Comparing the aggregation methods in the analytic hierarchy process when uniform distribution

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Abstract: The Analytic Hierarchy Process (AHP) is a popular methodology for group decision making. Individual judgments can be aggregated in several ways, with the most effective approach being the aggregation of individual judgments (AIJ) and Individual priorities (AIP). When the judgments are aggregated, regardless of whether AIJ or AIP are used, arithmetic and geometric means are selected by decision makers. Some articles have discussed these two methods and made relevant suggestions. But when they discuses the issue of these, the distribution from judges' opinion were not considering. This study performed simulation to generate the weights of the judgers, assuming that the opinion are distributed as Uniform distribution and then generated AHP weights using arithmetic and geometric means. Following performing statistical testing for the relative mean square errors between the parameter and estimator based on simulation, the results demonstrated no significant difference. Finally, based on the results of this study, we conclude the following: (1) if number of judges is not large both then both methods are applicable; (2) if number of judges is large then the geometric mean cannot be obtained, and the arithmetic mean is applicable; and (3) when the opinions of judges coincide, the arithmetic mean is applicable.

Keywords: Analytic hierarchy process; Aggregating individual judgments; Aggregating individual priorities; Arithmetic mean; Geometric mean

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1. Introduction

The Analytic Hierarchy Process (AHP) of Satty (1980) is a popular decision making method, and uses pairwise comparisons for priority setting. In AHP, several processes are used to aggregate the decision maker opinions, with the two most popular being: (1) aggregating individual judgments regarding each set of pairwise comparisons to produce an aggregate hierarchy; (2) synthesizing each of the individual hierarchies and aggregating the resulting priorities (Forman and Peniwati, 1998). These two processes are also termed the aggregation of individual judgments (AIJ), and the aggregation of individual priorities (AIP). In practice, researchers and decision makers use both AIJ and AIP, and there is no special reason for favoring one over the other, but the aggregation of expert opinion is a problem, because the different aggregation method may caused the different weight and influence the decision. Aczel and Satty (1983) and Aczel and Roberts (1989) have demonstrated that judgments regarding the AHP matrix were reciprocal and the property is assumed even for a single n-tuple, and moreover only the geometric mean satisfied the Pareto Principle (unanimity condition) and homogeneity condition. However, Forman and Peniwati (1998) explained that the optimal mathematical procedure for aggregation depends on whether the group is assumed to be a synergistic unit or merely a collection of individuals. Furthermore, Forman and Peniwati (1998) discussed the inapplicability of the Pareto Principle, since when aggregating individual judgments each individual is regarded as independent**.** From their perspective both geometric mean and arithmetic mean are appropriate procedures for ratio scales. But the concepts of these theories consider the experts in one condition which their weight toward the criteria was under consistency and transitivity. Actually the distribution of opinion from experts may different in different situations, when opinions from experts were around the specific value that means the priority of each criteria from every experts is close to each other, and the shape of the data may show as a bell shape distribution, such as normal distribution. But if the priority of each criteria from every experts is different, the weight may spread as uniform distribution. This study performed simulation works to generate the weights of the judgers, assuming that the opinion are distributed as Uniform distribution and then generated AHP weights using arithmetic and geometric means. The result may provide some suggestion when dealing the AHP to make decision.

2. Aggregation method

The AHP process includes five stages, namely: (1) establish a hierarchy; (2) design the questionnaire; (3) calculate the criteria weighting; (4) calculate the weigh and (5) rank the projects. This study focuses on stage 3, calculate the criteria weighting, because the aggregation methods are applied during this stage. The first step in calculating the criteria weighting is establishing the pair wise comparison matrix $A^k = \left[a_{ij}^k\right]_{n\times n}$. Let represent the set of elements or criteria, and let the comparison pairs for elements and be represented by a $C_1, C_2, \cdots, C_n C_i C_j n \times n$ matrix:

$$A^k = \begin{array}{c} \\ C_1 \\ C_2 \\ \vdots \\ C_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} 1 & a_{12}^k & \cdots & a_{1n}^k \\ 1/a_{12}^k & 1 & \cdots & a_{2n}^k \\ \vdots & \vdots & \vdots & \vdots \\ 1/a_{1n}^k & 1/a_{2n}^k & \cdots & 1 \end{bmatrix} \end{array}$$

Where represent the experts and $k k k = 1, 2, 3, \cdots, m$. Meanwhile, the weighting matrix is as follows:

$$\begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array}$$
$$W = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}$$

The various aggregating methods considered in this study are applied to calculate *W*.

