial in the entire range of the natural numbers.

The proposed formula has been derived starting from the discussion on the parameters of the correction functions associated to the classical Stirling's formula. Its final form it also contains a structural modification with respect to the classical Stirling's formula. The proposed formula is characterized by a relative approximation error nonnegative and with low maximum value (about 0.01%). Two specific examples of application of the proposed formula have been shown, in roder to obtain factorial-free formulations for the calculation of orthogonal Fourier-Mellin moments and Pseudo-Zernike moments. Some notes on the possible computational complexity reduction obtainable by exploiting the proposed formulation with respect to the computation of the same moments using the factorials, in analogy to what has been done in the literature by using a different type of approximation, have been shown to confirm the effectiveness of the possible application of the proposed formula.

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