Market Index Biases and Minimum Risk Indices

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Abstract: Markets, in the real world, are not efficient zero-sum games where hypotheses of the CAPM are fulfilled. Then, it is easy to conclude the market portfolio is not located on Markowitz's efficient frontier, and passive investments (and indexing) are not optimal but biased. In this paper, we define and analyze biases suffered by passive investors: *the sample, construction, efficiency and active biases and tracking error* are presented. We propose Minimum Risk Indices (MRI) as an alternative to deal with to market index biases, and to provide investors with portfolios closer to the efficient frontier, that is, more optimal investment possibilities. MRI (using a Parametric Value-at-Risk Minimization approach) are calculated for three stock markets achieving interesting results. Our indices are less risky and more profitable than current Market Indices in the Argentinean and Spanish markets, facing that way the Efficient Market Hypothesis. Two innovations must be outlined: an error dimension has been included in the backtesting and the Sharpe's Ratio has been used to select the 'best' MRI.

Key-Words: Index Biases, Passive Investing, Market Indices, VaR, Portfolio Optimization, Minimum Risk Indices.

1 Introduction

Finance has grown to include ideas such as the market, zero-sum games, efficiency or the CAPM.

Financial theory, as based on these four pillars, concludes that an investor can not consistently beat the total market. Active investment in traditional finance is useless, thus meaning that a passive investment, *a buy and hold* strategy is the optimal strategy to follow if we operate in an efficient zero-sum game market where CAPM hypotheses are fulfilled. To simplify this strategy, to help passive investors to *hold the market*, market indexes were created as references for passive investors, as proxies of a concrete analytical market. In the framework described by traditional finance, total market indices are appropriate benchmarks for passive investment. In [5] I proposed an extended revision of these four fundamental items achieving different conclusions that are summarized here:

a) Real markets are not really zero-sum games but minus-sum games. What some people lose does not go completely to others, because money is eliminated from the system through commissions and fees from brokers and dealers, advisors' and analysts' revenues, etc.

b) Markets are marginally efficient. Investors are not fully rational. They underprice and overprice securities, overreact to good and bad news or even to irrelevant information. Individuals systematically violate Bayes' rule and other maxims of probability theory in their investment decisions and suffer from psychological biases. Moreover, the psychological evidence shows irrationality is correlated in price bubbles and panic movements. Arbitrage is limited in these marginal efficient markets, sometimes because there are not perfect substitutes to securities, or because there are legal or operational limitations. In real markets, arbitrage can not always avoid mispricing.

c) The results of the basic CAPM are absolutely related to its assumptions, so if some of them are not fulfilled the results are in deep trouble. In particular, there are transaction costs that make market a minus-sum game and limit arbitrage; there are taxes, and investors' decisions include realization of losses and deferral of capital gains to avoid them; not all investors are well informed, and they do not have the same predictions for expected returns, volatilities and correlations, etc. When CAPM hypotheses are not fulfilled, and we have provided some examples that they are not always fulfilled, investors hold different portfolios, therefore, the market portfolio is not optimal and not located in the efficient frontier.

In that revised framework, passive investment (and indexing) is not optimal but biased. These biases can be seen in Fig. 1, in the traditional Markowitz's mean-variance analysis. In a concrete moment of time t, the Market Portfolio return R_{mt} is located below the Efficient Point return R_{et} and the Maximum Return available at the market R_{ct}. The market Index return R_{bt} (proxy of the Market Portfolio), and the Proxy Portfolio return R_{pt} (proxy of the index held by passive investors), are usually located even below in terms of return and probably suffering from more risk.

Paying attention to return[1] differences, market index biases are defined as follows: i) The sample bias (SB) is the difference between R_{bt} and R_{mt} due to the stock sample selection that make up an index. ii) The construction bias (CB) is the difference between R_{bt} and R_{mt} due to the use of different weighting criteria (price, equal, GDP, fundamental criterions, etc.) or methodologies (Laspeyres, Paasche or Geometric mean among others) in the construction of an index. iii) Tracking error (TE) is the difference between R_{bt} and R_{pt} due to commissions and turnover costs and is the only bias that has been studies in the literature. iv) Efficiency bias (EB) is the return difference between R_{mt} and R_{et} and is created by the lack of efficiency in the financial system. v) Active bias (AB) is defined as the difference between the Efficient Point (R_{et}) and the best investment opportunity (R_{ct}), that is, an opportunity cost.

Mathematically, these biases can be calculated using expressions in Table 1. In that table, it is possible to found biases defined in a moment of time (t), and also along a time period -Average Absolute Bias (AA) and Standard Deviation of a Bias (SD)-.

Concerned with this situation, the aim of this paper is to propose alternative Market Indices that solve, at least in part, some of the detected biases. There are three important reasons for creating new Market Indices. First, there is a huge interest in market risk management after the last bearish market context and financial disasters. This is clear from the Basel agreements, and from the concern regarding financial bankruptcies such as the Long Term Capital Management case or the more recent Bear Stearns or Lehman Brothers disasters. Because of their special characteristics, such interest is perhaps greater when we speak about emergent markets, where efficiency and development in financial markets is lower. Second, because of the traditional Fama's idea of efficient markets and models such as CAPM, market yield is still used as an essential parameter and trillions of dollars are invested by following the market. Third, market

[1] Market Index Biases could be defined using return, risk, or risk-adjusted returns. I defined them using return in [5].

index biases shows us that a capitalization-weighted Index (and even the market portfolio itself) is not always located in the efficient frontier, therefore, there are other portfolios (and also indexes) able to beat the market with lower assumed risk, offering investors better risk-adjusted investment alternatives.

In this paper, we propose a methodological approach to deal with this question using a parametric Value-at-Risk Minimization to build Minimum Risk Indices (MRI), benchmark portfolios with better risk-adjusted characteristics than nowadays market indices. The authors apply the method to the Spanish, Argentinian and American stock markets during the 2000-2004 period, extending the results of [6]. The first step in the calculation process is the estimation of Covariance matrices by different methodologies. The second step is to optimize VaR to obtain the weights each share must have within the index to minimize risk. And finally, the best alternative market index is selected among all alternatives using traditional backtesting and improved backtesting using a simple error dimension measure and the Sharpe's Ratio.

Although the estimation methods used here are very simple (Moving Averages), results are quite interesting. All indices are less risky than the Spanish IBEX 35® and the Argentinean Merval (current Market Indexes) and, surprisingly, more profitable. This does not happen in the American market. These results highlight one idea: similar investment strategies could beat some markets, thus questioning the Efficient Market Hypothesis, and reinforcing the market bias analysis. Possible applications of Minimum Risk Indices are clear: they could reduce the risk assumed by institutional and mutual funds that nowadays follow current Market Indices. They could also be used as a benchmark for risky assets or as a basis for developing derivatives.

