System-Theoretic Foundations of the Theory of Economic Policy

REINHARD NECK
Department of Economics
Klagenfurt University
Universitaetsstrasse 65–67, A-9020 Klagenfurt
AUSTRIA
reinhard.neck@uni-klu.ac.at http://www.uni-klu.ac.at/vwl

Abstract: - In this paper, we formulate a dynamic theory of economic policy using some concepts and axioms of mathematical system theory. First, the notion of a dynamic economic system is characterized axiomatically. Then the basic problem of the theory of economic policy as introduced by Tinbergen and Theil is expressed as a control problem for a dynamic economic system. In this way, a more general framework for the theory of economic policy than available so far can be developed in terms of mathematical system theory. Finally, we discuss some extensions of the framework, in particular to economic policy problems with more than one decision-maker.

Key-Words: Dynamic system; system theory; control theory; economics; economic policy; modeling.

1 Introduction

Can the government and other political authorities exert influence on the performance of the economy of a country? If so, how should they affect the economy in order to achieve results which are desirable in some specific sense? What do government interventions into the economy look like in practice, and how could they be improved with respect to their effectiveness? These and similar questions are obviously of great importance to policy-makers and social scientists alike, and to economists in particular.

Economic theory has been stimulated by practical and policy issues since its beginnings as a scientific discipline in the 18th century. Problems of economic policy proper, however, only became subject of a theory of their own in the 20th century. Building on earlier contributions by Austrian, Italian, and British writers, among others, in the 1950s the Dutch economists, Tinbergen and Theil, created what is now known as the “theory of economic policy”. It aims at providing a general theoretical framework for the analysis of various kinds of economic policy problems, both from a macroeconomic and a microeconomic point of view. Although this theory has been extended in several ways, so far its connection with mathematical system theory has not yet been fully displayed.

It is the objective of the present paper to recast a dynamic formulation of the theory of economic policy in terms of the notions and axioms of mathematical system theory.

The plan of the paper is as follows: In Section 2, we give a summary of the Tinbergen-Theil theory of economic policy in non-technical terms for readers not familiar with it. Section 3 provides an axiomatic characterization of the notion of a class of dynamic economic systems relevant for the theory of economic policy while special types of dynamic economic systems are discussed in Section 4. In Section 5, the basic problem of the theory of economic policy is formulated as a control problem for a dynamic economic system. Finally, Section 6 discusses some economic policy problems not covered by the formulation of the present paper and gives suggestions on how to incorporate them into a system-theoretic framework as well.

2 The Theory of Economic Policy

The theory of economic policy, which was mainly developed by Tinbergen [18], [19] and Theil [17] and is therefore sometimes also called the “Tinbergen-Theil paradigm” (of the theory of quantitative economic policy), is teleological in the sense of attempting to answer questions relating to the “best” achievement of given goals by political decision-makers. It assumes as a starting point the existence of (at least) one central policy-maker with well-defined preferences and a well-defined set of policy instruments at his (her) disposal. This implies elements of a decision-theoretic scheme of economic policy planning; moreover, the approach is “taxonomic” [10] in the sense of presupposing an a priori partition of the variables involved into several classes.

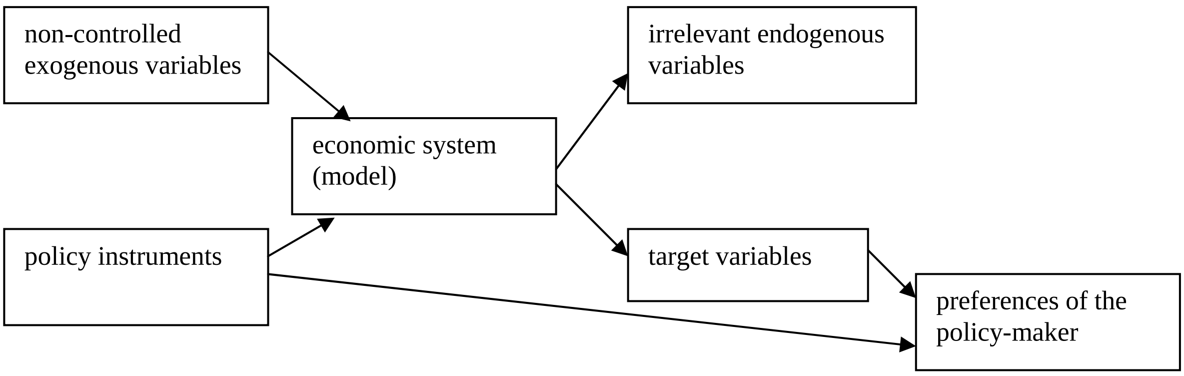
In particular, the following variables are distinguished:

(1) *Exogenous variables*, which are not explained by the model of the economic system under consideration.
   (a) *Policy instruments*: exogenous variables which are under the control of the policy-maker.
   (b) *Non-controlled exogenous variables* ("data"): exogenous variables which are not controllable by the policy-maker.
(2) *Endogenous variables*, which are explained by the model of the economic system.
   (a) *Target variables*: endogenous variables which are considered as goals (are evaluated) by the policy-maker.
   (b) "*Irrelevant*" *variables*: endogenous variables which are not evaluated by the policy-maker and express the side-effects of economic policies.

The above classification makes use of two other basic ingredients of the theory of economic policy which are assumed to be given within this theory: the model of the economic system and the preferences of the policy-maker. On the one hand, the *model of the economic system* describes the structure and the functioning of the economy under consideration. It may be a theoretical model or an empirical one, where the latter is usually obtained from econometric estimations; it may be a macroeconomic model for problems relating to stabilization policies, or a microeconomic model to deal with policy problems concerning allocation and distribution. Not only nationwide models may be considered, but also regional or international ones. In any case, the model transforms exogenous to endogenous variables in order to express the influence of the former on the latter.

On the other hand, the *preferences of the policy-makers* may be expressed either by an explicit objective function, which may be interpreted as an individual (for the policy-maker) or collective ("social") welfare, utility, or cost function, or by an incomplete scale, containing at least a most-preferred value for each target variable (and possibly for some policy instrument variables as well).

Following [8, p.21] and [9, p.58], the basic framework of the theory of economic policy can be displayed by the following scheme:



To be more specific, denote by U the set of possible policy instruments, with $\boldsymbol{u} \in U$, where $\boldsymbol{u}$ may (but need not) be an element of some vector space. The set U expresses institutional, political, physical, and other constraints on the values of the policy instrument variables. The number of elements contained in this set and the set's dimension are usually specific to the policy problem under consideration; this may not only be a question of factual evidence, but also one of inventing new instruments. Let Z be the set of non-controlled exogenous variables, with $\boldsymbol{z} \in Z$ being a deterministic or a random variable.

The economic system can be described by a model $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{u}, \boldsymbol{z})$, which may be a reduced-form econometric model, for instance, with $\boldsymbol{x} \in X$ denoting the state of the economy, i.e. everything which results from a combined "action" of the policy instruments and the other exogenous variables on the economic system. The set of endogenous variables X is constrained by U, Z, and the model (the function $\boldsymbol{f}$(..)). In particular, when $\boldsymbol{z}$ is a stochastic variable, then $\boldsymbol{x}$ is stochastic, too, and for each $\boldsymbol{u}$ there is a corresponding probability distribution over $\boldsymbol{x}$. In the latter case, which will be neglected in the following, policy-maker's preferences have to be defined over probability distributions. Target variables are separately denoted by $\boldsymbol{x}_1$, say.

