

A quadrature formula based on a spline quasi-interpolant

ANA MARIA ACU

"Lucian Blaga" University

Department of Mathematics

Str. Dr. I. Rațiu, No. 5-7, 550012 - Sibiu

ROMANIA

acuana77@yahoo.com

MUGUR ACU

"Lucian Blaga" University

Department of Mathematics

Str. Dr. I. Rațiu, No. 5-7, 550012 - Sibiu

ROMANIA

acu_mugur@yahoo.com

Abstract: In this paper we study a new simple quadrature rule based on integrating a spline quasi-interpolant operator on a bonded interval. We also give estimates of the quadrature error for smooth functions in the case of uniform partition by using the associated Peano kernel. We improve the degree of exactness of quadrature formulae with the knots depending on a parameter.

Key-Words: quadrature rule, numerical integration, spline quasi-interpolant

1 Introduction

Denote by

$$H^n[a, b] := \left\{ f \in C^{n-1}[a, b], f^{(n-1)} \text{ abs. cont.} \right\}.$$

Theorem 1 (Peano's theorem) Let $L(f)$ be an arbitrary linear functional defined in $H^n[a, b]$ such that the function $K(t) := L[(x-t)_+^{n-1}]$ is integrable over $[a, b]$. Suppose that $L(p) = 0$ for each polynomial $p \in \mathcal{P}_{n-1}$. Then $L(f) = \frac{1}{(n-1)!} \int_a^b K(t) f^{(n)}(t) dt$, for each $f \in H^n[a, b]$.

Definition 2 [2] The function $s(x)$ is called a spline function of degree n with knots $\{t_i\}_{i=1}^d$ if $-\infty := t_0 < t_1 < \dots < t_d < t_{d+1} := \infty$ and
i) for each $i = 0, \dots, d$, $s(x)$ coincides on (t_i, t_{i+1}) with a polynomial of degree not greater than n ;
ii) $s(x), s'(x), \dots, s^{(n-1)}(x)$ are continuous functions on $(-\infty, +\infty)$.

We shall denote by $S_n(t_1, \dots, t_d)$ the class of all spline functions of degree n with knots at t_1, \dots, t_d . For fixed $\{t_i\}_{i=1}^d$, $S_n(t_1, \dots, t_d)$ is a linear space and $\dim S_n(t_1, \dots, t_d) = n + d + 1$.

Let $x_0 \leq \dots \leq x_{n+1}$ be arbitrary points in $[a, b]$ such that $x_0 < x_{n+1}$.

Definition 3 [2] The spline function $B(x_0, \dots, x_{n+1}; x) = (\cdot - x)_+^n [x_0, \dots, x_{n+1}]$ is called a B-spline of degree n with knots x_0, \dots, x_{n+1} .

We denote by $(\cdot - x)_+^n [x_0, \dots, x_{n+1}]$ the divided difference of the function $(\cdot - x)_+^n$ at the points x_0, \dots, x_{n+1} .

A property of B -spline it is:

$$\int_a^b B(x_0, \dots, x_{n+1}; t) dt = \frac{1}{n+1}.$$

Given the sequence (finite or infinite) of points $\{x_i\}$, such that

$$\dots \leq x_i \leq x_{i+1} \leq \dots$$

and $x_i < x_{i+n+1}$ for all i , we shall denote by $B_{i,n}(t)$ the B -spline

$$B_{i,n}(t) = (\cdot - x)_+^n [x_i, \dots, x_{i+n+1}].$$

Theorem 4 [2] Let $a < x_{n+2} \leq \dots \leq x_m < b$ be fixed points such that $x_i < x_{i+n+1}$ for all admissible i . Choose arbitrary $2n+2$ additional points $x_1 \leq \dots \leq x_{n+1} \leq a$ and $b \leq x_{m+1} \leq \dots \leq x_{m+n+1}$ and define $B_{i,n}(t) = B(x_i, \dots, x_{i+n+1}; t)$. The B -spline $B_{1,n}(t), \dots, B_{m,n}(t)$ constitute a basis for $S_n(x_{n+2}, \dots, x_m)$.

