# A New N-factor Affine Term Structure Model of Futures Price for $CO_2$ Emissions Allowances: Empirical Evidence from the EU ETS

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*Abstract:* - In recent years, carbon emission markets have become liquid and promising markets within the European Union emissions trading scheme (EU ETS). In order to fit and forecast futures price for $CO_2$ emissions allowances, we propose a new N-factor affine term structure model for $CO_2$ futures price and estimate parameters in the new affine model by using the Kalman filter technique. Our empirical results show that $CO_2$ futures price follow significant mean-reversion process, and the estimated coefficients of mean-reversion speed, market risk premium, volatility and correlation among state variables are almost significant. Compared with one-factor model, mean absolute errors (MAE) and root mean square errors (RMSE) in prediction errors from two-factor and three-factor model are lower, accordingly two-factor and three-factor model can accurately describe the term structure of $CO_2$ futures price.

*Key-Words*: $CO_2$ emissions allowances; futures prices; new affine model; term structure; Kalman filter

## 1 Introduction

In recent years, $CO_2$ emission allowances markets have become liquid and promising markets within the European Union emissions trading scheme (EU ETS). The EU ETS dominates the global carbon markets, a variety of specialized financial products such as spot, futures and options are traded. The trading scale of global emissions markets has achieved 93 billion euros or about 120 billion dollars until 2010. The emission rights of greenhouse gas, called EU allowances (EUA), allow for the right to emit one ton of $CO_2$ in the EU ETS. Therefore carbon emissions right has given specific ownership, and it is significantly valuable credit assets for the investors, hedgers, other market practitioners. Benz and Truck (2006) propose emission allowances prices are directly determined by the expected market scarcity and their empirical results show futures prices have strongly time-varying trend [1].

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Stochastic models of commodity prices have received much attention among the scholars, hedgers, financial practitioners, and stochastic models are mainly about the pricing and hedging of commodities assets. Early studies in the field typically assume that storable commodities prices follow Brownian motion process. Gibson and Schwartz (1990) develop two-factor model of commodity pricing, where the spot price follows a geometric Brownian motion and the convenience yields follow mean-reverting Ornstein-Uhlenbeck (O-U) process [2]. Schwartz (1997), Miltersen and Schwartz (1998) add the third stochastic interest rates by assuming mean reversion process, and they propose the three-factor model of commodity futures pricing [3-4]. Therefore the commodity spot price, the instantaneous convenience yields, and the instantaneous interest rate are of important state variables for commodity futures price.

Lautier (2005) indicates that the term structure is defined as the relationship between spot price and futures prices with different maturities [5]. In many commodities markets, the concept of term structure becomes significant because it provides useful information for the hedging or investment decision of different assets. Many scholars consider that the term structure of commodity price is affected by many state variables. Schwartz and Smith (2000) develop a two-factor model of commodity price which allows mean-reversion in short-term prices and uncertainty in the equilibrium level [6]. Manoliu and Tompaidis (2002) present multi-factor stochastic model of term structure for energy futures price [7]. Cortazar and Naranjo (2006) propose an *N*-factor Gaussian model to explain the stochastic behaviour of oil futures prices [8]. Wang et al (2010) put forward an N-factor affine term structure model in terms of behavioural characteristics of copper futures prices in Shanghai Futures Exchange (SHFE) [9]. They propose that spot prices are composed of multi arbitrary state variables, and they estimate unobserved state variables and calibrated model parameters by using Kalman filter method. The above affine models can well simulate and forecast term structure of commodities futures prices, and they have become the significantly hedging and risk-managing tools for all market participants.

$CO_2$ emission allowances prices are directly determined by the expected market scarcity which is induced by the change of emissions regulation policy, extreme weather, energy prices, abatement technology progress etc. Benz and Truck (2009) analyze the short-term dynamics behaviour of spot price for carbon dioxide ($CO_2$) emission allowances [10]. Wagner and Homburg (2009), Daskalakis and Psychoyios (2009) propose the dynamics behaviour of futures price for $CO_2$ emission allowances. Benz and Truck (2009) reveal that these futures contracts for $CO_2$ emission allowances lead the price discovery process in the EU ETS [11-12]. Daskalakis and Psychoyios (2009) develop the empirically and theoretically valid framework for the pricing and hedging of intra-phase and inter-phase futures and options on emission futures contracts. Chevallier (2010) analyzes time-varying risk premium and positive relationship between risk premium and time-to-maturity in $CO_2$ spot and futures prices [13].