The preferences of the policy-maker can be defined over $\boldsymbol{x}_1$ and $\boldsymbol{u}$. For instance, if we define $\boldsymbol{y} = (\boldsymbol{x}_1' \; \boldsymbol{u}_1')$ with $\boldsymbol{u}_1$ being those policy instruments which are directly evaluated by the policy-maker, we may have an ordering, i.e. $\boldsymbol{y}' \succ \boldsymbol{y}''$ (meaning that $\boldsymbol{y}'$ is strictly preferred to $\boldsymbol{y}''$), or $\boldsymbol{y}' \sim \boldsymbol{y}''$ (the policy-maker is indifferent between $\boldsymbol{y}'$ and $\boldsymbol{y}''$), or $\boldsymbol{y}' \prec \boldsymbol{y}''$ ($\Leftrightarrow \boldsymbol{y}'' \succ \boldsymbol{y}'$), for all $\boldsymbol{y}'$, $\boldsymbol{y}'' \in Y$, the set of all possible $\boldsymbol{y}$. Under some further assumptions this may be represented by an objective function, say $J(\boldsymbol{y})$, with $\boldsymbol{y}' \sim \boldsymbol{y}'' \Rightarrow J(\boldsymbol{y}') = J(\boldsymbol{y}'')$, and $\boldsymbol{y}' \succ \boldsymbol{y}'' \Rightarrow J(\boldsymbol{y}') > J(\boldsymbol{y}'')$. Conditions under which this holds are provided by the theory of decisions under certainty and under uncertainty.

If this objective function $J(\boldsymbol{y})$ is given, then the policy *problem* is one *of flexible objectives*: the policy-maker has to delimit the sets U and Y, to construct a model $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{u},\boldsymbol{z})$, to define $\boldsymbol{y} = (\boldsymbol{x}_1' \, \boldsymbol{u}_1')'$, to establish $J(\boldsymbol{y})$, to predict $\boldsymbol{z}$, and finally to find $\boldsymbol{u}^*$ such that

$$J(\boldsymbol{y}^*) = J\left[\left(\boldsymbol{x}_1{}^{*\prime}\, \boldsymbol{u}_1{}^{*\prime}\right)'\right] = \max_{\boldsymbol{u}\in U} J\left[\boldsymbol{y} : \boldsymbol{y} = \left(\boldsymbol{x}_1' \, \boldsymbol{u}_1'\right)',\right.$$
$$\left. \boldsymbol{x} = \left(\boldsymbol{x}_1' \, \boldsymbol{x}_2'\right)', \; \boldsymbol{u} = \left(\boldsymbol{u}_1' \, \boldsymbol{u}_2'\right)', \; \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{u},\boldsymbol{z}), \; \boldsymbol{z} \text{ given}\right], \quad (1)$$

with corresponding notation for the starred values for $\boldsymbol{x}$, $\boldsymbol{u}$, and $\boldsymbol{y}$. Otherwise, in a *problem of fixed objectives*, we have given a most-preferred element $\boldsymbol{y}^* \in Y$, and the problem consists in determining that value of $\boldsymbol{u}^*$ (if it exists) which achieves the attainment of $\boldsymbol{y}^*$ exactly, i.e. $\boldsymbol{y}^* = (\boldsymbol{x}_1{}^{*\prime}\, \boldsymbol{u}_1{}^{*\prime})'$, $\boldsymbol{x}^* = (\boldsymbol{x}_1{}^{*\prime}\, \boldsymbol{x}'_2)'$, $\boldsymbol{u}^* = (\boldsymbol{u}_1{}^{*\prime}\, \boldsymbol{u}_2{}^{*\prime})'$, $\boldsymbol{x}^* = \boldsymbol{f}(\boldsymbol{u}^*, \boldsymbol{z})$ for given $\boldsymbol{z}$ and some $\boldsymbol{x}_2$.

The above scheme may be extended in various ways to deal with additional elements of economic policy problems arising in practical applications. For instance, the dependence of the set Y on the particular element $\boldsymbol{z} \in Z$ may be considered explicitly; uncertainty may be introduced; and more than one decision-maker may be modeled explicitly. For many economic policy problems, in particular for those of stabilization policy, a dynamic generalization is required: the economic system evolves over time, and decisions have to be made not only once and for all, but repeatedly in time. Economic policy applications of optimal control theory incorporate these dynamic aspects.

Dynamic system theory in a more general sense has been used to provide mathematical formulations of problems in the theory of economic policy by Preston and Pagan [15], but they confine themselves to linear time-invariant economic systems. An axiomatic characterization of the theoretical framework of the theory of economic policy in a dynamic setting using system theory for more general problems is still missing. Indeed, the purpose of the present paper is to provide such a formulation. To do so, we first have to define the notion of a dynamic economic system and then the dynamic policy problem can be formulated as a general control problem.

3 Formal Definition of a Dynamic Economic System

The mathematical theory of dynamic systems (e.g. [20], [3], [11], [5], [6]) considers a system as a "machine" that transforms some "inputs" during an observation interval to "outputs" (measurements) over that time interval. Hence the system can be represented as a set of pairs of inputs and outputs on the observation interval; this "external" system description considers the system itself as a "black box". Alternatively, an "internal" system description introduces the notion of the "state" of the system, which is intermediate between the inputs and the outputs. Although the state may not be directly measurable, this concept is useful to introduce notions of causality and internal structure into the description of a dynamic system. The state can be interpreted as the amount of "information" necessary to determine (together with the input) the state of the system at the next moment of its evolution; in general, it need not necessarily be interpreted in substantial (i.e. economic) terms.

In the following, we first provide the internal description of a dynamic economic system which is appropriate for the theory of economic policy. We restrict our attention to deterministic systems (without random elements) and non-anticipatory ones, i.e. inputs, outputs, and states do not depend on future values. The economic character of the system under consideration is a matter of the interpretation of the variables involved. Our formulation differs from that of a general dynamic system in several respects, e.g. by the presence of non-controlled exogenous variables.

We assume the existence of a *time-set* $T \subseteq \mathbb{R}$, which is an ordered subset of the real numbers. For the internal description, we need the existence of a *state set* X, which has a metric (distance) d defined upon it, i.e.

$$d : X \times X \mapsto \mathbb{R} \quad (2)$$

with (1) $d(\boldsymbol{x}_1, \boldsymbol{x}_2) \geq 0 \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in X$ and $d(\boldsymbol{x}_1, \boldsymbol{x}_2) = 0 \Leftrightarrow \boldsymbol{x}_1 = \boldsymbol{x}_2$,
(2) $d(\boldsymbol{x}_1, \boldsymbol{x}_2) = d(\boldsymbol{x}_2, \boldsymbol{x}_1) \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in X$,
(3) $d(\boldsymbol{x}_1, \boldsymbol{x}_3) \leq d(\boldsymbol{x}_1, \boldsymbol{x}_2) + d(\boldsymbol{x}_2, \boldsymbol{x}_3) \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in X$.