These three aggregating methods are: 1 calculate the AHP weight of every expert and then get the weight through arithmetic mean, which named method 1 in the article; 2. aggregate every experts' grades toward every criteria by arithmetic mean and then find the AHP weight, which named method 2 in the article; and 3. aggregate every experts' grades toward every criteria by geometric mean and then find the AHP weight, which named method 3 in the article.The process of weighting criteria generated using the method one is explained as follows:

Step1: obtain comparison matrixes from *m* experts, as follows: $A^k = \left[a_{ij}\right]_{n\times n}$ where; .
$k = 1, 2, 3, \cdots, m i = 1, 2, 3, \cdots, n$

Step2: calculate the maximum eigenvalue and its eigenvector of, and obtain the standard eigenvector as the AHP weight matrix of the k th expert, as follows $A^k W^k = \begin{bmatrix} w_1^k & w_2^k & \cdots & w_n^k \end{bmatrix}$

Step3: calculate the arithmetic mean of the expert weightings of the criteria in levelthus l

$$W = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} = \begin{bmatrix} \frac{\sum_{k=1}^m w_1^k}{m} & \frac{\sum_{k=1}^m w_2^k}{m} & \cdots & \frac{\sum_{k=1}^m w_n^k}{m} \end{bmatrix}$$

The process used for weighting the criteria in level *l* that were generated via method 2 is explained as follows:

Step1: obtain the pairwise comparison matrix for m experts, as follows $A^k = \left[a_{ij}\right]_{n\times n}$, where;
$k = 1, 2, 3, \cdots, m i = 1, 2, 3, \cdots, n$ and.
$j = 1, 2, 3, \cdots, n$

Step2: find the aggregation pair wise comparison matrix $A^l = \left[a_{ij}\right]_{n\times n}$ by using the arithmetic mean:

$$A^k = \left[a_{ij}\right]_{n\times n} = \begin{bmatrix} 1 & \frac{\sum_{k=1}^m a_{12}^k}{m} & \cdots & \frac{\sum_{k=1}^m a_{1n}^k}{m} \\ \frac{m}{\sum_{k=1}^m a_{12}^k} & 1 & & \frac{\sum_{k=1}^m a_{2n}^k}{m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m}{\sum_{k=1}^m a_{1n}^k} & \cdots & \frac{m}{\sum_{k=1}^m a_{2n}^k} & 1 \end{bmatrix}$$

Step3: calculate the maximum eigenvalue and its eigenvector of $A = \left[a_{ij}\right]_{n\times n}$ and obtain the standard eigenvector as an criteria of the expert weighting of criteria in level. l

The process of weighting the criteria in level *l*, which are generated via method 3 almost the same with method 2, the different step is step 2, it is explained as folling

Step2: find the aggregation pair wise comparison matrix $A = \left[a_{ij}\right]_{n\times n}$ based on arithmetic mean:

$$A = \left[a_{ij}\right]_{n \times n} = \begin{bmatrix} 1 & \sqrt[m]{\prod_{k=1}^{m} a_{12}^{k}} & \cdots & \sqrt[m]{\prod_{k=1}^{m} a_{1n}^{k}} \\ \frac{1}{\sqrt[m]{\prod_{k=1}^{m} a_{12}^{k}}} & 1 & \vdots & \sqrt[m]{\prod_{k=1}^{m} a_{2n}^{k}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt[m]{\prod_{k=1}^{m} a_{1n}^{k}}} & \cdots & \frac{1}{\sqrt[m]{\prod_{k=1}^{m} a_{2n}^{k}}} & 1 \end{bmatrix}$$