Therefore futures prices for $CO_2$ emissions allowances are affected by many state variables, especially non-observable state variables. Understanding term structure of futures prices and accurately forecasting futures prices for emission allowances are of crucial importance for all market participants. The paper has three major objectives. Firstly on basis of understanding the behaviour feature of $CO_2$ futures price, we propose a new multi-factor affine term structure model of futures price for $CO_2$ emissions allowances. Secondly we also show how to estimate model parameters of unobservable state variables by using the Kalman filter and maximum likelihood methods in the whole period. Thirdly we compare with measurements errors in observable futures price by making an optimal use of all market prices available.

The remainder of this paper is organized as follows. The second section describes the data samples and the statistical analysis results of $CO_2$ futures prices. Section 3 proposes a new *N*-factor affine term structure model of futures price for emissions allowances. Section 4 presents the Kalman filter technique. Section 5 shows estimated model parameters by using the Kalman filter and maximum likelihood methods. Section 6 compares the evaluation of model Robustness. Section 7 concludes the paper.

# 2 Data description

## 2.1 Data description

The EU ETS is the largest greenhouse gas (GHG) emissions trading system in the world. European Climate Exchange (ECX, now it is merged by ICE) is the most liquid platform for $CO_2$ futures market in European. The EU ETS has the existing two phrases: the pilot phase (2005-2007) and the Kyoto phase (2008-2012). Montagnoli and Vries (2010) indicate that market efficiency was inefficient in the trial and learning period, then it shows restoring signs in the Phase II [14]. Since EU government banned out-of-phrase banking and borrowing, then the spot price for $CO_2$ emissions allowances fell down to zero from October 2006 until December 2007 (see Chevallier, 2010). Therefore emissions allowances prices had lost their real value.

The minimum trading volumes for each futures contract are 1,000 tons of $CO_2$ equivalents. We select that date samples are time-varying daily settlement prices for EUA futures contracts with varying maturities going from December 2010 to December 2014. The trading of futures contracts with vintages December 2013 and 2014 were introduced on April 8, 2008.Considering the availability and continuity of EUA futures price, we choose that date samples cover the period going from April 8, 2008 to December 20, 2010 in the ECX market.

## 2.2 Descriptive of statistical evidence for emissions futures price

In the following figure 1, $F_1,F_2,F_3,F_4,F_5$ denote the traded EUA futures contracts with varying maturities going from December 2010 to December 2014.Among them, $F_1$ is the closest to maturity for

EUA futures contract, $F_2$ is the second closest to maturity for EUA futures contract, and so on. In the figure 1, we see the prices of EUA futures contracts with varying delivery dates have strongly time-varying trend in the whole sample period. We find that futures prices for emissions allowances show strongly time-varying volatility, and they have the higher upward and downward jump.