Next, we have a *set of policy instrument values U* together with a metric $\hat{d}$, and a set of piecewise continuous functions with values in U, the *set of policy functions* $\Omega = \{\boldsymbol{u}(.) : T \mapsto U\}$. Similarly, we assume the existence of a *set of non-controlled exogenous variables Z* with metric $\hat{\hat{d}}$, and a set of piecewise continuous functions with values in Z, the *set of* *exogenous-variables functions* $\Theta = \{\boldsymbol{z}(.) : T \mapsto Z\}$. The *state variable* $\boldsymbol{x}(t)$ is defined on T and takes values in X, i.e.

$$\boldsymbol{x}(.) : T \mapsto X. \qquad (3)$$

For the set of policy functions, we assume $\Omega \neq \emptyset$ and the following property: let the segment of the policy instrument $\boldsymbol{u}(\tau)$, $t_1 < \tau \le t_2$, in Ω be restricted to $(t_1, t_2] \cap T$, where $t_1, t_2 \in T$. If $\boldsymbol{u}(.), \boldsymbol{u}'(.) \in \Omega$ and $t_1 < t_2 < t_3$, then $\boldsymbol{u}''(.) \in \Omega$ exists such that $\boldsymbol{u}''(\tau) = \boldsymbol{u}(\tau)$, $t_1 < \tau \le t_2$, and $\boldsymbol{u}''(\tau) = \boldsymbol{u}'(\tau)$, $t_2 < \tau \le t_3$. The same is assumed for the set of exogenous-variables functions Θ. Then we may denote by $\boldsymbol{u}(t_1, t_2]$ and by $\boldsymbol{z}(t_1, t_2]$ the segment of $\boldsymbol{u}(.)$ and $\boldsymbol{z}(.)$, respectively, on $(t_1, t_2]$, to be called policy and exogenous input to the system on the interval $(t_1, t_2]$, respectively.

Finally, we have a *set of output values Y* and a *set of output functions* $\Gamma = \{\boldsymbol{y}(.) : T \mapsto Y\}$. Then an *output mapping* is defined as

$$\boldsymbol{g} : X \times U \times Z \times T \mapsto Y \qquad (4)$$

with

$$\boldsymbol{y}(t) = \boldsymbol{g}[\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{z}(t), t] \qquad (5)$$

for $t \in T$, $\boldsymbol{u}(.) \in \Omega$, $\boldsymbol{z}(.) \in \Theta$, $\boldsymbol{u}(t) \in U$, $\boldsymbol{z}(t) \in Z$, $\boldsymbol{x}(t) \in X$, $\boldsymbol{y}(.) \in \Gamma$, $\boldsymbol{y}(t) \in Y$. $\boldsymbol{y}(t)$ is called the *output* of the system, and its segment on $(t_1, t_2]$ is denoted by

$$\boldsymbol{y}(t_1, t_2] = \hat{\boldsymbol{g}}[\boldsymbol{x}(t_1), \boldsymbol{u}(t_1, t_2], \boldsymbol{z}(t_1, t_2]], \qquad (6)$$

where (6) is called the output equation of the system.

This formulation assumes the following *axiom* to hold: $\forall \boldsymbol{x}(t_0) \in X$, $\forall t_0 \in T$, $\forall t \ge t_0$ with $t \in T$, $\forall \boldsymbol{u}(t_0, t]$ with $\boldsymbol{u}(.) \in \Omega$, $\forall \boldsymbol{z}(t_0, t]$ with $\boldsymbol{z}(.) \in \Theta$, $\boldsymbol{y}(t_0, t]$ is uniquely determined by $\boldsymbol{x}(t_0)$, $\boldsymbol{u}(t_0, t]$, and $\boldsymbol{z}(t_0, t]$. In particular, $\boldsymbol{u_1}(t_0, t] = \boldsymbol{u_2}(t_0, t]$ and $\boldsymbol{z_1}(t_0, t] = \boldsymbol{z_2}(t_0, t]$ imply that $\hat{\boldsymbol{g}}[\boldsymbol{x}(t_0), \boldsymbol{u_1}(t_0, t], \boldsymbol{z_1}(t_0, t]] = \hat{\boldsymbol{g}}[\boldsymbol{x}(t_0), \boldsymbol{u_1}(t_0, t], \boldsymbol{z_2}(t_0, t]] = \hat{\boldsymbol{g}}[\boldsymbol{x}(t_0), \boldsymbol{u_2}(t_0, t], \boldsymbol{z_1}(t_0, t]] = \hat{\boldsymbol{g}}[\boldsymbol{x}(t_0), \boldsymbol{u_2}(t_0, t], \boldsymbol{z_2}(t_0, t]]$. This means that from the knowledge of the initial state at t_0 and the inputs in $(t_0, t]$, we can uniquely determine the output in $(t_0, t]$. If the state $\boldsymbol{x}(t_0)$ is known, no knowledge of inputs prior to t_0 is required. This assumption implies the non-anticipatory character of the system: no future inputs (neither policy nor exogenous) have an influence upon $\boldsymbol{y}(t_0, t]$.

Now consider $t_0 < \hat{t} < t \in T, \boldsymbol{x}(t_0) \in X$, and denote $X[\boldsymbol{x}(t_0), \boldsymbol{u}(.), \boldsymbol{z}(.), \hat{t}] = \{\boldsymbol{x}(\hat{t}) \in X$ such that $\boldsymbol{y}(\hat{t}, t] = \hat{\boldsymbol{g}}[\boldsymbol{x}(t_0), \boldsymbol{u}(t_0, t], \boldsymbol{z}(t_0, t]]$ in $(\hat{t}, t]$ be equal to $\hat{\boldsymbol{g}}[\boldsymbol{x}(\hat{t}), \boldsymbol{u}(\hat{t}, t], \boldsymbol{z}(\hat{t}, t]]\}$.

Let $\boldsymbol{u^*}(.) \in \Omega$, $\boldsymbol{z^*}(.) \in \Theta$. Then another system-theoretic *axiom* requires

$$\bigcap_{\substack{\boldsymbol{u}(.) \in \Omega \text{ with } \boldsymbol{u}(t_0, \hat{t}] = \boldsymbol{u^*}(t_0, \hat{t}] \\ \boldsymbol{z}(.) \in \Theta \text{ with } \boldsymbol{z}(t_0, \hat{t}] = \boldsymbol{z^*}(t_0, \hat{t}]}} X\left[\boldsymbol{x}(t_0), \boldsymbol{u}(.), \boldsymbol{z}(.), \hat{t}\right] \neq \emptyset. \qquad (7)$$

This means that at least one $\boldsymbol{x}(.) \in X$ exists which generates each combination of inputs $\boldsymbol{u}(\hat{t}, t]$, $\boldsymbol{z}(\hat{t}, t]$ and outputs $\boldsymbol{y}(\hat{t}, t]$, or there are enough states of the system to "explain" every such combination. For the purposes of the theory of economic policy, the assumption that the intersection over all $\boldsymbol{u}(.) \in \Omega$ with $\boldsymbol{u}(t_0, \hat{t}] = \boldsymbol{u^*}(t_0, \hat{t}]$ in (7) be non-empty for any given $\boldsymbol{z}(.) \in \Theta$ can be used as an axiom instead of (7).

From the mathematical theory of dynamic systems it is well known that the two above axioms imply that a function exists

$$\boldsymbol{\varphi} : T \times T \times X \times \Omega \times \Theta \mapsto X \qquad (8)$$

such that

$$\boldsymbol{x}(t) = \boldsymbol{\varphi}[t, t_0, \boldsymbol{x}(t_0), \boldsymbol{u}(t_0, t], \boldsymbol{z}(t_0, t]], \qquad (9)$$

i.e. its value is $\boldsymbol{x}(t) \in X$ at $t \in T$, resulting from the initial state $\boldsymbol{x}(t_0) \in X$ at initial time $t_0 \in T$ under the effects of policy inputs $\boldsymbol{u}(.) \in \Omega$ and exogenous inputs $\boldsymbol{z}(.) \in \Theta$ in $(t_0, t]$. This means that knowledge of the initial state at t_0 and inputs applied over $(t_0, t]$ determines not only the output $\boldsymbol{y}(\tau), t_0 < \tau \le t$, but also the state $\boldsymbol{x}(\tau), t_0 < \tau \le t$. Thus the state contains all information about the past which is required to "predict" future outputs and states for given policy and exogenous inputs.