The structure of the paper is as follows. In Section 2 we discuss the theoretical framework of Minimum Risk Indices. This analysis is completed in Section 3 with an empirical study, where the theoretical framework is applied to the Spanish, American and Argentinean Stock Markets during the 1999–2004 period. Performance and risk of MRI are reconstructed and the 'best index' is selected using traditional backtesting and improved backtesting using performance parameters. Section 4 gives the conclusions, Section 5 establishes future lines of research, and Sections 6 and 7 contain the appendix and references.

2 Minimum Risk Indices (MRI)

Financial risk has historically been analyzed by multiple measures [21, 26] and models as Markowitz's or Sharpe's approaches [17, 25, 23]. However, the increasing volatility in financial markets, derivatives and technological advances force academics to now treat market risk from other perspectives and conceptions about performance and risk must be modified [7, 11, 18]. One widely accepted measure is Value-at-Risk [24] with its evolutions [22, 14, 1, 15], a measure of risk that has been rapidly and widely accepted since it was introduced in 1995. All the VaR-based methods have several problems regarding leptokurtosis or skewness, so complementary techniques as Stress Testing, Conditional VaR or Extreme Value Theory with the Expected Shortfall have been analysed [19 or 20]. VaR has several problems, but if we can determine a controlled scenario with some interesting conditions, traditional VaR methods are reliable enough [12] and easier to calculate than Extreme Value methods and more sophisticated approaches.

In this paper we pay attention to VaR as a tool for market risk management and portfolio optimization through the creation of new Market Indices by VaR minimization. If we want to use VaR as a risk management tool, we have to find a method that institutions and investors find easy to follow. These characteristics mean that parametric VaR is the most suitable method, however, it is necessary to take into account the weaknesses of the parametric approach: if returns are not normal, the VaR measure will not be coherent [8]. In this study, the Central Limit Theorem should make Market Index returns similar to a Gaussian distribution if the number of shares forming the Index is high enough. Indeed, as in the portfolio we do not include non-linear positions, and we use weekly data at a 5% significance level to calculate the VaR, the parametric Gaussian approach is considered reasonably good [12].

The problem then is reduced to minimize Parametric VaR subject to non-negativity of stock weights and non-leveraged possibilities, selecting the desired significance level. One of the most important steps in this procedure is to estimate return covariance matrix, and this could be done using:

i) The Historical Volatility Method: this method has problems with time-varying volatility.

ii) The Moving Average Method: this method provides better estimations.

iii) The Exponential Weighted Moving Average Method: here last observations receive higher weight, which solves some problems of the Moving Average Method. The decay factor election is critical.

iv) The GARCH and E-GARCH Methods: it is possible to deal with heteroskedastic time dependent variance with these methodologies. Covariance estimations are clearly better than those from simple methods, but GARCH and E-GARCH are not very used in the professional world.

After the Covariance matrix has been estimated using one of these methods, the minimization process can be applied to obtain the optimal weights each share must have within the Index to minimize the Index's market risk. With the historical data available, we can reconstruct the performance and evolution of Minimum Risk Indices to compare returns and risks between Minimum Risk Indices and current Market Indices.

Once the reconstruction of Minimum Risk Indices is available for a certain market, the validity of each approximation must be checked by a backtesting process. This process will establish how well the model applied to the data fits the real market. Here we can see the importance of selecting the 'best' Index from all the approximations. In our opinion the 'best' model should be selected in accordance with two key ideas: The model's capacity to be accepted by a periodic backtesting process, and the relationship between return and risk in each index.

3 Minimum Risk Indices in Real Markets

Using the theoretical framework developed in part 2, we can generate Minimum Risk Indices for each Stock Market we chose. As an example of how our methodology reacts to different Market Indices, in this section we apply the MRI construction to the Spanish Stock Market, to the American Stock Market, and to the Argentinean Stock Market. These examples have two objectives. First, it is interesting to test how Minimum Risk Indices work in Stock Markets with different volatility and efficiency. Second, each of the Indices represents a different way to build a Market Index, and using them in our approximation is a first step to determine the importance of different sampling strategies, weightings and construction rules in the calculation of a market index. The sample and construction and efficiency bias must be taken into account when analyzing the performance of an index. In the Argentinean and American markets, the MERVAL and the DowJones show important construction and sample biases; first because they are weighted using negotiation and price, and second because they are not calculated using a Laspeyres capitalization approach.

The aim is to create Minimum Risk Indices based on the historical composition of the IBEX35®, the Dow Jones Industrial Average^{SM} and the MERVAL for the 2000-2004 period. To put it more simply: Minimum Risk Indices would be developed by taking into account only the shares contained in each Index in each period. In this way it is possible to determine whether a different weighting in the components of the actual Indices using a VaR Minimization criterion can reduce risk and to analyze how this affects the profitability of Market Indices. We call our Indices IndexVaR35 (IVaR35) for the Spanish Market, IndexVaR30 (IVaR30) for the American Market, and IndexVaRM (IVaRM) for the Argentinean Market. As we have mentioned, there is not just one Minimum Risk Index for each market, because with each estimation criterion we can create a Minimum Risk Index. The Covariance matrix was estimated in all the markets by the simplest estimation methods (the Historical method and the Moving Average Method using lengths from 4 to 100 weeks) in order to explain the method's potential benefits, although the authors know these estimates can be improved by more complex methods. In the end we decided to present IVaR35, IVaR30 and IVaRM Indices calculated only by some of these Moving Averages as being representative of the short, medium and long terms.

Covariance matrices estimated with a few data (4-30 weeks) are problematical because the minimization process is difficult or rather unstable in some cases. Short-length Moving Averages change quickly in response to financial data but they consistently underestimate the VaR value and cause problems inside the minimization process because the positive and semi-defined Variance-Covariance Matrix condition is sometimes not fulfilled. Medium-length Moving Averages (30-52 weeks) are more stable and VaR measures closer to real values. Finally, long-length Moving Averages (e.g. 60-100 weeks) are the most stable but are less able to adapt to volatile short-term changes. Despite the limited prediction capacity of Moving Averages, results with these approximations are quite interesting.

3.1 Volatility Analysis

The basic objective of the study is, by VaR minimization, to create Minimum Risk Indices that are less risky than current ones. Table 2 shows how our Indices are less risky than the current ones. From the data, it is easy to see how the reduction in volatility is greater in the Spanish Market than in the American or Argentinean Markets. It also shows that, in general, the longer the moving average, the less volatile, which means that risk is reduced. This seems not to be true in all the cases with the longest moving averages (52 and 78 in the IVaR35, 78-100 in the IVaR30 and 52-78 in the IVaRM) for which volatility is more or less the same or increases slightly. As with longer lengths, it is more difficult to estimate short changes in volatility, which could mean that there is an optimal moving average length beyond which it is impossible to reduce risk using the moving average method. Improved Moving Averages (in the IVaR35 and in IVaRM) are a little more risky than those with no improvements. This result is rational because, firstly, multiple-step estimation is applied to avoid underestimating the risk and, secondly, the 0.01% weighting restricts one asset so that the portfolio can be less diversified and risk rises.