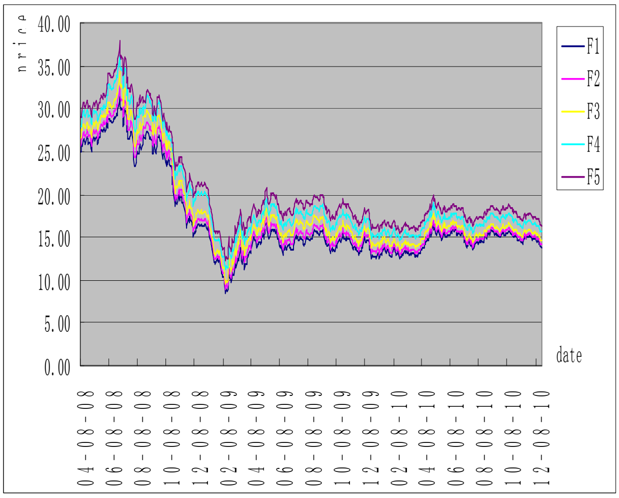


*Fig. 1.* EUA futures price for $CO_2$ emissions allowances

*Table 1* Descriptive statistical evidence for EUA futures prices

| *futures* | *mean* | *max* | *min* | *Std.dev* | *skew* | *kurt* |
|---|---|---|---|---|---|---|
| $F_1$ | 16.87 | 31.71 | 8.43 | 5.15 | 1.27 | 3.25 |
| $F_2$ | 17.48 | 32.90 | 8.90 | 5.31 | 1.28 | 3.25 |
| $F_3$ | 18.32 | 34.38 | 9.43 | 5.49 | 1.28 | 3.24 |
| $F_4$ | 19.63 | 36.43 | 11.30 | 5.66 | 1.29 | 3.29 |
| $F_5$ | 20.67 | 37.78 | 12.30 | 5.71 | 1.30 | 3.34 |

Seen from the above table 1, mean statistics shows mean values for EUA futures price gradually step up with time-to-maturity increase. Emissions futures price with long-term delivery date is higher than the recent futures contract, therefore term structure of $CO_2$ futures price is contango. To some extent, the volatility can measure changing speed in the market price. The increasing volatility in futures price for emissions allowances can be observed with the rising of time-to-maturity, and it denotes mean-reverting speed in the price of futures contract with long-term maturity is faster than those futures contract with short-term maturity. With the increasing of time-to-maturity, higher positive kurtosis and skewness denote the greater deviation degree and long tail dragging on the right of Gaussian distribution.

# 3 Affine term structure model for $CO_2$ futures prices

Futures prices for $CO_2$ emissions allowances are affected by the observable and non-observable state variables, for example emissions regulation policy, energy prices and energy efficiency, low-carbon technologies progress and application, extreme climate change and short-time deviation in emissions allowances price etc. Regulation policy, energy efficiency, low-carbon technologies progress and application promote long-term equilibrium between demand and supply in $CO_2$ emissions allowances markets, they directly determine long-term trend in emissions allowances prices. Extreme climate change, interest rate fluctuation, energy price and the other factors induce expected changes between demand and supply in $CO_2$ emissions allowances markets, they push short-term deviation in emissions allowances prices. Therefore futures prices are affected by many state variables.

Manoliu and Tompaidis (2002), Wang et al (2010) propose that the commodities spot prices are composed of general N-factor state variables $\ln S_t = \sum_{i-1}^{N} x_{it}$ , and they provide simple analytical valuation formulas for futures prices. Cotarzar and Naranjo (2006) assume that the commodity spot price can be described as the arbitrary number of state variables and the long-term constant growth rate, $\log S_t = h' x_t + u_t$ .In our model, Let $S_t$ denote the spot prices for emissions allowances at time *t*, and spot prices are assumed as non-observable state variables. We consider that the logarithmic process of spot prices for $CO_2$ emissions allowances can be more precisely expressed as a sum of N-factor state variables[7][9]:

$$\ln S_t = \sum_{i-1}^{N} x_{it} \tag{1}$$

where the vector of state variables $x_t$ follows mean-reverting process with the Ornstein- Uhlenbeck type. We assume that a constant market risk premium $\lambda$ , the risk-adjusted process for the vector of state variables is equal to [8-9]:

$$dx_t = -(\mathrm{K}x_t + \lambda) + \Sigma dZ_t \tag{2}$$

where market risk premium $\lambda = [\lambda_1 \;\; \lambda_2 \;\; \cdots \;\; \lambda_n]^T$ is an $n \times 1$ vector of real constants, $K = \begin{bmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_n \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix}$ are $n \times n$ diagonal matrices with entries that are positive constants and term-wise different. $dZ_t$ is a $n \times 1$ vector of correlated Brownian motion increments, such that $(dZ_t)(dZ_t)^T = \Omega dt$ ,where the $(i, j)$ element of $\Omega$ is $\rho_{ij} \in [-1,1]$ , the instantaneous correlation between state variables $i$ and $j$[8].