The main focus of this investigation is comparing three generated using three different methods.

$$W = \left[w_1 \quad w_2 \quad \cdots \quad w_n\right]$$ [redacted]

3. Experimental Simulation and Results

In the simulation work, the real weights of the criteria of every expert are generated during the first stage and then the three methods were applied to aggregate the AHP weights and determine them separately. Finally, the differences between the real weights and AHP weights are compared, and consequently the relative mean square error (RMSE) of these methods. During the simulation, the number of experts, mean of weight, variance and the number of criteria are the important parameters. The manipulating of them is described as following:

1. The number of experts: 20 to 300 experts were simulated.
2. Mean of weight: the mean of experts' weight toward the criteria were set and varied by 0.1 each time.
3. Variance: variance of Uniform distribution were operated and 4 situation were setting, they were $1/5$, $1/10$, $1/15$ and $1/20$ mean.
4. Criteria: according to the Satty(1980), the criteria is from 3to 7.

The real weights can be used to generate the pairwise comparison matrix of every expert in every situation, and then the three AHP weights can be obtained using the three methods being compared. After comparing the differences of real weight and AHP weights obtained using the three methods, the relative mean square errors (RSM) between the parameter and estimators were derived. Following the statistical testing, the p-values for the source of variance, including method, variance of simulation data and interaction between method and variance of simulation data, were listed in Table 1.

First, from the p-value when the source of variance is method, the result indicated that the differences were significant when the number of expert exceeded 200, the number of criterias are three or four; and when the number of expert exceeded 200, the number of criterias are three four or five. However, from Table 2, the differences between means are not easy found. For example, when the criteria is 5, and the number of experts was 300, the p-value was 0.005, while the mean of methods 1, 2 and 3 was 0.034.

Second, all of the p-values were 0 when the source of variance is the variance of the simulation data in Table 1. Moreover, differences really exist in the mean of RSM listed in Table 3, and they varied with the variation of the variance of simulation data, and the RSM decreased by the increasing of variance. And the mean decreased by the increasing of number of experts.

Third, the p-value of interaction of variance and method indicated that when the number of experts exceeded 300, and the criterias are three, four and five the differences were significant. Tables 5 show the differences in the means of individual situations when the criterias are three, four and five, and from those result, the differences are not easy found. For example, when the source of variance was interaction of method and variance, the indictors are 5, and the number of experts was 300, the p-value in Table 1 was 0.002. In comparison, Table 4 lists that the means obtained using the three methods with variance of 1 were, (0.050, 0.049, 0.049); while for variance of 2 the means were(0.034, 0.034, 0.034); for variance of 3 they were (0.027,0.027,0.027) and for variance of 4 they were (0.024, 0.024, 0.024). Figure 1 shows these results more clearly.

Table 1 P-value of Every Situation

Number of Criterias	Number of Experts / Source of Variance	20	30	40	50	60	70	80	90	100	200	300
3	1	0.950	0.963	0.804	0.884	0.711	0.883	0.623	0.728	0.404	0.006	0.012
	2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3	1.000	1.000	0.997	0.999	0.981	1.000	0.859	0.996	0.755	0.093	0.032
4	1	0.912	0.947	0.913	0.944	0.900	0.810	0.589	0.560	0.461	0.035	0.003
	2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

	3	1.000	1.000	1.000	1.000	1.000	0.996	0.930	0.901	0.847	0.096	0.003
5	1	0.921	0.914	0.862	0.846	0.753	0.682	0.586	0.000	0.600	0.062	0.005
	2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3	1.000	1.000	0.999	0.997	0.990	0.978	0.920	0.003	0.941	0.155	0.002
6	1	0.916	0.883	0.823	0.880	0.828	0.719	0.673	0.783	0.681	0.200	0.06
	2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3	1.000	0.999	0.997	0.998	0.997	0.989	0.973	0.987	0.989	0.356	0.102
7	1	0.905	0.849	0.860	0.854	0.905	0.891	0.914	0.849	0.835	0.654	0.363
	2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	3	0.999	0.995	0.998	0.997	0.999	0.998	0.999	0.995	0.998	0.966	0.731