Volatility reduction is clearer in Fig. 2, which shows cumulative volatilities. The first picture shows the Spanish Market. The first line represents the riskiest Index, which in our case is the current Market Index (IBEX 35®). The second group of lines is made up of the MA10, MA10a and MA10b approximations. The third group, with half the IBEX 35® risk, is made up of the MA25, MA52, MA78 approximations and all their modifications. The most stable approximations are the modifications a, especially MA52a, which is less risky than MA25a and MA78a. This again indicates the existence of an optimal length for moving averages beyond which it is impossible to better estimate the covariance matrix and reduce risk with moving average methods. The second picture shows the American Market. The first line again represents the riskiest Index, which is now the MA10 approximation due to problems with the positive and semi-defined covariance matrix condition. Then, and before observation 75, the second riskiest Index is the current Dow Jones Industrial Average[SM]. Below the current Market Index, and with less risk, we find all the other MA approximations. The least risky approximations are MA25 and MA30 and the more data is used to build the MA, the riskier the Moving Average approximation seems to be, which supports the idea of the optimal length for moving averages. Finally, the third picture represents the Argentinean Market. The first line (the riskiest Index) is again the current Market Index (MERVAL). Below we can see the MA10a and MA10b as the second group of riskiest approximations, and below this group, with less risk, the other MAs. Again we can establish the idea of optimal length for moving averages.

3.2 Analysis of extreme losses in VaR minimization

[9, 16] show that not allowing agents to assume more risk than a certain VaR value or to develop VaR minimizations can increase extreme losses, especially when return distributions are very different from Normal distributions. These results appear basically when distributions are heavily skewed or have long fat tails. In our case, the problems noticed by the above authors are not excessively important (see Table 3). For the shortest moving average, extreme losses are similar to those of IBEX 35® and lower in the American and Argentinean market and decrease when we increased the moving average length. There is a certain moving average length when extreme losses start to rise again (MA78 in IVaR35, MA70 in IVaR30 and more difficult to define in IVaRM), which again supports the existence of an optimal length moving average.

3.3 Return Analysis

Fig. 3 shows all the returns of Moving Average approximations. In the Spanish market, all our Indices have higher returns than the IBEX 35®. These results are surprising but not unique. Other authors have constructed portfolios able to beat the market [e.g. 3, 4, 10]. There are two reasons for these data. First, the Spanish Stock Market is suffering efficiency bias and Minimum Risk Indices are harvesting part of it. Second, the way the IBEX35® is constructed generates sample and construction bias, even they should be relatively small comparing with these biases in other markets. In contrast, in the American Stock Market, no Minimum Risk Index beats the market, reporting more efficiency in this market. In the IVaR30, the approximations with the worst returns are MA10. The other approximations performed quite well during the bearish market, being near the actual index or beating it in some periods, but they performed worse than the current index during and after the Iraq war in the bullish market. In the end, the best approximation in terms of profitability is the MA60, with a 20% lower return than the Dow Jones Industrial Average[SM]. The same two reasons could be put forward in this case. The efficiency bias in the American market is clearly lower than in the Argentinean or Spanish case, but the sample and construction biases in the Dow Jones are theoretically important. How these biases seem to compensate among them avoiding the outperformance of our indices is an interesting idea to be analyzed in the future. Finally, in the Argentinean Market, all approximations (except 78b) are able to beat the market. The same reasons put forward for the IBEX 35® are valid here. The efficiency biases seem to be especially important knowing a little about the Argentinean market. The construction and sample biases are very important taking into account that the sample in the MERVAL index is not totally representative of the market and the index is constructed using a negotiation weighting.

It is necessary to take into account some considerations. Firstly, it is essential to discover how efficiency affects these conclusions. This would mean calculating efficiency tests for each market and comparing results, but we must leave this for further research. Secondly, we must deeply analyze how other index biases affect risk and return. Finally, it is essential to analyze how results could be improved using more powerful techniques to estimate variance and covariance.

3.4 VaR Analysis

Each approximation has a different VaR measure that evolves over time. Fig.4 shows how great changes in volatility that are common in moving average approximations are greater in short length moving averages than in long length ones. On the other hand, as these types of averages do not attach different weights to more recent data than to older data, moving averages are indicators of ‘past’ volatility, regarding inappropriate Covariance estimations when price trends change. This problem decreases when the lengths are longer. Finally, we should point out that short moving averages usually underestimate VaR, so the losses beyond the VaR will be more frequent in those cases.

3.5 Normality Analysis and Backtesting

Normality analysis of logarithmic returns is not very positive, as it can be seen in Tables 4, 5 and 6. In all cases, the Normality Hypothesis has been rejected except in the case of MERVAL. In the case of IVaR35, the distributions have leptokurtosis and are slightly negatively skewed. In the case of IVaR30, the distributions are more leptokurtical, and negative skewness is especially important, affecting, as we have said, the profitability results. Finally, in the case of IVaRM, distributions are extremely leptokutical and skewness is positive because of the evolution of MERVAL during the period analyzed. If we look at the backtesting results, though real errors are more frequent than the 5% significance level expected, they are not very large (around 2% higher than the VaR value in the IVaR35, 2.80% higher in the IVaR30, and 2% higher in the IVaRM). Errors are more controlled in terms of frequency in the case of IVaR30 than for the IVaR35 or IVaRM, but they are less controlled in terms of magnitude (the mean error in the American and Argentinean cases is higher than in the Spanish case). [9] observed that setting VaR limits on institutions could lead to higher extreme losses than when these limits are not set. We can see from results, however, that this theoretical result is not clear here.

3.6 Model Selection

We have seen how parametric VaR minimization could create Minimum Risk Indices with less risk and, in the Spanish and Argentinean case, with greater profitability than current market indices. In this paper we construct 12 approximations using Moving Averages of different lengths for the Spanish and Argentinean market and 9 approximations for the American market. It is necessary now to decide which is the ‘best’ approximation to use in each market. In our opinion this selection should be done on the basis of two ideas:

(1) The model's capacity to explain reality or, in other words, its capacity to be accepted by a regular backtesting process. After determining the number of returns lower than the VaR value (classic backtesting), it is important to also measure the error magnitude. This type of backtesting has not yet been developed and here we only propose a very simple method that deals with error magnitude using the Excess Total Loss (ETL) measure, which is defined as the total sum of all returns lower than the VaR value over the studied period. We will choose those approximations with the lowest ETL in order to take into account the risk 'out of the model'. It is then necessary to select those approximations with less mean VaR or with less risk 'within the model'. As we can see in Table 7, in the Spanish market, using ETL the best approximations are MA52a, MA52b, MA78a and MA78b. Moreover, studying the 'controlled' risk within the model, we conclude that the MA52a and MA52b approximations are the least risky. In the American Market, (see Table 8), the best approximations using the ETL are MA60, MA70, MA78, MA85 and MA100. Using the 'controlled' risk, we conclude that the best approximations are MA60, MA78 and MA85. Finally, in the Argentinean market, the best approximations by ETL are MA52a, MA52b, MA78a and MA78b. After using the 'controlled risk' measured by the VaR, we can conclude that the best approximations among the four proposed are MA52a and MA52b.