The function $\boldsymbol{\varphi}$ is called the *state transition function* of the system and determines its trajectory (motion), namely $\{\boldsymbol{x}(\tau) : \boldsymbol{x}(\tau) = \boldsymbol{\varphi}[\tau, t_0, \boldsymbol{x}(t_0), \boldsymbol{u}(t_0, \tau], \boldsymbol{z}(t_0, \tau]]$, $\tau \in [t_0, t] \cap T\} \subseteq X$, generated by inputs $\boldsymbol{u}(t_0, t]$ and $\boldsymbol{z}(t_0, t]$. Equation (9) is called the state equation of the dynamic economic system, a pair $(t, \boldsymbol{x})$ with $t \in T$, $\boldsymbol{x} \in X$ is called an event of that system, and $T \times X$ is the phase space of the system. It is assumed that $\boldsymbol{\varphi}$ is well-defined for all $t \ge t_0$, though not necessarily for all $t < t_0$.

Another *axiom* requires the functions $\boldsymbol{g}$, $\hat{\boldsymbol{g}}$, and $\boldsymbol{\varphi}$ to be continuous with respect to all of their arguments. In particular, we demand for all $\varepsilon > 0$

$$\begin{aligned} &\sup_{\tau \in [t_0, t] \cap T} \left\{\hat{d}\left(\boldsymbol{u_1}(\tau), \boldsymbol{u_2}(\tau)\right)\right\} < \varepsilon \Rightarrow \\ &\Rightarrow \begin{matrix} d\left(\boldsymbol{\varphi}\left[t, t_0, \boldsymbol{x}(t_0), \boldsymbol{u_1}(t_0, t], \boldsymbol{z}(t_0, t]\right],\right. \\ \left.\boldsymbol{\varphi}\left[t, t_0, \boldsymbol{x}(t_0), \boldsymbol{u_2}(t_0, t], \boldsymbol{z}(t_0, t]\right]\right) < \delta_1(\varepsilon) \end{matrix} \end{aligned} \qquad (10)$$

and

$$\sup_{\tau\in[t_0,t]\cap T} \{\| \hat{\boldsymbol{g}}[\boldsymbol{x}(t_0),\boldsymbol{u_1}(t_0,\tau],\boldsymbol{z}(t_0,\tau]] - \hat{\boldsymbol{g}}[\boldsymbol{x}(t_0),\boldsymbol{u_2}(t_0,\tau],\boldsymbol{z}(t_0,\tau]]\|\} < \delta_2(\varepsilon) \quad (11)$$

for all $\boldsymbol{z}(t_0,t]$ with $\boldsymbol{z}(.)\in\Theta$; here we define

$$\hat{\boldsymbol{g}}[\boldsymbol{x}(t_0),\boldsymbol{u}(t_0,t_0],\boldsymbol{z}(t_0,t_0]] = \boldsymbol{g}[\boldsymbol{x}(t_0),\boldsymbol{u}(t_{0+}),\boldsymbol{z}(t_{0+}),t_0].$$

The same is required for the exogenous-variables functions $\boldsymbol{z_1}(\tau)$, $\boldsymbol{z_2}(\tau)$ with given policy input $\boldsymbol{u}(t_0,t]$. This means that "small" changes in the policy input, the exogenous input, or the initial state lead to "small" changes in the state and the output of the system.

Some further assumptions have to be imposed on the state transition function in order to define a dynamic economic system. In particular, for all $t,t_0\in T$, $\boldsymbol{x}(t_0)\in X$, $\boldsymbol{u}(.)\in\Omega$, $\boldsymbol{z}(.)\in\Theta$, we must have

$$\boldsymbol{\varphi}[t_0,t_0,\boldsymbol{x}(t_0),\boldsymbol{u}(t_0,t_0],\boldsymbol{z}(t_0,t_0]] = \boldsymbol{x}(t_0), \quad (12)$$

where we define $\boldsymbol{\varphi}[t_0,t_0,\boldsymbol{x}(t_0),\boldsymbol{u}(t_0,t_0],\boldsymbol{z}(t_0,t_0]] = \lim_{t\to t_0,\, t>t_0} \boldsymbol{\varphi}[t,t_0,\boldsymbol{x}(t_0),\boldsymbol{u}(t_0,t],\boldsymbol{z}(t_0,t]]$. This means that a unique trajectory starts from every initial state, and the initial condition $\boldsymbol{x}(t_0)$ is the starting-point of the trajectory. Next, we have the *semigroup property* of the function $\boldsymbol{\varphi}$: for all $t_0<\hat{t}\le t\in T$, $\boldsymbol{x}(t_0)\in X$, $\boldsymbol{u}(.)\in\Omega$, $\boldsymbol{z}(.)\in\Theta$, we have

$$\boldsymbol{\varphi}[t,t_0,\boldsymbol{x}(t_0),\boldsymbol{u}(t_0,t],\boldsymbol{z}(t_0,t]] = \boldsymbol{\varphi}[t,\hat{t},\boldsymbol{\varphi}[\hat{t},t_0,\boldsymbol{x}(t_0),\boldsymbol{u}(t_0,\hat{t}],\boldsymbol{z}(t_0,\hat{t}]],\boldsymbol{u}(\hat{t},t],\boldsymbol{z}(\hat{t},t]]. \quad (13)$$

This generalizes the uniqueness assumption for the solution of the state equation of the system and means the following: if some policy and exogenous inputs transfer the system from an initial state $\boldsymbol{x}(t_0)$ to some state $\boldsymbol{x}(t)$, and if some state, say $\hat{\boldsymbol{x}}\in X$, occurs along that trajectory, then these inputs transfer the system from $\hat{\boldsymbol{x}}$ to $\boldsymbol{x}(t)$. Finally, for all τ,t_0,t with $\tau\in[t_0,t]\cap T$ and for all $\boldsymbol{x}(t_0)\in X$ we must have

$$\boldsymbol{u}_1(.),\boldsymbol{u}_2(.)\in\Omega \text{ with } \boldsymbol{u_1}(t_0,t]=\boldsymbol{u_2}(t_0,t] \Rightarrow \boldsymbol{\varphi}[\tau,t_0,\boldsymbol{u_1}(t_0,t],\boldsymbol{z}(t_0,t]] = \boldsymbol{\varphi}[\tau,t_0,\boldsymbol{u_2}(t_0,t],\boldsymbol{z}(t_0,t]] \quad (14)$$

$\forall\tau\in[t_0,t]\cap T$ for any given $\boldsymbol{z}(t_0,t]$ with $\boldsymbol{z}(.)\in\Theta$, and the analogous condition for exogenous-variables functions. That is, applying identical inputs generates identical state trajectories, which again implies a non-anticipatory system.

A *dynamic economic system* $\mathfrak{S}$ is now given as

$$\mathfrak{S} = (T,X,\Omega,U,\Theta,Z,\boldsymbol{x}(t),\Gamma,Y,\boldsymbol{g}) \quad (15)$$

such that the above axioms are fulfilled. *T* is also called the domain of the system, *X* is the state space, $U\times Z$ is the input space, and *Y* is the output space. The system is described by the state equation (9) and the output equation

$$\boldsymbol{y}(t) = \hat{\boldsymbol{g}}[\boldsymbol{x}(t_0),\boldsymbol{u}(t_0,t],\boldsymbol{z}(t_0,t]]. \quad (16)$$

The functions $\boldsymbol{\varphi}$ and $\hat{\boldsymbol{g}}$ are intended to describe the economy under consideration, i.e. they have to fit the data of the real economy to be influenced by the policy-maker. Alternatively, the system may be given in external (input-output) description, if sets $T,U,\Omega,\Theta,Z,Y,\Gamma$ exist with all the properties of the internally defined system and a family of input-output functions relating *Y* to Ω and Θ for given parameters from an index set. This allows for omission of the state; however, the internal description results in a more informative characterization of the dynamic behavior of $\mathfrak{S}$.