$F(t, T)$ denotes $CO_2$ emissions allowances price at time t for futures contract with the delivery date T. The futures price for emissions allowances $F(t, T)$ can be defined as the expected value of spot price at the delivery date $T$ under the risk-neutral measure Q [15].

$$F(x_t, t, T) = E_t^Q(S_T) \tag{3}$$

As shown in the Appendix A, the futures price for emissions allowances at time $t$ and maturing at $T$ in Equation (3) can be described as:

$$F(x_t, t, T) = \exp\left(\sum_{i=2}^{N} e^{-k_i(T-t)} x_i + \mathrm{A}(T-t)\right)$$

$$A(T-t) = -\sum_{i=1}^{N} (1 - e^{-k_i(T-t)}) \frac{\lambda_i}{k_i} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j \rho_{ij} \frac{1 - e^{-(k_i+k_j)(T-t)}}{k_i + k_j} \tag{4}$$

One important advantage of this model is tractable to obtain simple analytical futures price formula for emissions allowances. The logarithm of the futures price is a linear function of N-factors state variables, it is useful when estimating the parameters of term structure model by using the Kalman filter method. Because the state variables have a multivariate normal distribution, any linear combination of state variables will also distribute normal by allowing maximum likelihood technique [8].

# 4 The Kalman filter

As the above mention, the most difficulty in the empirical implementation for EUA futures price is the arbitrary and non-observable state variables in the affine model. The state space form is the appropriate procedure to deal with situations in which the state variables are not directly observable. The Kalman filter is the most appropriate estimation methodology which recursively calculates the model parameters and the time series of unobservable state variables. The form of state variables is applied to a multivariate time series of state variables. The measurement equation relates a vector of observable variables $y_t$ with a vector of state variables $x_t$. In our affine model, EUA futures prices as inputted observable variables are the time series at several varying maturities. Measurement equation in the affine model is then given by equation (4):

$$y_t = \mathrm{H}_t x_t + d_t + v_t \quad v_t \in (0, \mathrm{R}_t) \tag{5}$$

where $y_t = [\ln F(t, T_i)]_{m\times1}$ is a $m \times 1$ vector of EUA futures contracts with varying maturities, $x_t = [x_{1t}, x_{2t}, \cdots\cdots, x_{nt}]^T$ *is* a $n \times 1$ vector of state variables, $H = \begin{bmatrix} e^{-k_1\tau_1} & e^{-k_2\tau_1} & \cdots & e^{-k_n\tau_1} \\ e^{-k_1\tau_2} & e^{-k_2\tau_2} & \cdots & e^{-k_n\tau_2} \\ \vdots & \vdots & \cdots & \vdots \\ e^{-k_1\tau_m} & e^{-k_2\tau_m} & \cdots & e^{-k_n\tau_m} \end{bmatrix}$ is a $m \times n$ matrix, $d_t = [A(T_i)]_{m\times1}$ is a $m \times 1$ vector, and $v_t$ is a $n \times 1$ vector of serially uncorrelated Gaussian disturbances with $E(v_t)=0$ and $cov(v_t)=R^2$ [8-9].

Based on equation 2, the transition equation in the affine model is described as the stochastic process followed by the state variables:

$$x_t = G_t x_{t-1} + c_t + w_t, \; w_t \in (0, \mathrm{Q}_t) \tag{6}$$

where $G_t = \begin{bmatrix} e^{-k_1\tau} & 0 & \cdots & 0 \\ 0 & e^{-k_2\tau} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{-k_n\tau} \end{bmatrix}$ is a $n \times n$ vector of diagonal matrix, we assume $\tau = T - t$ , $c_t = [0, 0, \cdots\cdots, 0]^T$ is a $n \times 1$ vector, and $w_t$ is a $n \times 1$ vector of serially uncorrelated Gaussian disturbances with $E(w_t)=\boldsymbol{0}$ and $\mathrm{cov}(w_t)=\boldsymbol{Q_t}$ [9] .

# 5 Parameters coefficients estimation

The data samples used in this empirical study are observable daily settlement price for EUA futures contracts with the varying maturities going from December 2010 to December 2014, and five futures contracts are used in the parameters estimation. The empirical period of data samples covers the historical time series going from April 8, 2008 to

December 20, 2010. EUA futures prices as observable variables are inputted into the measurement and transition equations, the estimated parameters in the affine model are obtained by using the Kalman filter and maximum likelihood techniques. When the number of state variables *n* is equal to one, two and three, the following estimated parameters coefficients in the affine model are listed in the following table 2, 3 and 4 by using the same date samples.