(2) The relationship between return and risk, since [13] criticize not attaching importance to that point in VaR calculations. Here the authors use Sharpe's ratio to analyze this relationship and leave Reward-to-VaR ratio [2] or more complex approaches for future research. In the Spanish market, Sharpe's ratio in the MA52b approximation is bigger than in the MA52a approximation so we can conclude that MA52b is the best approximation with which to construct the Spanish Minimum Risk Index. In the American market, Sharpe's ratio in the MA60 approximation is the lowest of the selected MAs, so MA60 is the best approximation with which to construct the American Minimum Risk Index. Finally, in the Argentinean market, Sharpe's ratio in MA52a is higher than in MA52b, so it is reasonable to conclude that MA52a is the best approximation to construct the Argentinean Minimum Risk Index.

4 Conclusions

In this article we propose using the VaR as an active risk measure to construct Minimum Risk Indices to solve market index biases. We have used the parametric VaR approach to construct a very simple minimization problem in which the Covariance matrix among asset returns has to be estimated. Covariance matrix estimation can be done using many methods, and the Moving Average method has been chosen in our approaches. Different Moving Average lengths have been used in the empirical part of our study, building a group of alternative Minimum Risk Indices. There are, therefore, many ways of constructing a Minimum Risk Index—one for each way of estimating the Covariance matrix—so a method of selecting the best model among MRI is needed. This selection method must be based on two key ideas: i) the model's capacity to be accepted by a regular backtesting process, taking into account not also the error frequency but also the error magnitude (we propose the ETL as a error magnitude measure), and ii) the return-risk relationship of each Minimum Risk Index, analyzed in our empirical part though the Sharpe's ratio.

We apply this method to the Spanish, American and Argentinean markets to create different Minimum Risk Indices for the 2000-2004 period. Using the simplest Covariance matrix estimation methods, we achieve interesting results: our indices are less risky than the current ones (half the risk in the Spanish Market). Also, thanks to their optimal portfolio characteristics, the Spanish and Argentinean cases achieved bigger returns than those of the current market Indices, contrary to what is expected from the Efficient Market Hypothesis. These results show that both markets suffer from efficiency biases, and that Minimum Risk Indices could partially solve this. Part of the results, in the Argentinean case, can be due to the existence of an important sample and construction bias created by how the Argentinean index is built. This highlights an interesting discussion that needs to be dealt with care in future research and which must be based on the following ideas: (i) the ability to moving averages to estimate future covariance matrices and the possibility of obtaining better results with more complex estimation methods; (ii) the influence of the weighting process and other construction rules on market indices (basically, the sample and construction bias, that can be found deeply analyzed in [5]; (iii) the influence of market index biases in the performance and risk of indexes and how biases are additive or can be compensated among them; (iv) the Minimum Risk Index approximation in order to prove the efficiency of a market and to solve the efficiency bias; and (v) whether it is possible to obtain better results by not limiting our Minimum Risk Index shares to the current Market Index components and to the particular and 'legal' timing of changes in components.

The potential uses of Minimum Risk Indices are clear. Firstly, they are less risky and in some cases more profitable than current ones, which makes them a suitable benchmark of risky assets for mutual funds that currently follow market indices or a suitable base for derivatives. Secondly, Minimum Risk Indices may generate more stable Betas in the CAPM model, which is a possibility that must be developed in the future.

5 Future Lines of Research

The results achieved by very simple methods in the examples presented are interesting but it must also be said that there is still a lot to do. First it is necessary to determine whether better Covariance estimations using EWMA or GARCH or HAC methods can achieve better results in terms of risk and profitability. We also need to determine how the sample and construction biases affect efficiency tests, market index performance and the possibility of beating it. Finally, methods for selecting the 'best' model must be further developed since here we have only provided some general guidelines.

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6 Appendix

6.1. Minimum VaR Indices

The problem to solve can be written as follows:

$$\begin{aligned} &Min\{\ Z_\alpha \sqrt{x'\Sigma x}\ \} \\ &s.t. \\ &\qquad x_i \geq 0;\ i = 1..N \\ &\qquad \sum_{i=1}^{N} x_i = 1 \end{aligned} \qquad (1)$$

Where Z_α is the Normal distribution value at the desired significance level. The x vector contains the weights of each share within the alternative Market Index we are trying to build, N is the number of shares that make up the Index, and Σ is the logarithmic return Covariance matrix assumed to be a Multivariate Normal. This problem is easy to simplify and solve using Lagrangian optimization in which the optimal weights are the objective of our study and λ is a positive constant:

$$MínL = x'\Sigma x + \lambda(1 - x'\vec{1}) \qquad (2)$$

As the Covariance matrix is positive and semi-defined, if the number of observations is bigger than the number of assets, first order conditions are enough for a minimum[2].

$$\frac{\partial L}{\partial x} = \Sigma x - \lambda\vec{1} = \vec{0} \qquad (3)$$

$$\frac{\partial L}{\partial \lambda} = 1 - x'\vec{1} = 0 \qquad (4)$$

To solve the solution:

$$\begin{aligned} x^* &= \lambda(\Sigma^{-1}\vec{1}) \\ \lambda &= (\vec{1}'\Sigma^{-1}\vec{1})^{-1} \end{aligned} \qquad (5)$$

The solution gives the optimal weight each share must have within the index to minimize market risk.

[2] Second-order conditions are also necessary in the other case.

In the empirical case for the Spanish, American and Argentinean Stock Markets, we use an iterative algorithm based on Newton's method to make the minimization process and the command of the quadratic programming problem in Gauss with similar results.

6.2. Market Indices and their Minimum Risk Indices

In this section we will briefly explain how the Spanish IBEX35®, the American Dow Jones Industrial Average^SM and the Argentinean MERVAL are built. It is also important to explain certain characteristics and problems we found and solved by creating the Minimum Risk Index for each Market.

The Spanish IBEX35®:

The IBEX35® is built using the 35 largest companies in the Spanish Stock Market in terms of market capitalization and liquidity. Every six months the components of the Index are checked, some shares are included or excluded but the total number of assets is maintained. The Index is calculated using a market capitalization weighting criterion.