4 Some Special Dynamic Economic Systems

The definition of a dynamic economic system given in the previous section is rather general. In order to obtain mathematically meaningful results in the theory of economic policy, and also in view of the requirements of economic models usually considered in that theory, it is often necessary to restrict attention to more special classes of dynamic economic systems. In particular, *finite-dimensional systems* are generally assumed, where the state space, the input space and the output space are finite-dimensional vector spaces. If these spaces are Euclidean spaces, the metric is defined as a Euclidean norm, i.e. for example, $d(\boldsymbol{x_1},\boldsymbol{x_2}) = \|\boldsymbol{x_1}-\boldsymbol{x_2}\| = \sqrt{\sum_{i=1}^{n}(x_{1i}-x_{2i})}$ for $\boldsymbol{x_1},\boldsymbol{x_2}\in\mathbb{R}^n$, $\boldsymbol{x}_j=(x_{j1},x_{j2}...x_{jn})'$, $j=1,2$. For $X=\mathbb{R}^n$, *n* is called the dimension of the dynamic system. Also we have $U=\mathbb{R}^m$, with $m\le n$, $Z=\mathbb{R}^s$, $Y=\mathbb{R}^k$, in this case.

Another special class of dynamic economic systems is that of *continuous systems* (systems in continuous time), where $T=(t_1,t_2)\subseteq\mathbb{R}$, $-\infty<t_1<t_2<\infty$, i.e. *T* is an open interval in $\mathbb{R}$. Many economic systems are formulated in discrete instead of continuous time, because economic data are usually available only for discrete time points or time intervals; however, for a theoretical formulation in general a continuous-time system is to be preferred.

A continuous system is called *smooth* if $T=\mathbb{R}$; *X*, Ω and Θ are topological vector spaces (in particular, open sets can be defined in these spaces, e.g. by Euclidean metrics), and the transition map $\boldsymbol{\varphi}$ is such that

$$(t_0,\boldsymbol{x}(t_0),\boldsymbol{u}(.),\boldsymbol{z}(.)) \mapsto \boldsymbol{\varphi}[.,t_0,\boldsymbol{x}(t_0),\boldsymbol{u}(t_0,.],\boldsymbol{z}(t_0,.]]$$

defines a continuously differentiable map

$\boldsymbol{f}: T \times X \times \Omega \times \Theta \mapsto$ {continuously differentiable functions from $T \mapsto X$ }. From the latter property it follows that the state transition function of a smooth dynamic system can be represented by the solution of the differential equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}[t, \boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{z}(t)]. \quad (17)$$

An important subset of this class of systems is composed of *differential systems*. More specifically, for a differential system the state equation (9) is the solution of the system of differential equations (17) with initial condition $\boldsymbol{x}(t_0)$, where $\boldsymbol{f}$ fulfils the following conditions:

(1) $\boldsymbol{f}: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^s \mapsto \mathbb{R}^n$ is a continuous function;
(2) $\partial \boldsymbol{f} / \partial \boldsymbol{x}(.)$ is a continuous function;
(3) $\boldsymbol{x}(t_0) \in \mathbb{R}^n, t_0 \in \mathbb{R}$;
(4) $\boldsymbol{u}(.)$ is a piecewise continuous function: $\mathbb{R} \mapsto \mathbb{R}^m$.

These conditions imply local (around t_0) existence and uniqueness of the solution of (17). Actually, condition (2) above may be substituted by the Lipschitz condition: $\exists K > 0$ such that $\| \boldsymbol{f}[t, \boldsymbol{x_1}, \boldsymbol{u}(t), \boldsymbol{z}(t)] - \boldsymbol{f}[t, \boldsymbol{x_2}, \boldsymbol{u}(t), \boldsymbol{z}(t)] \| < K \| \boldsymbol{x_1} - \boldsymbol{x_2} \|$ $\forall \boldsymbol{x_1}, \boldsymbol{x_2} \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^m$, $\boldsymbol{z}(t) \in \mathbb{R}^s$, $t \in \mathbb{R}$. Moreover, for a differential system the output equation (5) is such that $\boldsymbol{g}: \boldsymbol{R}^n \text{ x } \mathbb{R}^m \text{ x } \mathbb{R}^s \text{ x } \mathbb{R} \mapsto \mathbb{R}^k$ is continuous with respect to all of its arguments.

Another important class of dynamic economic systems is that of *linear systems*. A system $\mathcal{S}$ is linear if $X, U, \Omega, Z, \Theta, Y$, and Γ are vector spaces over $\mathbb{R}$, the mapping $\boldsymbol{\varphi}[t, t_0, ...]: X \times \Omega \times \Theta \mapsto X$ is $\mathbb{R}$-linear for all $t, t_0 \in T$, and the mapping $\boldsymbol{g}[..., t]: X \times \Omega \times \Theta \mapsto X$ is $\mathbb{R}$-linear for all $t \in T$.

In this case, $\boldsymbol{\varphi}$ and $\hat{\boldsymbol{g}}$ are linear operators on $\{\boldsymbol{x}(t_0)\} \times \{\boldsymbol{u}(t_0, t]\} \times \{\boldsymbol{z}(t_0, t]\}$, or, equivalently: for $\alpha, \beta \in \mathbb{R}$, $\boldsymbol{x_1}(t_0), \boldsymbol{x_2}(t_0) \in X$, $\boldsymbol{u_1}(t_0, t], \boldsymbol{u_2}(t_0, t] \in \Omega$, $\boldsymbol{z_1}(t_0, t], \boldsymbol{z_2}(t_0, t] \in \Theta$ with corresponding outputs $\boldsymbol{y_1}(t_0, t], \boldsymbol{y_2}(t_0, t]$, $t \geq t_0$, we have

(1) $\boldsymbol{x_3}(t_0) = \alpha \boldsymbol{x_1}(t_0) + \beta \boldsymbol{x_2}(t_0)$,
$\boldsymbol{y_3}(t_0, t] = \alpha \boldsymbol{y_1}(t_0, t] + \beta \boldsymbol{y_2}(t_0, t]$,
$\boldsymbol{u_3}(t_0, t] = \alpha \boldsymbol{u_1}(t_0, t] + \beta \boldsymbol{u_2}(t_0, t]$,
$\boldsymbol{z_3}(t_0, t] = \alpha \boldsymbol{z_1}(t_0, t] + \beta \boldsymbol{z_2}(t_0, t]$;
(2) $\boldsymbol{x_3}(t_0), \boldsymbol{u_3}(t_0, t], \boldsymbol{z_3}(t_0, t]$ and $\boldsymbol{y_3}(t_0, t]$ are possible elements of the system;
(3) $\boldsymbol{y_3}(t_0, t]$ and $\boldsymbol{x_3}(t_0, t]$ correspond to $\boldsymbol{x_3}(t_0)$, $\boldsymbol{u_3}(t_0, t]$, and $\boldsymbol{z_3}(t_0, t]$.