The B -spline basis for the space $S_n(x_{n+2}, \dots, x_m)$ was constructed by Curry and Schoenberg in [7].

The spline function $N_{i,n}(t) = (x_{i+n+1} - x_i) B_{i,n}(t)$ is called normalized B -spline and satisfy the relation

$$\begin{aligned} N_{i,n}(t) &= \frac{x_{i+n+1} - t}{x_{i+n+1} - x_{i+1}} N_{i+1,n-1}(t) \\ &+ \frac{t - x_i}{x_{i+n} - x_i} N_{i,n-1}(t), \\ N_{i,0}(t) &= \begin{cases} 1, & t \in [x_i, x_{i+1}), \\ 0, & t < x_i \text{ and } t \geq x_{i+1}. \end{cases} \end{aligned}$$

From the above relation, we have

$$N_{i,1}(t) = \begin{cases} \frac{x_{i+2}-t}{x_{i+2}-x_{i+1}}, & x_{i+1} \leq t \leq x_{i+2}, \\ \frac{t-x_i}{x_{i+1}-x_i}, & x_i \leq t \leq x_{i+1}, \end{cases} \quad (1)$$

and

$$N_{i,2}(t) = \begin{cases} \frac{t-x_i}{x_{i+2}-x_i} \cdot \frac{t-x_i}{x_{i+1}-x_i}, & x_i \leq t \leq x_{i+1}, \\ \frac{x_{i+3}-t}{x_{i+3}-x_{i+1}} \cdot \frac{t-x_{i+1}}{x_{i+2}-x_{i+1}} + \\ \frac{t-x_i}{x_{i+2}-x_i} \cdot \frac{x_{i+2}-t}{x_{i+2}-x_{i+1}}, & x_{i+1} \leq t \leq x_{i+2}, \\ \frac{x_{i+3}-t}{x_{i+3}-x_{i+1}} \cdot \frac{x_{i+3}-t}{x_{i+3}-x_{i+2}}, & x_{i+2} \leq t \leq x_{i+3}. \end{cases} \quad (2)$$

Given a function f , the basic problem of quasi-spline approximation is to determine B -spline coefficients $(c_i)_{i=1}^m$ such that

$$Pf = \sum_{i=1}^m c_i N_{i,n}$$

is a reasonable approximation to f .

Interesting results about spline quasi-interpolants were obtain by P. Sablonière in [12], [13], [14], [15], [16], T. Lyche and K. Mørken in [10], D. Barrera, M.J. Ibáñez, P. Sablonière, D. Sbibih in [3], [4], [5], B.G. Lee, T. Lyche and L.L.Schumaker in [8].

In [10] is given the following procedure for determining the B -spline coefficients.

Let $x = (x_j)_{j=1}^{m+n+1}$ be arbitrary points in $[a, b]$, nondecreasing with $x_{n+1} = a$ and $x_{m+1} = b$. We assume that f is defined on $[a, b]$. We fix j and propose the following procedure for determining c_j :

1) Choose a local interval $I = (x_\mu, x_\nu)$ with the property that I intersects the support of $N_{j,n}$:

$$I \cap (x_j, x_{j+n+1}) \neq \emptyset.$$

Denote the restriction of the space $S_n(x_{n+2}, \dots, x_m)$ to the interval I by $S_{n,I}$, namely

$$S_{n,I} = \text{span} \{N_{\mu-n,n}, \dots, N_{\nu-1,n}\}.$$

2) Choose some local approximation method P^I with the property that $P^I g = g$ for all $g \in S_{n,I}$.

3) Let f^I denote the restriction of f to the interval I . Then there exist B -spline coefficients $(b_i)_{i=\mu-n}^{\nu-1}$

such that $P^I f^I = \sum_{i=\mu-n}^{\nu-1} b_i N_{i,n}$. Note that $\mu - n \leq j \leq \nu - 1$ since $\text{supp } N_{j,n}$ intersects I .
4) Set $c_j = b_j$.