*Table 2*: Estimated parameters coefficients in one-factor affine model

| *Model parameters* | *Coefficient value* | *Std. Error* | *Z-Statistic* | *Probability* |
|---|---|---|---|---|
| $k_1$ | 0.0214 | 0.0007 | 31.75 | 0.0000 |
| $\lambda_1$ | -0.116 | 0.0019 | -60.63 | 0.0000 |
| $\ln\sigma_1^2$ | -7.602 | 0.0452 | -168.21 | 0.0000 |
| *Log likelihood* | | 8163.769 | | |

*Table 3*: Estimated parameters coefficients in two-factor affine model

| *Model parameters* | *Coefficient value* | *Std. Error* | *Z-Statistic* | *Probability* |
|---|---|---|---|---|
| $k_1$ | 0.093 | 0.0048 | 19.48 | 0.0000 |
| $\lambda_1$ | -0.437 | 0.0295 | -14.84 | 0.0000 |
| $\ln\sigma_1^2$ | -6.225 | 0.1684 | -36.97 | 0.0000 |
| $k_2$ | 0.229 | 0.0137 | 16.80 | 0.0000 |
| $\lambda_2$ | 0.138 | 0.0209 | 6.61 | 0.0000 |
| $\ln\sigma_2^2$ | -7.373 | 0.338 | -21.81 | 0.0000 |
| cov($x_1$, $x_2$) | -0.001 | 0.0003 | -3.91 | 0.0001 |
| *Log likelihood* | | 10574.52 | | |

As is shown in the above table 2, 3 and 4, all mean-reversion speed parameters $k_{1,}$, $k_2$, $k_3$ are significantly unequal to zero at the significant level 1%, and their corresponding standard deviations of the measure errors approximately go to zero when the number of unobservable state variables are equal to one, two and three, those signs show that the state variables $x_{1t}$, $x_{2t}$, $x_{3t}$ follow mean-reverting process. The higher value of mean-reversion speed parameters *k* indicates the existence of strong mean-reversion process and the shorter time-variant of the corresponding state variables for EUA futures price. The parameters of market risk premium $\lambda_1, \lambda_2, \lambda_3$ and volatility $\sigma_1, \sigma_2 . \sigma_3$ are almost significant at the significant level 1%, and their standard deviations of the measure errors are lower. Covariance statistics describes that the state variables $x_{1t}$ and $x_{2t}$ have negative correlation, and the state variables $x_{1t}$ and $x_{3t}$ have positive correlation, however the state variables $x_{2t}$ and $x_{3t}$ have negative correlation.

*Table 4:* Estimated parameters coefficients in three-factor affine model

| *Model parameters* | *Coefficient value* | *Std. Error* | *Z-Statistic* | *Probability* |
|---|---|---|---|---|
| $k_1$ | 0.0462 | 0.0014 | 34.16 | 0.0000 |
| $\lambda_1$ | -0.0017 | 0.0001 | -33.80 | 0.0000 |
| $\ln\sigma_1^2$ | -7.196 | 0.1071 | -67.19 | 0.0000 |
| $k_2$ | 1.001 | 0.061 | 16.46 | 0.0000 |
| $\lambda_2$ | 0.2955 | 0.0903 | 3.273 | 0.0011 |
| $\ln\sigma_2^2$ | -8.073 | 0.2310 | -34.955 | 0.0000 |
| cov($x_1$, $x_2$) | -0.0001 | 0.0001 | -2.201 | 0.0277 |
| $k_3$ | 0.1384 | 0.0055 | 24.94 | 0.0000 |
| $\lambda_3$ | -0.6032 | 0.0174 | -34.68 | 0.0000 |
| $\ln\sigma_3^2$ | -7.247 | 0.1013 | -71.55 | 0.0000 |
| cov($x_1$, $x_3$) | 0.0003 | 0.0002 | 3.712 | 0.0002 |
| cov($x_2$, $x_3$) | -0.0003 | 0.0001 | -3.98 | 0.0001 |
| *Log likelihood* | | 11181.16 | | |

## 6 Parameters robustness

The performance and estimation procedure in the affine model are measured by estimating term structure of observable futures price and the empirical volatility of term structure in futures returns [8]. These daily prediction errors represent the difference between observable futures prices and predictable futures prices by using the Kalman filter technique. The prediction errors for EUA futures contracts with varying delivery dates are shown in the following Figure 2, 3, 4. $e_1, e_2, e_3 . e_4 . e_5$ denote the prediction errors for $CO_2$ futures contracts with varying maturities, $e_1$ is the prediction error for $CO_2$ futures contract with the closest to maturity, $e_2$ is the prediction error for $CO_2$ futures contract with the second closest to maturity, and so on.