Note: Source of variance
1: Method
2:Variance
3: The interaction of Method and Variance

Table 2 Mean of Relative Mean Square Errors of Methods

Number of Criterias	Number of Experts Method	20	30	40	50	60	70	80	90	100	200	300
3	1	0.064	0.058	0.054	0.051	0.049	0.047	0.046	0.044	0.043	0.037	0.033
	2	0.064	0.057	0.053	0.050	0.048	0.047	0.045	0.044	0.042	0.036	0.033
	3	0.064	0.057	0.053	0.051	0.048	0.047	0.045	0.044	0.042	0.036	0.033
4	1	0.066	0.059	0.055	0.052	0.050	0.048	0.047	0.045	0.044	0.038	0.034
	2	0.066	0.059	0.055	0.052	0.050	0.048	0.047	0.045	0.044	0.037	0.034
	3	0.066	0.059	0.055	0.052	0.050	0.048	0.047	0.045	0.044	0.037	0.034
5	1	0.066	0.060	0.056	0.053	0.050	0.048	0.047	0.046	0.044	0.037	0.034
	2	0.066	0.060	0.056	0.053	0.050	0.048	0.047	0.046	0.044	0.037	0.034
	3	0.066	0.060	0.056	0.053	0.050	0.048	0.047	0.048	0.044	0.037	0.034
6	1	0.065	0.059	0.055	0.052	0.050	0.048	0.046	0.045	0.044	0.037	0.033
	2	0.065	0.059	0.055	0.052	0.050	0.048	0.046	0.045	0.044	0.037	0.033
	3	0.065	0.058	0.055	0.052	0.050	0.048	0.046	0.045	0.044	0.037	0.033
7	1	0.064	0.058	0.054	0.051	0.049	0.047	0.046	0.044	0.043	0.036	0.033
	2	0.064	0.058	0.054	0.051	0.049	0.047	0.046	0.044	0.043	0.036	0.033
	3	0.064	0.058	0.054	0.051	0.049	0.04	0.046	0.044	0.043	0.036	0.033

Table 3 Mean of Relative Mean Square Errors of Variances

Number of Criterias	Number of Expers Variance	20	30	40	50	60	70	80	90	100	200	300
3	1	0.091	0.083	0.078	0.071	0.070	0.067	0.066	0.064	0.061	0.053	0.048
	2	0.065	0.059	0.053	0.051	0.049	0.048	0.046	0.043	0.043	0.037	0.033
	3	0.053	0.048	0.045	0.043	0.041	0.038	0.037	0.036	0.035	0.029	0.027
	4	0.046	0.041	0.038	0.037	0.035	0.034	0.033	0.031	0.031	0.025	0.023
4	1	0.095	0.086	0.080	0.074	0.073	0.069	0.068	0.065	0.063	0.054	0.049
	2	0.067	0.060	0.056	0.053	0.050	0.049	0.047	0.046	0.045	0.038	0.034
	3	0.054	0.049	0.046	0.043	0.042	0.040	0.039	0.038	0.036	0.031	0.028

	4	0.048	0.043	0.039	0.038	0.036	0.034	0.034	0.033	0.031	0.027	0.024
5	1	0.095	0.087	0.080	0.075	0.072	0.070	0.068	0.067	0.064	0.054	0.049
	2	0.067	0.060	0.056	0.054	0.051	0.049	0.048	0.047	0.044	0.038	0.034
	3	0.055	0.049	0.046	0.044	0.042	0.040	0.039	0.039	0.037	0.031	0.027
	4	0.047	0.043	0.040	0.038	0.036	0.035	0.034	0.033	0.032	0.027	0.024
6	1	0.094	0.085	0.079	0.075	0.071	0.069	0.066	0.065	0.063	0.053	0.048
	2	0.067	0.060	0.056	0.053	0.050	0.049	0.047	0.045	0.044	0.037	0.034
	3	0.054	0.049	0.046	0.043	0.041	0.040	0.038	0.037	0.036	0.031	0.028
	4	0.047	0.042	0.040	0.037	0.036	0.034	0.033	0.032	0.031	0.026	0.024
7	1	0.092	0.084	0.078	0.073	0.070	0.060.	0.066	0.064	0.061	0.053	0.048
	2	0.065	0.059	0.055	0.052	0.049	0487	0.046	0.045	0.043	0.037	0.033
	3	0.053	0.048	0.044	0.042	0.040	0.038	0.038	0.037	0.036	0.030	0.027
	4	0.046	0.042	0.039	0.037	0.035	0.040	0.033	0.032	0.031	0.026	0.023