2 A quadrature formula with degree of exactness equal to 1

Let $S_1(x_3, \dots, x_m)$ be the space of spline functions of degree 1 with knots at x_3, \dots, x_m . Let f be a function defined by $[x_2, x_{m+1}]$. The spline quasi-interpolant operator is

$$P_1 f = \sum_{j=1}^m c_j N_{j,1}.$$

To determine B -spline coefficients, c_j , by choosing the local interval $I = [x_j, x_{j+1}]$ and the local approximation method

$$P_1^I f(x) = \sum_{i=j-1}^j b_i N_{i,1}(x). \quad (3)$$

We consider that local approximation method is the polynomial interpolation at knots

$$\begin{cases} x_j^{(1)} = (1-\alpha)x_j + \alpha x_{j+1} \\ x_j^{(2)} = \alpha x_j + (1-\alpha)x_{j+1} \end{cases}$$

where $\alpha \in [0, 1] \setminus \left\{ \frac{1}{2} \right\}$. We have

$$\begin{cases} P_1^I f(x_j^{(1)}) = f(x_j^{(1)}), \\ P_1^I f(x_j^{(2)}) = f(x_j^{(2)}), \end{cases}$$

namely

$$\begin{cases} b_{j-1} N_{j-1,1}(x_j^{(1)}) + b_j N_{j,1}(x_j^{(1)}) = f(x_j^{(1)}), \\ b_{j-1} N_{j-1,1}(x_j^{(2)}) + b_j N_{j,1}(x_j^{(2)}) = f(x_j^{(2)}). \end{cases} \quad (4)$$

Using relations (1) and (4) we obtain

$$\begin{cases} (1-\alpha)b_{j-1} + \alpha b_j = f(x_j^{(1)}) \\ \alpha b_{j-1} + (1-\alpha)b_j = f(x_j^{(2)}) \end{cases} \quad (5)$$

From (5) we have

$$b_j = \frac{1}{2\alpha-1} [\alpha f(x_j^{(1)}) - (1-\alpha)f(x_j^{(2)})]$$

and the spline quasi-interpolant operator will be

$$P_1 f = \sum_{j=1}^m \frac{1}{2\alpha-1} [\alpha f(x_j^{(1)}) - (1-\alpha)f(x_j^{(2)})] N_{j,1}. \quad (6)$$

Let $x_1 = x_2 = a$, $x_{m+1} = x_{m+2} = b$ and $(x_i)_{i=3}^m$ are the equidistant nodes from the interval $[a, b]$. If we integrate the approximation formula of function f

$$\begin{aligned} f(x) &= \sum_{j=1}^m \frac{1}{2\alpha-1} \left[\alpha f(x_j^{(1)}) - (1-\alpha)f(x_j^2) \right] N_{j,1}(x) \\ &+ r_m[f] \end{aligned}$$

to obtain following quadrature formula with the exactness degree 1

$$\begin{aligned} \int_a^b f(x)dx &= \frac{1}{2\alpha-1} \sum_{j=1}^m \frac{x_{j+2}-x_j}{2} \left[\alpha f(x_j^{(1)}) \right. \\ &\quad \left. - (1-\alpha)f(x_j^{(2)}) \right] + \mathcal{R}_m[f]. \end{aligned} \quad (7)$$

If we choose $\alpha = 1$ or $\alpha = 0$, then the quadrature formula (7) have degree of exactness equal to 1 and can be written:

$$\int_a^b f(x)dx = \sum_{j=1}^m \frac{x_{j+2}-x_j}{2} f(x_{j+1}) + \mathcal{R}_m[f]. \quad (8)$$

For $m = 2$ we have trapezoid quadrature formula. If we consider $m = 3$, to obtain the following quadrature formula

$$\int_a^b f(x)dx = \frac{b-a}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] + \mathcal{R}_3[f].$$