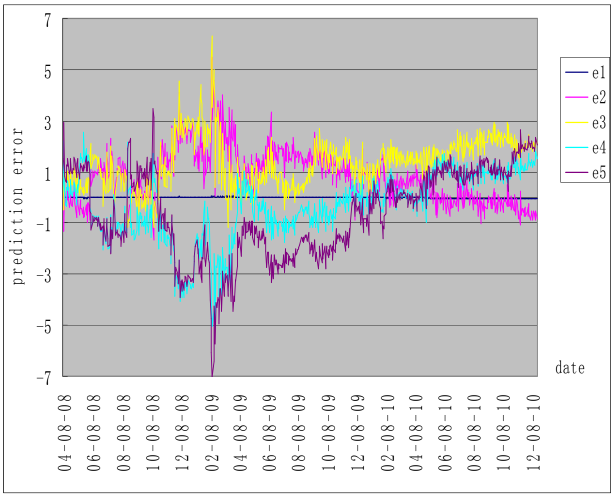


*Fig. 2.* The prediction errors in EUA futures price (one-factor model, ×100%)

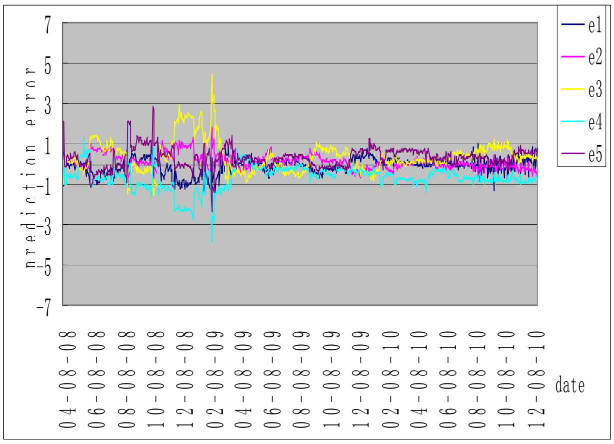


*Fig. 3.* The prediction errors in EUA futures price (two-factor model, ×100%)

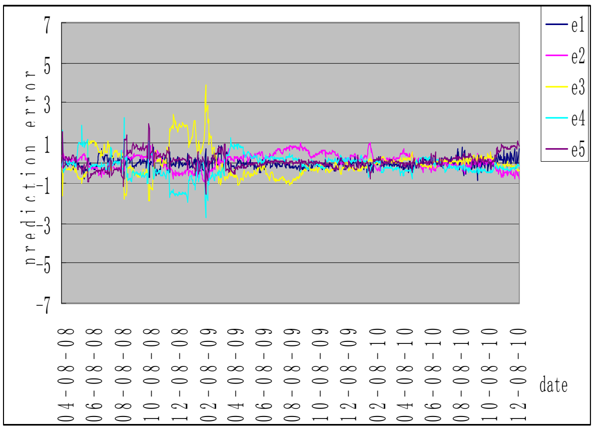


*Fig. 4.* The prediction errors in EUA futures price (three-factor model, ×100%)

Based on historical time series for EUA futures contracts, we compare the capabilities of fitting in the affine model. Seen from the above figures 2, 3, 4, we can obviously find the prediction errors in $CO_2$ futures price are the biggest in the one-factor model, and the prediction errors in $CO_2$ futures price are significantly lower in the two-factor and three-factor model. Mean absolute errors (MAE) and root mean square errors (RMSE) in prediction errors for five EUA futures contracts are shown in the above Table 5. The MAE and RMSE in prediction errors from the one-factor model are the largest, they reflect that the very large deviations from the one-factor model cannot accurately fit observable futures price in time-variant date samples. Compared with one-factor model, MAE and RMSE in prediction errors from the two-factor model is lower, and the lower deviations from the two-factor model appear to significantly forecast for the observable futures price. Compared with two-factor model, MAE and RMSE in prediction errors from the three-factor model is the lowest, and their lowest deviations indicate that their fitting ability slightly increases. Therefore MAE and RMSE for five $CO_2$ futures contracts from the two-factor and three-factor model are less than 1%, they indicate that two-factor and three-factor model can more accurately describe the term structure of EUA futures prices.