The Minimum Risk Indices (MRI) we created for this market were named IvaR35, and comprise the 35 shares of the IBEX 35® at each moment with the optimal weight established by the VaR minimization process. We must point out one problem with the IBEX 35® Spanish Market Index. In the six-month revision of the composition of the IBEX 35®, it is normal to include shares and companies with very little history on the Stock Exchange because it is relatively easy to be both new and one of the biggest 35 companies in the Spanish Market. During the period of our analysis we sometimes encountered this problem—especially in 1999-2000 because of the Internet and .com companies that grew quickly at that time. This makes it difficult to obtain complete data for all the IBEX 35® components in some periods and has important consequences in Covariance matrix estimation. After April 2000 we solved this problem with the following techniques:

a) Covariance matrix estimation using a multiple-step method: when we did not have complete data on the 35 shares, Covariance matrix estimation was done using a multiple-step method, estimating each individual value in the covariance matrix with all the available data.

b) 0.01% Weighting: the above solution improved the results, but shares with short historical data tended to underestimate risk and therefore received high weights because of their 'artificial' low risk. With this approximation we forced these shares to have the minimum weight accepted for our study.

The approximations we finally developed are shown in Table 10.

The American Dow Jones Industrial Average^SM:

The DJIA^SM is built using the 30 biggest companies in the American Stock Market and, for the sake of continuity, composition changes are rare. Inclusions and exclusions of shares are therefore rare and basically related to corporate acquisitions or dramatic business events. The Index is calculated using a price-weighting criterion.

Using available data for the DJIA^SM we did not need to apply improvements to the Covariance matrix estimation. The good quality of these data means that we used the methodology with a greater number of moving average lengths. Minimum Risk Indexes we created to this market were named IVaR30.

The approximations we developed are shown in Table 11.

The Argentinean MERVAL®:

The MERVAL is built using the most traded companies in the Argentinean Stock Market. The weights of each share in the index are calculated using the number of transactions of these shares in the Stock Market and the Volume of these transactions, so the Index is calculated using a negotiation weighting criterion.

The Minimum Risk Indices we created for this market were named IVaRM and comprise the shares of the MERVAL at each moment with the optimal weight established by the VaR minimization process. Every three months, the MERVAL composition is changed, and it is possible, as in the IBEX 35®, to find companies with very little historical data. Calculations must then be improved using the same techniques as for the IBEX 35®. Table 12 shows the approximations we used in this paper. It is important to point out that approximation b is especially influenced in the MERVAL by the fact that there are a lot of stocks with a short or no history when they enter the Index, this affects the performance and backtesting of the approximation.

6.3. IvaR35, IVaR30 and IVaRM Composition.

Each Minimum Risk Index has a different optimal composition. With short-term Moving Averages, the optimal composition changes frequently over the weeks, whereas it is more stable with long-term Moving Averages. This is important if we bear in mind that our index approximation needs a weekly adjustment, which means higher turnover and transaction costs with shorter moving averages. As optimal compositions are calculated using a risk measure and the Covariance matrix estimation, they are not like current compositions of Market Indices.

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Table 1
Biases in Passive Investment Strategies

Bias	Measure		
	Definition	Average Absolute (AA)	Standard Deviation (SD)
TE	$TE_t = R_{pt} - R_{bt}$	$AA_{TE} = \frac{\sum_{t=1}^{n} \lvert TE_t \rvert}{n}$	$SD_{TE} = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(TE_t - \overline{TE})^2}$
SCB	$SCB_t = R_{bt} - R_{mt}$	$AA_{SCB} = \frac{\sum_{t=1}^{n} \lvert SCB_t \rvert}{n}$	$SD_{SCB} = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(SCB_t - \overline{SCB})^2}$
EB	$EB_t = R_{mt} - R_{et}$	$AA_{EB} = \frac{\sum_{t=1}^{n} \lvert EB_t \rvert}{n}$	$SD_{EB} = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(EB_t - \overline{EB})^2}$
AB	$AB_t = R_{et} - R_{ct}$	$AA_{AB} = \frac{\sum_{t=1}^{n} \lvert AB_t \rvert}{n}$	$SD_{AB} = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(AB_t - \overline{AB})^2}$

Nota: (TE)=Tracking Error; (SCB)=Sample and Construction Bias; (EB)= Efficiency Bias; (AB)=Active Bias.
Source: Author's own.

Table 2
Market Indices' Standard Deviation

Approximation		Standard Deviation		Standard Deviation		Standard Deviation
Current Market Index	IBEX35®	0.02958	DJIA[SM]	0.02610	MERVAL	0.05817
Minimum Risk Index	IVaR35		IVaR30		IVaRM	
MA10	MA10	0.02387		0.02727		0.04970
	MA10a	0.02397				0.04989
	MA10b	0.02393				0.04983
MA25	MA25	0.01811		0.02347		0.04655
	MA25a	0.01835				0.04591
	MA25b	0.01884				0.04587
MA30				0.02354		
MA52	MA52	0.01736		0.02415		0.04796
	MA52a	0.01760				0.04855
	MA52b	0.01807				0.04838
MA60				0.02383		
MA70				0.02452		
MA78	MA78	0.01742		0.02433		0.04805
	MA78a	0.01835				0.04822
	MA78b	0.01867				0.04971
MA85				0.02451		
MA100				0.02533		

Table 3
Extreme Losses

	Highest Extreme Loss (%)				
IBEX35®		DJIA^{SM}		MERVAL	
IVaR35	11.1	IVaR30	15.4	IVaRM	15.3
MA10	11.2		10.8		12.1
MA10a	11.2				12.1
MA10b	11.2				12.1
MA25	6.6		9.2		11.4
MA25a	6.6				9.8
MA25b	6.6				9.8
MA30			8.4		
MA52	6.3		10.4		15.3
MA52a	6.8				15.3
MA52b	6.8				15.3
MA60			9.2		
MA70			10.6		
MA78	6.4		10.5		11.4
MA78a	7.6				10.9
MA78b	7.6				10.0
MA85			10.7		
MA100			10.9		

Table 4
Backtesting process in the IVaR35

	Normality		Backtesting			
	Jarque-Bera	Probability	Mean VaR(%)	Errors	% Errors	Mean Error(%)
IBEX35®	12.22	0.002				
MA10	102.4	0.000	0.654	75	31	1.95
MA10a	101.4	0.000	0.671	75	31	1.89
MA10b	103.3	0.000	0.665	75	31	1.89
MA25	32.9	0.000	1.446	43	18	1.35
MA25a	40.86	0.000	1.557	37	15	1.35
MA25b	48.3	0.000	1.582	36	15	1.37
MA52	43.3	0.000	1.746	33	13	1.41
MA52a	65.8	0.000	1.943	28	11	1.34
MA52b	56.9	0.000	1.996	24	10	1.52
MA78	45.6	0.000	1.843	33	13	1.41
MA78a	116.2	0.000	2.136	25	10	1.46
MA78b	124.95	0.000	2.192	22	9	1.63

Note: VaR value calculated at 5% significance level using data available from 242 weeks. Errors: losses worse than the VaR value. % Errors: ‘real’ significance level. Mean Error (%) shows the mean loss exceeding the VaR value.