Table 4 Relative Mean Square Errors of 300 Experts

Variance	1			2			3			4		
Method Number of Criterias	1	2	3	1	2	3	1	2	3	1	2	3
3	0.050	0.047	0.048	0.033	0.033	0.033	0.027	0.027	0.027	0.023	0.023	0.023
4	0.050	0.048	0.048	0.034	0.034	0.034	0.028	0.028	0.028	0.024	0.024	0.024
5	0.050	0.049	0.049	0.034	0.034	0.034	0.027	0.027	0.027	0.024	0.024	0.024

Estimated Marginal Means

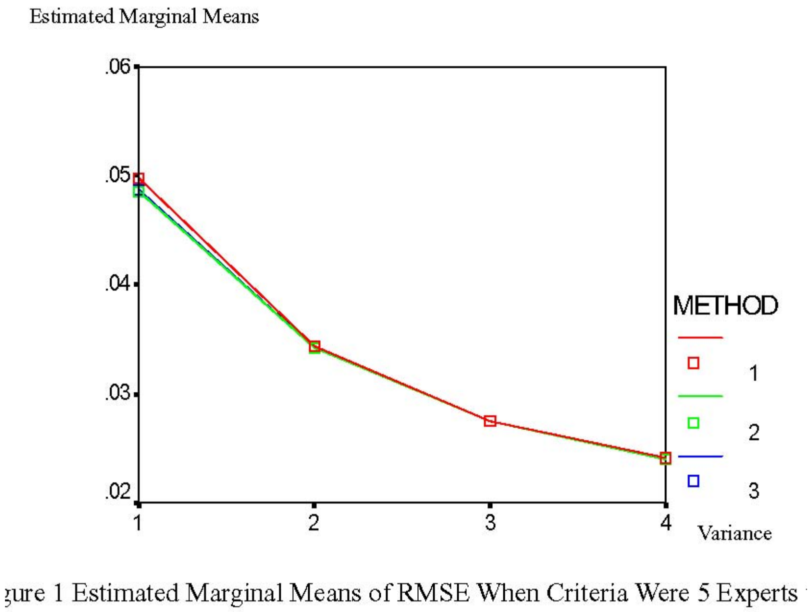

Figure 1 Estimated Marginal Means of RMSE When Criteria Were 5 Experts were 300

Variance

4. Conclusion

This investigation obtained five important findings. First, variance of experts' opinion may influence the result of AHP weight, and the RMSE will decrease when the variance decrease. Second, from the p-value, there is no difference among the three methods until the number of experts exceeds 200 and the criteria is less than 4; and when experts exceeds 300 and the criteria is less than 5. But the differences between the real and simulated weights are extremely small, which demonstrates that methods of aggregation will not directly influence the result. Third, although no differences were observed among the methods, the find the AHP of every expert and then get their arithmetic mean is inefficient. Forth, the number of experts should be considered when decision makers are selecting the aggregation method; if the number of experts is large, a geometric mean is inappropriate, because it cannot be calculated; and thus the arithmetic mean is a better method in this situation. Finally, variance should be considered, decision maker need to check the variance before dealing the AHP, if there is outlier in the opinion set, it should be considered to drop. The applications of AHP or fuzzy AHP (Hsu and Chen, 2006; Wu et al., 2006a; Wu et al., 2006b) are very popular in this day, clarifying the aggregation method and contents of AHP will help researchers and decision makers when using this method.

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