*Table 5* Comparison results of the prediction errors statistics ( $\times 10^{-4}$ )

| ln*F* | *One-factor* | | *Two-factor* | | *Three-factor* | |
|---|---|---|---|---|---|---|
| | *MAE* | *RMSE* | *MAE* | *RMSE* | *MAE* | *RMSE* |
| $\ln F_1$ | 177 | 225 | 30.9 | 42.1 | 19.1 | 25.0 |
| $\ln F_2$ | 100 | 130 | 38.2 | 48.0 | 33.2 | 39.5 |
| $\ln F_3$ | 140 | 166 | 58.6 | 83.4 | 48.9 | 70.2 |
| $\ln F_4$ | 105 | 172 | 73.2 | 89.4 | 39.1 | 54.3 |
| $\ln F_5$ | 153 | 240 | 43.4 | 56.9 | 27.3 | 38.9 |

## 7 Conclusions

In recent years, futures markets for emissions allowances have become liquid and potential markets within the EU emissions trading scheme (EU ETS). Although the $CO_2$ spot price is a tradable and observable asset in the EU ETS, we assume that spot price is an unobservable state variable, and our affine model provides that futures price can be expressed as multi-factor arbitrary state variables. Based on the affine model of futures price [8-9], we propose a new N-factor affine term structure model of EUA futures price and the corresponding futures valuation. Based on the state space formulation of futures price, we can estimate parameters in the affine model by using the Kalman filter and maximum likelihood techniques.

We find that futures prices for $CO_2$ emission allowances show strongly time-varying motion trend. Our empirical results show that all unobservable state variables follow significantly mean-reversion process, therefore futures prices for $CO_2$ emissions allowances also follow mean-reverting process. The parameters coefficients of mean-reversion speed, market risk premium, volatility and correlation are of almost significant level. Compared with one-factor model, MAE and RMSE in prediction errors from two-factor and three-factor model are lower; accordingly two-factor and three-factor model can accurately describe the term structure of EUA futures price. In general, our affine model can work well in estimating the term structure of EUA futures prices. The direction of future work is to study term structure of volatility and the implications of affine model in future options pricing for futures contracts.

# Appendix A

This Appendix deduces Equation (4) with the use of Equation (3). Because the conditional normal distribution of the spot price $S_T$ is the lognormal, it follows that

$$\mathrm{E}^Q(S_T) = \exp(E_t^Q(\ln S_T) + \frac{1}{2}VAR_t^Q(\ln S_T)) \quad \text{(A1)}$$

Where

$$\mathrm{E}_t^Q(\ln S_T) = l^T E_t^Q(x_T)$$

$$VAR_t^Q(\ln S_T) = l^T \operatorname{cov}_t^Q(x_T) l$$

Where $l = [1,1,\cdots\cdots,1]^T$ is a $n \times 1$ vector, $E_t^Q(x_T)$ is the expected value of the state variable $x_T = [x_{1t}, x_{2t}, \cdots\cdots, x_{nt}]^T$, $\operatorname{cov}_t^Q(x_T)$ is a covariance matrix of the state variable $x_T$

From equation (2), the conditional moments of $\boldsymbol{x}_t$ are

$$\mathrm{E}_t^Q(x_T) = e^{-K(T-t)}x_t - \lambda\int_0^{T-t} e^{-K\tau}d\tau \quad \text{(A2)}$$

$$\operatorname{cov}_t^Q(x_T) = \int_0^{T-t} e^{-K\tau}\Sigma Q \Sigma^T (e^{-K\tau})^T d\tau \quad \text{(A3)}$$