Table 5
Backtesting process in the IVaR30

	Normality		Backtesting			
	Jarque-Bera	Probability	Mean VaR(%)	Errors	% Errors	Mean Error(%)
DJIASM	271.4	0.000				
MA10	85.85	0.000	1.208	67	25.5	2.23
MA25	62.09	0.000	2.221	38	14.5	2.03
MA30	49.04	0.000	2.365	33	12.6	2.05
MA52	124.2	0.000	2.825	22	8.3	2.78
MA60	75.17	0.000	2.944	23	8.7	2.39
MA70	123.32	0.000	3.062	20	7.6	2.80
MA78	117.74	0.000	3.141	19	7.2	2.81
MA85	130.40	0.000	3.205	22	8.3	2.41
MA100	150.14	0.000	3.329	24	9.1	2.45

Note: VaR value calculated at 5% significance level using data available from 262 weeks. Errors: losses worse than the VaR value. % Errors: 'real' significance level. Mean Error (%) shows the mean loss exceeding the VaR value.

Table 6
Backtesting process in the IVaRM

	Normality		Backtesting			
	Jarque-Bera	Probability	Mean VaR(%)	Errors	% Errors	Mean Error(%)
MERVAL	4.28	0.111				
MA10	664.02	0.000	3.08	61	23.4	1.98
MA10a	648.56	0.000	3.10	60	22.9	2.07
MA10b	652.00	0.000	3.12	60	22.9	2.03
MA25	534.82	0.000	4.47	38	14,5	2.02
MA25a	619.22	0.000	4.64	33	12.64	2.04
MA25b	623.62	0.000	4.69	31	11.87	2.02
MA52	292.01	0.000	4.57	39	14.94	1.92
MA52a	243.82	0.000	5.24	29	11.11	1.98
MA52b	264.28	0.000	5.48	26	9.96	1.95
MA78	276.73	0.000	4.38	42	16.09	1.85
MA78a	207.70	0.000	5.59	29	11.11	1.90
MA78b	177.41	0.000	6.15	26	9.96	1.99

Note: VaR value calculated at 5% significance level using data available from 261 weeks. Errors: losses worse than the VaR value. % Errors: 'real' significance level. Mean Error (%) shows the mean loss exceeding the VaR value.

Table 7
Model Selection in the Spanish Stock Market

	Backtesting			Mean VaR (%)	Sharpe's Ratio
	Errors	Mean Error (%)	ETL in 2000-2004		
MA10	75	1.95	146.25	0.654	11.71
MA10a	75	1.89	141.75	0.671	15.83
MA10b	75	1.89	141.75	0.665	15.69
MA25	43	1.35	58.05	1.446	4.18
MA25a	37	1.35	49.95	1.557	7.75
MA25b	36	1.37	49.32	1.582	10.70
MA52	33	1.41	46.53	1.746	6.52
MA52a	28	1.34	37.52	1.943	6.96
MA52b	24	1.52	36.48	1.996	11.56
MA78	33	1.41	46.53	1.843	8.03
MA78a	25	1.46	36.5	2.136	6.25
MA78b	22	1.63	35.86	2.192	10.54

Note: in Sharpe's ratio non-risk return has been considered equal to zero.

Table 8
Model Selection in the American Stock Market

	Backtesting			Mean VaR (%)	Sharpe's Ratio
	Errors	Mean Error(%)	ETL in 2000-2004		
MA10	67	2.23	149.42	1.208	-18.46
MA25	38	2.03	77.14	2.221	-16.58
MA30	33	2.05	67.80	2.365	-11.75
MA52	22	2.78	61.21	2.825	-10.39
MA60	23	2.39	55.03	2.944	-6.34
MA70	20	2.80	56.18	3.062	-8.80
MA78	19	2.81	53.56	3.141	-8.17
MA85	22	2.41	53.14	3.205	-7.94
MA100	24	2.45	59.02	3.329	-13.22

Note: in Sharpe's ratio non-risk return has been considered equal to zero.

Table 9
Model Selection in the Argentinian Stock Market

	Backtesting			Mean VaR (%)	Sharpe's Ratio
	Errors	Mean Error(%)	ETL in 2000-2004		
MA10	61	1.98	121.1	3.08	27.73
MA10a	60	2.07	124.2	3.10	26.31
MA10b	60	2.03	122.2	3.12	26.62
MA25	38	2.02	77.1	4.47	32.67
MA25a	33	2.04	67.4	4.64	28.08
MA25b	31	2.02	62.7	4.69	28.22
MA52	39	1.92	75.2	4.57	27.03
MA52a	29	1.98	57.7	5.24	29.69
MA52b	26	1.95	50.8	5.48	24.78
MA78	42	1.85	77.8	4.38	27.02
MA78a	29	1.90	55.2	5.59	27.67
MA78b	26	1.99	51.7	6.15	17.92

Note: in Sharpe's ratio non-risk return has been considered equal to zero.

Table 10

Approximations used for Covariance matrix estimation in the IVaR35

Approximation	Method	Length	Improvements Applied.
MA10	Moving Average	10 weeks	None
MA10a	Moving Average	10 weeks	Multiple-step Method
MA10b	Moving Average	10 weeks	Multiple-step Method and 0.01% Weighting
MA25	Moving Average	25 weeks	None
MA25a	Moving Average	25 weeks	Multiple-step Method
MA25b	Moving Average	25 weeks	Multiple-step Method and 0.01% Weighting
MA52	Moving Average	52 weeks	None
MA52a	Moving Average	52 weeks	Multiple-step Method
MA52b	Moving Average	52 weeks	Multiple-step Method and 0.01% Weighting
MA78	Moving Average	78 weeks	None
MA78a	Moving Average	78 weeks	Multiple-step Method
MA78b	Moving Average	78 weeks	Multiple-step Method and 0.01% Weighting