Basically, for a linear system in the sets of admissible policy and exogenous inputs and of outputs, the operations of addition and scalar multiplication are defined, and the input-output mapping is a linear transformation. For a linear differential system, both the vector differential equation for $\boldsymbol{x}(t)$ and the output equation for $\boldsymbol{y}(t)$ are linear in $\boldsymbol{x}(t)$, $\boldsymbol{u}(t)$ and $\boldsymbol{z}(t)$:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) + \boldsymbol{C}(t)\boldsymbol{z}(t), \quad (18)$$

$$\boldsymbol{y}(t) = \boldsymbol{D}(t)\boldsymbol{x}(t) + \boldsymbol{E}(t)\boldsymbol{u}(t) + \boldsymbol{F}(t)\boldsymbol{z}(t), \quad (19)$$

where $\boldsymbol{A}(t)$, $\boldsymbol{B}(t)$, $\boldsymbol{C}(t)$, $\boldsymbol{E}(t)$, and $\boldsymbol{F}(t)$ are matrix-valued functions of dimension $(n \times n)$, $(n \times m)$, $(n \times s)$, $(k \times n)$, $(k \times m)$, and $(k \times s)$, respectively. In this case, the properties of the state transition function required for general dynamic systems in the axioms of Section 3 correspond to those of the fundamental matrix of the linear dynamic system. The notion of equivalence of systems can then be introduced by defining a similarity transformation for the linear system.

Finally, we may consider *constant (time-invariant) systems*. A system is constant if all functions involved are constant, i.e. a translation of the time axis results in an equivalent system; otherwise, the system is called time-dependent. More precisely, a system $\mathcal{S}$ is constant if

(1) $T \subseteq \mathbb{R}$ is an additive group, i.e. for $t_1, t_2, t_3 \in T$ we have
$t_1 + t_2 \in T$; $(t_1 + t_2) + t_3 = t_1 + (t_2 + t_3)$; $0 \in T$;
$0 + t = t \quad \forall t \in T$; and
$\forall t \in T \ \exists (-t) \in T : t + (-t) = 0$;
(2) Ω and Θ are closed under the shift operator L^τ:
$L^\tau: \ \boldsymbol{u} \mapsto \boldsymbol{u}'$ with $\boldsymbol{u}'(t) = \boldsymbol{u}(t + \tau)$
$\boldsymbol{z} \mapsto \boldsymbol{z}'$ with $\boldsymbol{z}'(t) = \boldsymbol{z}(t + \tau) \quad \forall t, \tau \in T$;
(3) $\boldsymbol{\varphi}[t, t_0, \boldsymbol{x}(t_0), \boldsymbol{u}(t), \boldsymbol{z}(t)] = \boldsymbol{\varphi}[t + \tau, t_0 + \tau,$ $\boldsymbol{x}(t_0 + \tau), L^\tau \boldsymbol{u}(t), L^\tau \boldsymbol{z}(t)] \quad \forall \tau \in T$;
(4) $\boldsymbol{g}[..., t]: X \times \Omega \times \Theta \mapsto Y$ is independent of t.

Hence, for constant differential systems, we have autonomous state and output equations, i.e.

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{z}(t)], \quad (20)$$

$$\boldsymbol{y}(t) = \boldsymbol{g}[\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{z}(t)]. \quad (21)$$

In practical applications of the theory of economic policy, linear constant dynamic economic systems are the most common ones (e.g. [15]); but the preceding discussion should have made it clear that these constitute only a special class, and the theory of economic policy is applicable to much more general economic systems.

5 The Basic Problem of the Theory of Economic Policy as a Control Problem

So far, we have defined the notion of a dynamic economic system in a manner following classical dynamic system theory, which analyses its object of scientific inquiry in a "passive" way, so to say. This corresponds to the point of view of economic theory, which is interested in uncovering the internal mechanism and functioning of the economic system. The theory of economic policy, on the other hand, is guided by the idea of a decision-maker who changes the behavior of the economic system in an "active" way. This is similar to the more modern approach of system theory, particularly control theory, where influencing the system under consideration by controller's inputs is of central concern. Although not all theorists of economic policy agree with this attitude, the activist approach can be also justified by its greater generality: "doing nothing", i.e. exerting no influence on the economic system (zero policy input), is obviously a special case of an economic policy in the activist sense.

Apart from the dynamic economic system to be influenced, the basic problem of the theory of economic policy in a dynamic setting requires a specification of the desired output, i.e. the target variables, of the set of admissible policy inputs, and of some measure of the effectiveness of a given policy action. The dynamic economic system is considered to be given by the state transition function (9) (or, for a differential system, the n-th order system equation (17)) and the output equation (5).

For the formulation of the basic problem of the theory of economic policy as a control problem for a dynamic system, we next have to define the admissible policy inputs. We assume that the policy-maker (e.g. the government) is able to determine each element of the vector $\boldsymbol{u}(t)$ at each point of time t according to his (her) discretion within a given set of possible values, i.e. the policy instrument variables have to fulfill given restrictions or constraints.

The *set of admissible policy instruments* may be defined for the dynamic problem of the theory of economic policy as follows: let $U_t \subseteq U$ be a closed, bounded and convex subset of U or all of U (in particular, for $U = \mathbb{R}^m$) for each given $t \in T$. Let $\Phi = \{U_t : t \in T\}$. U_t is called the policy constraint set at time t, and Φ is called the policy constraint. Let Ω' be the set of all bounded piecewise continuous functions $\boldsymbol{u}(.)$ defined on T such that $\boldsymbol{u}(t) \in U_t$ for all $t \in T$. Then Ω' may be called the set of admissible policy functions fulfilling the policy constraint Φ, $\boldsymbol{u}(.) \in \Omega'$ is called an admissible policy function, taking values in Φ, which may be called admissible policy instrument values.

The policy-maker wants to influence the economic system in such a way as to obtain some desired response or dynamic behavior of the economy. This may include a given state or a given set of states or a given output or set of output values; these may or may not depend on time. Since the output in our formulation of the dynamic economic system is the actual response of that system, it seems natural to define desired objectives of the policy-maker in terms of the output rather than the state variables; in addition, output variables are observed by the economic policy-maker (and the public) and usually have a definite economic meaning.

The output vector $\boldsymbol{y}(t)$ may be decomposed into target and irrelevant variables in a straightforward way: let $\boldsymbol{y}(t) = (\boldsymbol{y_1}(t)' \; \boldsymbol{y_2}(t)')'$ be the output, with $\boldsymbol{y_1}(t)$ denoting the k_1-vector of target (evaluated) variables, and $\boldsymbol{y_2}(t)$ the k_2-vector of irrelevant variables; $k_1 + k_2 = k$, for $Y = \mathbb{R}^k$. Then we have

$$\boldsymbol{y_1}(t) = [\boldsymbol{I} \quad \boldsymbol{0}]\boldsymbol{y}(t), \tag{22}$$

with $\boldsymbol{I}$ being the k_1-dimensionsal identity matrix, and $\boldsymbol{0}$ a ($k_1 \times k_2$)-matrix of zeroes. (22) selects the target variables among the output variables. One possible policy objective is to bring $\boldsymbol{y_1}(t)$ to the zero vector for some t or for some (finite or infinite) time interval, if $\boldsymbol{y_1}(t)$ is measured in terms of deviations from desired target values. This corresponds to the various dynamic versions of the problem of fixed objectives.

Since the problem of fixed objectives presumes the specification of some desired (i.e. "optimal") values of the target variables, it may be considered as logically subordinate to a problem of flexible objectives, i.e. an optimization problem. To formulate such a problem within the system-theoretic framework of this paper, it is necessary to introduce a performance functional (an objective function) measuring the degree of goal attainment achieved by any admissible policy action.