Where $(dZ_t)(dZ_t)^T = \Omega dt$, thus

$$E_t^Q(x_T) = e^{-k_i(T-t)}x_{it} - (1-e^{-k_i(T-t)})\frac{\lambda_i}{k_i}, i = 1,2,\cdots\cdots,n \quad \text{(A4)}$$

$$\operatorname{cov}_t^Q(x_T) = \rho_{ij}\sigma_i\sigma_j \frac{1-e^{-(k_i+k_j)(T-t)}}{k_i+k_j}, i,j = 1,2,\cdots\cdots,n \quad \text{(A5)}$$

The futures price for $CO_2$ emissions allowances *F(t, T)* can be defined as the expected value of spot price at the delivery date *T* under the risk-neutral measure *Q*, $F(x_t, t, T) = E_t^Q(S_T)$. The valuation of formula (4) is obtained by inserting equations (A1), (A4) and (A5) into Equation (4).

$$F(x_t, t, T) = \exp(\sum_{i=2}^{N} e^{-k_i(T-t)}x_i + \mathrm{A}(T-t))$$

$$A(T-t) = -\sum_{i=1}^{N}(1-e^{-k_i(T-t)})\frac{\lambda_i}{k_i} + \quad \text{(A6)}$$

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j\rho_{ij}\frac{1-e^{-(k_i+k_j)(T-t)}}{k_i+k_j}$$

*References:*


[1] Benz E., Truck S. $CO_2$ emission allowances trading in Europe: Specifying a new class of assets, Problems and Perspectives in Management, Vol. 4, Issue 3, 2006, pp. 30-40.

[2] Gibson R., Schwartz E.S. Stochastic convenience yield and the pricing of oil contingent claims, Journal of Finance, Vol. 45, No.3, 1990, pp. 959-976.

[3] Schwartz E.S. The stochastic behaviour of commodity prices: implications for valuation and hedging, Journal of Finance, Vol. 52, No.3, 1997, pp. 923- 973.

[4] Miltersen K.R., Schwartz E.S. Pricing of options on commodity futures with stochastic term structures of convenience yields and interest rates, Journal of Financial and Quantitative Analysis, Vol. 33, No.1, 1998, pp. 33-59.

[5] Lautier D. Term structure models o f commodity prices: a review, Journal of Alternative Investments, Vol. 8, No. 1, 2005, pp. 42- 63.

[6] Schwartz E.S., Smith J.E. Short-term variations and long-term dynamics in commodity prices, Management Science, Vol. 46, No.7, 2000, pp. 893- 911.

[7] Manoliu M., Tompaidis S. Energy futures prices: term structure models with Kalman filter estimation, Applied Mathematical Finance, Vol. 9, Issue 1, 2002, pp. 21-43.

[8] Cortazar G., Naranjo L. An n-factor Gaussian model of oil futures prices, Journal of Futures Markets, Vol. 26, No. 3, 2006, pp. 243–268.

[9] Wang S.S, Wang L., Li Z.C, Xiang J. Affine term structure models of futures prices based

on Kalman filter, Journal of Systems Engineering, Vol. 25, No. 3, 2010, pp. 346-353.
[10] Benz E., Truck S. Modeling the price dynamics of $CO_2$ emission allowances, Energy Economics, Vol. 31, 2009, pp. 4–15.
[11] Wagner M.W., Homburg M.U. Futures price dynamics of $CO_2$ emission certificates–An empirical analysis, Journal of Derivatives, Vol.17, No.2, 2009, pp. 73-88.
[12] Daskalakis G., Psychoyios D., Markellos R.N. Modeling $CO_2$ emission allowance prices and derivatives: Evidence from the European trading scheme, Journal of Banking & Finance, Vol. 33, 2009, pp. 1230–1241.
[13] Chevallier J. Modeling risk premia in CO2 allowances spot and futures prices, Economic Modelling, Vol. 27, 2010, pp. 717–729.
[14] Montagnoli A., Vries F.P. Carbon trading thickness and market efficiency, Energy Economics, Vol. 4, 2010, pp. 1-6.
[15] Cox J.C., Ingersoll J., Ross S. The relation between forward prices and futures prices, Journal of Financial Economics, Vol. 9, 1981, pp. 321–346.