Table 11

Approximations used for Covariance matrix estimation in the IVaR30

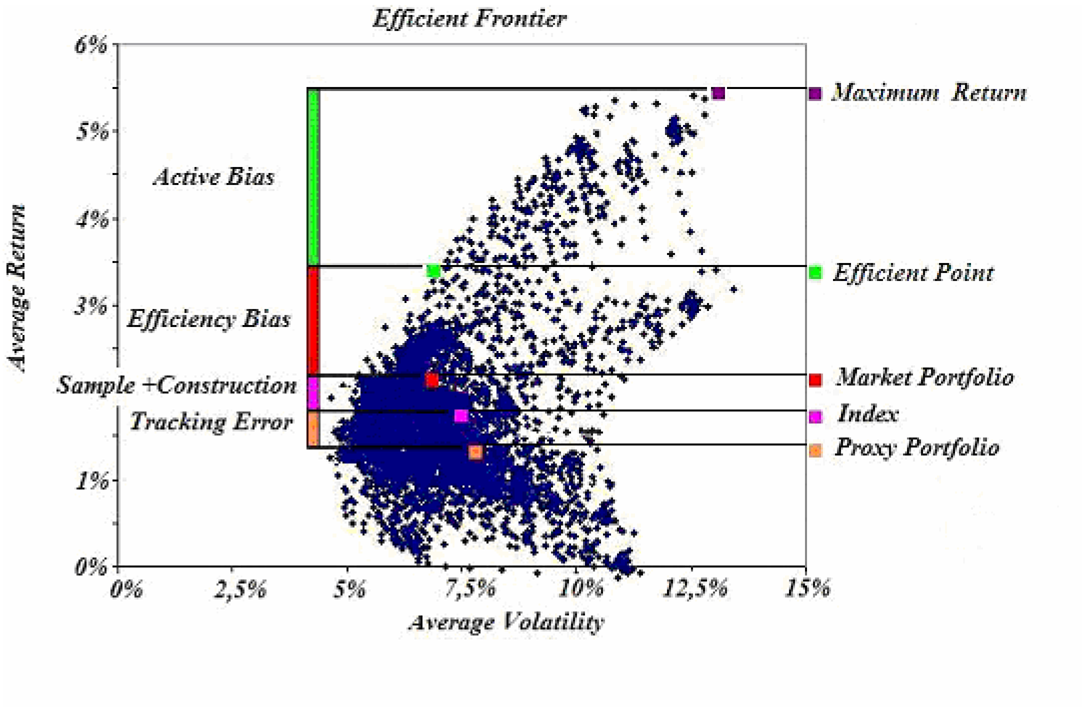
Approximation	Method	Length	Improvements Applied.
MA10	Moving Average	10 weeks	None
MA25	Moving Average	25 weeks	None
MA30	Moving Average	30 weeks	None
MA52	Moving Average	52 weeks	None
MA60	Moving Average	60 weeks	None
MA70	Moving Average	70 weeks	None
MA78	Moving Average	78 weeks	None
MA85	Moving Average	85 weeks	None
MA100	Moving Average	100 weeks	None

Table 12

Approximations used for Covariance matrix estimation in the IVaRM

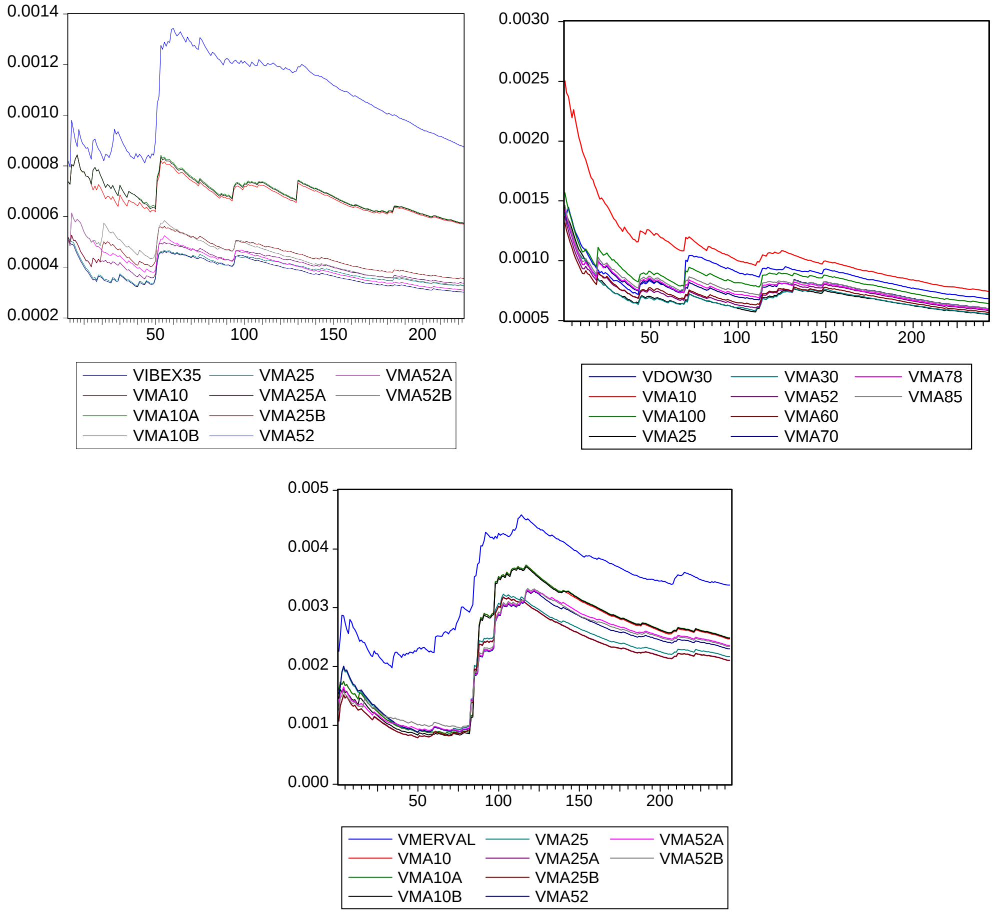
Approximation	Method	Length	Improvements Applied.
MA10	Moving Average	10 weeks	None
MA10a	Moving Average	10 weeks	Multiple-step Method
MA10b	Moving Average	10 weeks	Multiple-step Method and 0.01% Weighting
MA25	Moving Average	25 weeks	None
MA25a	Moving Average	25 weeks	Multiple-step Method
MA25b	Moving Average	25 weeks	Multiple-step Method and 0.01% Weighting
MA52	Moving Average	52 weeks	None
MA52a	Moving Average	52 weeks	Multiple-step Method
MA52b	Moving Average	52 weeks	Multiple-step Method and 0.01% Weighting
MA78	Moving Average	78 weeks	None
MA78a	Moving Average	78 weeks	Multiple-step Method
MA78b	Moving Average	78 weeks	Multiple-step Method and 0.01% Weighting