For that purpose, let T be an open interval in $\mathbb{R}$ and $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Z = \mathbb{R}^s$, and $Y = \mathbb{R}^k$, and define continuous functions

$$L : \mathbb{R}^n \times \mathbb{R}^m \times T \mapsto \mathbb{R}, \tag{23}$$

$$K : \mathbb{R}^n \times T \mapsto \mathbb{R}, \tag{24}$$

and a given target set $S \subseteq \mathrm{R}^n \times T$. Let $t_0 \in T$ and $\boldsymbol{x_0} \in \mathbb{R}^n$ be given. Let

$$\boldsymbol{x}(\tau) = \boldsymbol{\varphi}[\tau, t_0, \boldsymbol{x_0}, \boldsymbol{u}(t_0, \tau], \boldsymbol{z}(t_0, \tau]] \tag{25}$$

be the unique solution of the system equation for initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x_0}$, given policy and exogenous inputs, and $\boldsymbol{u}(.) \in \Omega'$. The following function is assumed to be well-defined for given $\boldsymbol{z}(t_0, t]$:

$$\begin{aligned} &\hat{J} : \mathbb{R}^n \times T \times \Omega \times \mathbb{R}^n \times T \mapsto \mathbb{R} : \\ &\hat{J}[\boldsymbol{x_0}, t_0, \boldsymbol{u}(.), \boldsymbol{x}(.), t] = \\ &K[\boldsymbol{x}(t), t] + \int_{t_0}^{t} L[\boldsymbol{x}(\tau), \boldsymbol{u}(\tau), \tau] \, \mathrm{d}\tau. \end{aligned} \tag{26}$$

Then we say that the policy function $\boldsymbol{u}(.)$ takes ($\boldsymbol{x_0}$, t_0) to S for given exogenous input if the corresponding trajectory meets S, i.e. if

$$\{(\boldsymbol{\varphi}[t,t_0,\boldsymbol{x_0},\boldsymbol{u}(t_0,t],\boldsymbol{z}(t_0,t]],t): t \geq t_0\} \cap S \neq \emptyset \quad (27)$$

for given $\boldsymbol{z}(t_0,t]$.

If $\boldsymbol{u}(.)$ takes ($\boldsymbol{x_0}$, t_0) to S for given exogenous input, if $t_f > t_0$ is the first instant of time after t_0 where $\boldsymbol{x}(t)$ meets S, and if

$$\boldsymbol{x_f} = \boldsymbol{x}(t_f) = \boldsymbol{\varphi}[t_f,t_0,\boldsymbol{x_0},\boldsymbol{u}(t_0,t_f],\boldsymbol{z}(t_0,t_f]], \quad (28)$$

then $J[\boldsymbol{x_0},t_0,\boldsymbol{u}(.)] = \hat{J}[\boldsymbol{x}_0,t_0,\boldsymbol{u}(.),\boldsymbol{x_f},t_f]$ may be called the value of the *objective function* (performance functional) of the economic policy problem for policy input $\boldsymbol{u}(.)$ with respect to the target set S. t_f is called the terminal time, $\boldsymbol{x_f}$ the terminal state, and $K[\boldsymbol{x_f},t_f]$ the terminal cost. If $\boldsymbol{u}(.) \in \Omega'$ does not take ($\boldsymbol{x_0}$, t_0) to S, then we may define $J[\boldsymbol{x_0},t_0,\boldsymbol{u}(.)] = \infty$. Thus the objective function of the economic policy problem is a function

$$J[\boldsymbol{x_0},t_0,\boldsymbol{u}(.)]: \mathbb{R}^n \times T \times \Omega' \mapsto \mathbb{R} \cup \{\infty\}. \quad (29)$$

For fixed ($\boldsymbol{x_0}$, t_0), $J[...,\boldsymbol{u}(.)]$ becomes a function of the policy instrument trajectory only. Note that a given (and known) trajectory of exogenous variables has been presumed in our formulation. At first sight it might be more natural to define the cost functions L, K, and $\hat{J}$ in terms of the target variables $\boldsymbol{y_1}(t)$ instead of the state variables $\boldsymbol{x}(t)$. However, when the dynamic economic system under consideration is observable in the sense of dynamic system theory, the state can be recovered from the output in a unique way. Under this additional assumption, which makes sense for economic policy problems with flexible objectives (see, e.g., [2]), it is possible to deal with policy optimization problems involving the target variables (or their deviations from the desired values) as arguments of the objective function (e.g. output-regulator problems), by reducing them to those of the above formulation (e.g. state-regulator problems).

For the economic policy problem with flexible objectives, the dynamic economic system, the set of admissible policy instruments, the initial state $\boldsymbol{x_0}$ at initial time t_0, and the exogenous-variables trajectory $\boldsymbol{z}(t_0,t]$ are assumed to be given. The policy-maker has to specify the target set S and the objective function $J[\boldsymbol{x_0},t_0,\boldsymbol{u}(.)]$.

The optimization problem consists in determining an admissible trajectory of policy instrument variables, i.e. $\boldsymbol{u^*}(.) \in \Omega'$, such that ($\boldsymbol{x_0}$, t_0) is taken to S and $J[\boldsymbol{x_0},t_0,\boldsymbol{u}(.)]$ is minimized. A policy function $\boldsymbol{u^*}(.)$ that solves this problem is called an optimal policy function (or trajectory), and its values are the optimal policy instrument values. Maximization problems can be subsumed under this formulation by considering $-J[...,\boldsymbol{u}(.)]$ to be maximized. Optimal policy trajectories need not always exist, and if they do, they need not be unique. The theory of economic policy is concerned with the questions of existence, uniqueness, design, and stability of optimal policy trajectories for given economic policy problems.

Some extensions of the economic policy problem can be easily introduced into the preceding formulation. For instance, so far we have assumed that the state can take every possible value in $\mathbb{R}^n$. If we have a given closed set $\mathcal{X} \subset \mathbb{R}^n$ and $S \subseteq \mathcal{X} \times T$, then we can formulate an economic policy problem with flexible objectives and a *state constraint*: given the dynamic economic system, $\boldsymbol{x_0} \in \mathcal{X}, t_0, S, \Omega'$ and $\boldsymbol{z}(.) \in \Theta$, find $\boldsymbol{u^*}(.) \in \Omega'$ such that $\boldsymbol{x_0}$ is taken to S along a trajectory which lies entirely in $\mathcal{X}$ and minimizes $J[...,\boldsymbol{u}(.)]$ over all admissible policy functions.

On the other hand, the set of admissible policy instruments may be unconstrained, i.e. $U_t = \mathbb{R}^m$ for all t. Then every bounded piecewise continuous function $\boldsymbol{u}(.): T \mapsto \mathbb{R}^m$ is an admissible policy function. It should be noticed that this *unconstrained (free)* policy problem with flexible objectives may not have a solution even if a corresponding constrained policy problem has one, and vice versa.

Further generalization could relax on the assumptions of U, Ω, Z, Θ, Y, and Γ being Euclidean vector spaces, for example, although in these cases more general definitions of the objective function will have to be introduced as well, and meaningful solutions to the resulting optimization problems in general will be very difficult to achieve. Furthermore some of the technical assumptions on the functions involved may be relaxed; for instance, $\boldsymbol{u}(.)$ need not be required to be bounded, in which case impulse control policies may become optimal.

On the other hand, the above formulation of the basic problem of the theory of economic policy covers some special cases which are important in practical policy problems. For instance, the *free-time* policy problem demands that a point or a subset of the state space, which may be moving over time, shall be met. In this case, we have $S_t \subseteq \mathbb{R}^n, S_t \neq \emptyset$ for all $t \in T$, and $S = \bigcup_{t \in T} S_t \times \{t\}$. A particular case of this is a policy problem with a fixed state objective, where $S_t = \{\boldsymbol{x^*}(t)\}$ or $S = \{(\boldsymbol{x^*}(t),t), t \in T\}$ with $\boldsymbol{x^*}(t): T \mapsto \mathbb{R}^n$ given. A generalization to the fixed objective problem is obvious.