Figure 1
Market Index Biases



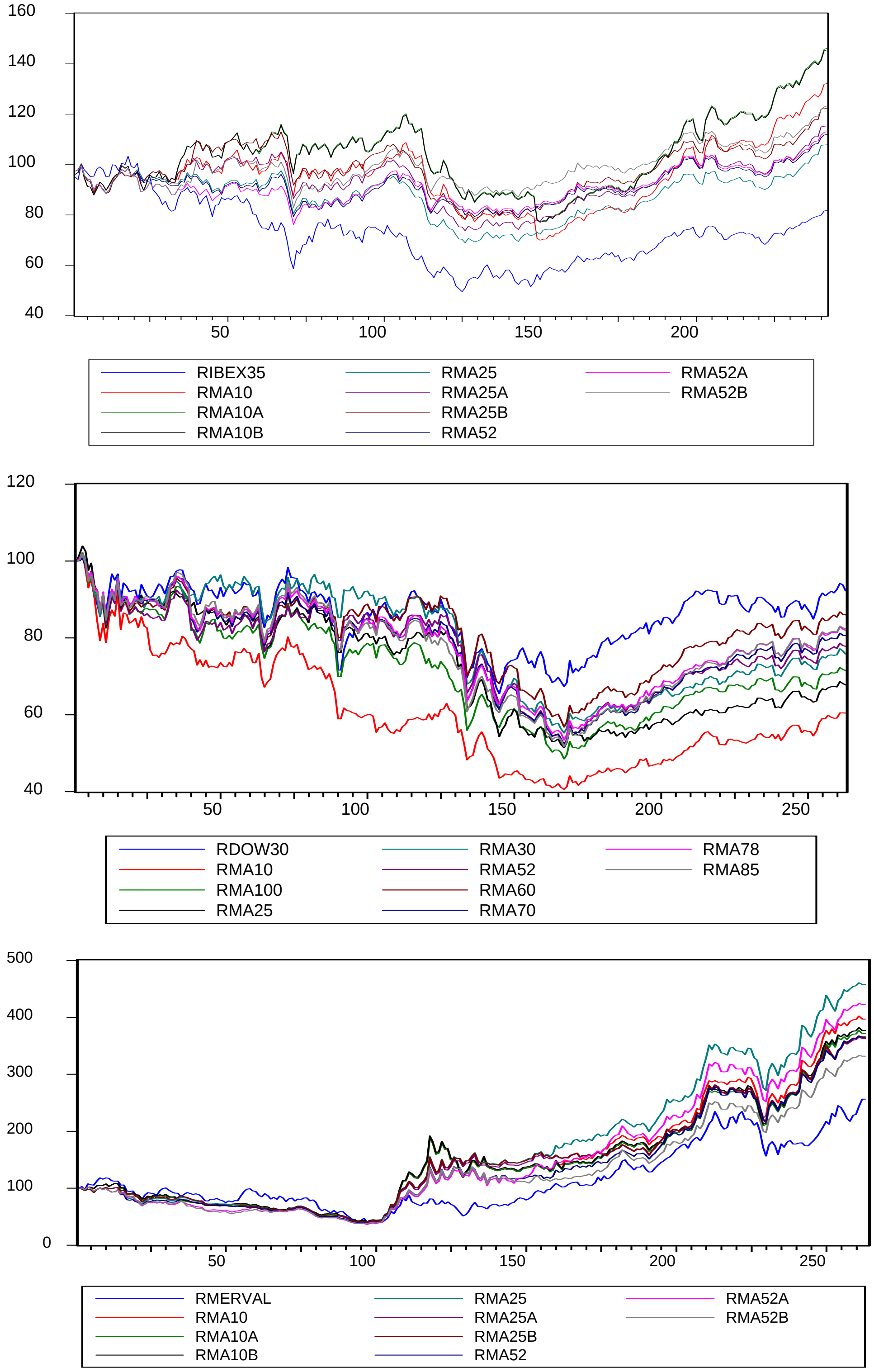
Note: Market Portfolio=Total Market; Proxy Portfolio= passive investor's portfolio used to proxy an Index.
Source: Authors' own

Figure 2
Market Indices' cumulative volatility (IVaR35, IVaR30, IVaRM)



Note: here, volatility is variance. In the first, second and third graphs, VIBEX35, VDOW30 and VMERVAL are the cumulative volatilities of the IBEX35®, the DowJones Industrial AverageSM and the MERVAL, respectively, and VMA are the cumulative volatilities of each moving average approximation used for each market.

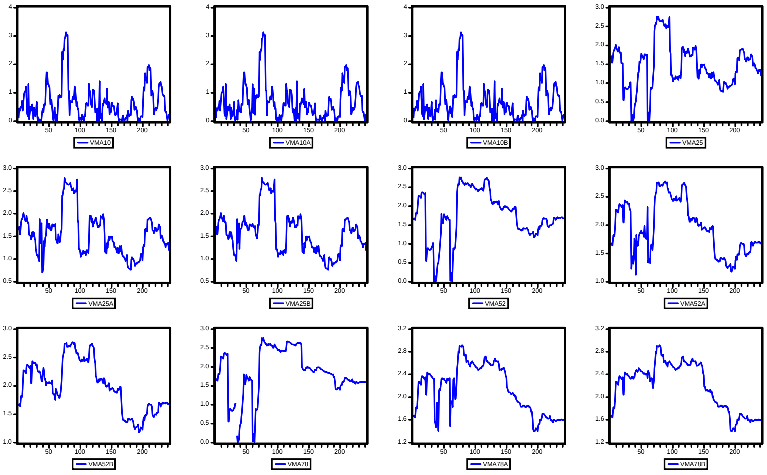
Figure 3
Evolution of the indices (IVaR35, IVaR30, IVaRM)



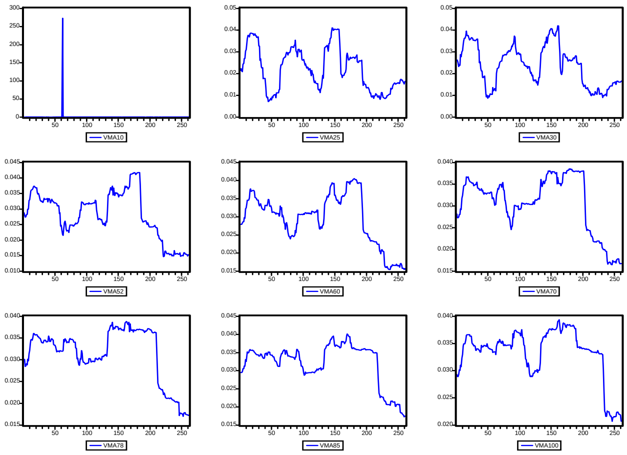
Note: 100 based. In the first, second and third graph RIBEX35, RDOW30 and RMERVAL are the evolution of IBEX35®, DJIASM and MERVAL, and RMA are the evolutions of each moving average approximation in each market.

Figure 4
VAR

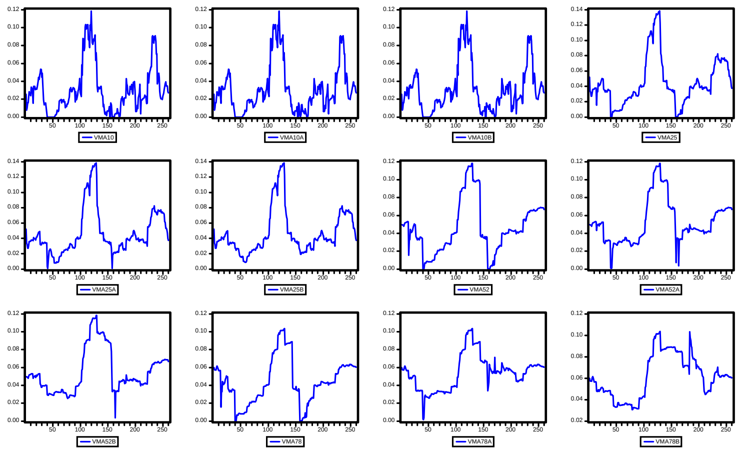
IVaR35



IVaR30



IvaRM



Note: The figures in IVaR35, IvaR30 and IvaRM, (left to right and top to bottom) are the VaR evolution for each approximation in each market.