Another special case is the *fixed-time* policy problem, where a given set $S_1 \neq \emptyset$, $S_1 \subseteq \mathbb{R}^n$ shall be reached at a given (fixed) time $\hat{t} \in T$, i.e. $S = S_1 \times \{\hat{t}\}$; if $T = (t_1, t_2)$ with $t_2 = \infty$, then $S = S_1 \times \{t_0 + \hat{t}\}$, $\hat{t} > t_0$, defines a fixed-time policy problem.

Policy problems with a fixed end-point are also covered by our formulation, for example, the free-time policy problem with a fixed end-point, $S = \{\boldsymbol{x}^*\} \times T$, where $\boldsymbol{x}^* \in \mathbb{R}^n$ is given (e.g. $\boldsymbol{x}^* = \mathbf{0}$), or, more general, fixed-end-point policy problems where $\{\boldsymbol{x}^*\} = \{\boldsymbol{x} \in \mathbb{R}^n : \exists t \text{ with } (\boldsymbol{x}, t) \in S\}$, i.e. there is only one point in $\mathbb{R}^n$ in the target set. A special case of this problem in turn, and of the free-time policy problem, is the state-regulator problem, where $\boldsymbol{x}^*$ is an equilibrium (stationary) point of the system under zero policy input for given exogenous input $\boldsymbol{z}(t)$, that is, in (17) we have $\mathbf{0} = \boldsymbol{f}[t, \boldsymbol{x}^*, \mathbf{0}, \boldsymbol{z}(t)] \;\; \forall t \in T$.

6 Some Possible Extensions

In this paper, we have tried to provide a general formulation of the basic problem of the theory of economic policy, using concepts of the mathematical theory of dynamic systems. It has been shown that a more general class of policy problems could be defined in this way than is available so far in the literature on the theory of economic policy; thus dynamic system theory can be helpful in generalizing the scope of the theory of economic policy. This is also a prerequisite for applying this theory to practical policy problems. Linear-quadratic optimal control theory is a tool that allows a direct application of the concepts developed here and has already proved to be useful in a variety of economic policy contexts (e.g., [7], [13], [4]). Applications include macroeconomic stabilization policies [13], sustainable development ([14], [12]), policies improving administrative efficiency [16], etc.

On the other hand, several aspects of economic policy problems are not covered by the present formulation. For lack of space, we will only give some hints on how the theory of economic policy might be extended to deal with such problems as well.

(1) Our formulation is deterministic throughout. Actual policy problems, however, are characterized by uncertainty and incomplete information. A stochastic formulation could incorporate additive error terms in the system transition equation (stochastic differential equations) or measurement errors in the output variables (stochastic output equations) relatively easily. Stochastic parameters might also be introduced into these functions, although this complicates the solution of optimization problems enormously. On the other hand, specification errors in the dynamic economic system or uncertainty of the policy-maker about the structure of that system seem extremely difficult to be treated within the system-theoretic framework chosen here.

(2) In our formulation, the behavior of the economy is described by a set of differential and static equations, or, more generally, by an input-output transformation. This corresponds to assuming that there are invariant modes of behavior on the part of the private economic agents determining the overall behavior of the dynamic economic system. The economy is “passive” in the sense of not reacting on policy-maker’s actions in an intelligent and deliberate way. This will no longer be the case if economic agents have rational expectations and react strategically on government actions. In particular, if future policies are anticipated by the private-sector agents and taken into account when planning their behavior, a non-anticipatory system will no longer be appropriate, and the causal structure of the dynamic economic system assumed in our formulation does not hold. This will necessitate a major revision of the axioms defining that system.

(3) The assumption of rational economic agents also has to be introduced if there is more than one agent big enough so as not to behave in a way which can be appropriately represented by a passive structure. The same is true if there are different agents with either conflicting interests or divergent information about the economic system. This may be the case when several policy-making institutions are introduced, for example in international economic policy problems. In these cases, the theory of economic policy has to be extended to incorporate the possibility of policy inputs from several decision-makers into the dynamic economic system. Dynamic game theory and decentralized control theory formulations will be helpful for such a purpose. Some results involving dynamic economic policy-making with more than one policy-maker are given in [1]. Further extensions along these lines will be the next task for our efforts aimed at formulating system-theoretic foundations for the theory of economic policy.

7 Acknowledgment

Financial support from the Jubiläumsfonds der Oesterreichischen Nationalbank (project no. 12166) is gratefully acknowledged.

References:

[1] N. Acocella and G. Di Bartolomeo, Towards a New Theory of Economic Policy: Continuity and Innovation, in: R. Neck, Ch. Richter, P. Mooslechner, eds., *Quantitative Economic Policy*, Springer, Berlin, 2008, pp. 15–34.

[2] A. Aoki, *Optimal Control and System Theory in Dynamic Economic Analysis*, North-Holland, New York, 1976.

[3] M. Athans and P.L. Falb, *Optimal Control*, McGraw-Hill, New York, 1966.

[4] C. Botanu and F. Ostafi, A Solution to the Continuous and Discrete-Time Linear Quadratic Optimal Problems, *WSEAS Transactions on Systems and Control*, 3, 2008, pp. 71–78.

[5] J.L. Casti, *Dynamical Systems and Their Applications: Linear Theory*, Academic Press, New York, 1977.

[6] J.L. Casti, *Nonlinear System Theory*, Academic Press, New York, 1985.

[7] G.C. Chow, *Analysis and Control of Dynamic Economic Systems*, Wiley, New York, 1975.

[8] K.A. Fox, J.K. Sengupta and E. Thorbecke, *The Theory of Quantitative Economic Policy*, North-Holland, Amsterdam, 2nd ed., 1973.

[9] L. Johansen, *Lectures on Macroeconomic Planning*, Part 1, North-Holland, Amsterdam, 1977.

[10] H.G. Johnson, The Taxonomic Approach to the Study of Economic Policy, *Economic Journal*, 61, 1951, pp. 812–832.

[11] R.E. Kalman, P.L. Falb and M. Arbib, *Topics in Mathematical System Theory*, McGraw-Hill, New York, 1969.

[12] D. Kralj and M. Markic, Processes Innovation and Sustainable Development, *WSEAS Transactions on Environment and Development*, 4, 2008, pp. 99–108.

[13] R. Neck, Control Theory and Economic Policy: Balance and Perspectives, *Annual Reviews in Control*, 2009, in print.

[14] R. Neck and J.-E. Sturm (eds.), *Sustainability of Public Debt*, MIT Press, Cambridge, MA – London, 2008.

[15] A.J. Preston and A.R. Pagan, *The Theory of Economic Policy*, Cambridge, Cambridge University Press, 1982.

[16] J. Simurina, D. Hruska and M. Markovic, E-Government in Croatia: A Comparative Analysis, *WSEAS Transactions on Business and Economics*, 5, 2008, pp. 222–232.

[17] H. Theil, *Optimal Decision Rules for Government and Industry*, North-Holland, Amsterdam, 1968.

[18] J. Tinbergen, *On the Theory of Economic Policy*, North-Holland, Amsterdam, 1952.

[19] J. Tinbergen, *Economic Policy: Principles and Design*, North-Holland, Amsterdam, 1967.

[20] L.A. Zadeh and C.A. Desoer, *Linear System Theory*, McGraw-Hill, New York, 1963.