For $m = 4$ we have the following quadrature formula

$$\begin{aligned} \int_a^b f(x)dx &= \frac{b-a}{6} \left[f(a) + 2f\left(\frac{2a+b}{3}\right) \right. \\ &\quad \left. + 2f\left(\frac{a+2b}{3}\right) + f(b) \right] + \mathcal{R}_4[f]. \end{aligned}$$

Next to study the quadrature formulae for $m > 4$. For simplicity of calculations we choose $a = 0$, $b = 1$. If denote $h = \frac{1}{m-1}$, we have $x_i = (i-2)h$, $i = \overline{3, m}$ and the quadrature formula (8) can be written

$$\int_0^1 f(x)dx = \frac{h}{2} \left\{ f(0) + 2 \sum_{j=1}^{m-2} f(jh) + f(1) \right\} + \mathcal{R}_m[f]. \quad (9)$$

The exactness degree of quadrature formula (9) is equal with 1 and from Theorem 1, the remainder term has the form

$$\mathcal{R}_m[f] = \int_0^1 K(t)f''(t)dt, \text{ where } f \in H^2[0, 1]$$

$$\begin{aligned} K(t) &= \mathcal{R}_m[(\cdot - t)_+] \\ &= \frac{(1-t)^2}{2} - \frac{h}{2} \left[2 \sum_{j=1}^{m-2} (jh - t)_+ + (1-t) \right]. \end{aligned} \quad (10)$$

Lemma 5 *The Peano's kernel defined in relation (10) verifies*

$$K(t) = K(1-t), \text{ any } t \in [0, 1]; \quad (11)$$

$$K(t) \leq 0, \text{ any } t \in [0, 1]; \quad (12)$$

$$\max_{t \in [0, 1]} |K(t)| = \frac{h^2}{8}; \quad (13)$$

$$\int_0^1 K(t)dt = -\frac{1}{12(m-1)^2}. \quad (14)$$

Proof: Using the symmetry of nodes and coefficients we obtain

$$\begin{aligned} K(1-t) &= \frac{t^2}{2} - \frac{h}{2} \left[2 \sum_{j=1}^{m-2} (t - (m-1-j)h)_+ + t \right] \\ &= \frac{t^2}{2} - \frac{h}{2} \left[2 \sum_{j=1}^{m-2} (t - jh)_+ + t \right]. \end{aligned} \quad (15)$$

If in the quadrature formula (9), we choose $f(x) = x - t \in \mathcal{P}_1$ to obtain the following relation

$$\frac{(1-t)^2}{2} - \frac{t^2}{2} = \frac{h}{2} \left[-t + 2 \sum_{j=1}^{m-2} (jh - t) + (1-t) \right]. \quad (16)$$

From the relation (10), (15), (16) and the formula

$$(t_i - t)_+ - (t - t_i)_+ = (t_i - t)$$

we have $K(t) = K(1-t)$. We denote

$$K(t) = K_i(t) \text{ for } t \in [(i-1)h, ih], i = \overline{1, m-1}.$$

From the relation (15) we obtain

$$\begin{aligned} K_1(t) &= \frac{t^2}{2} - \frac{h}{2}t, t \in [0, h] \\ K_i(t) &= \frac{t^2}{2} - \frac{h}{2} \left\{ 2 \sum_{j=1}^{i-1} (t - jh) + t \right\} \\ &= \frac{t^2}{2} - (2i-1)\frac{h}{2}t + \frac{h^2}{2}i(i-1), \end{aligned}$$

for $t \in [(i-1)h, ih]$, $i = \overline{2, m-1}$.

We have

$$\begin{aligned} K'_1(t) &= t - \frac{h}{2}, & t \in [0, h] \\ K'_i(t) &= t - (2i-1)\frac{h}{2}, & t \in [(i-1)h, ih], \\ && i = \overline{2, m-1}. \end{aligned} \quad (17)$$

From relation (17) we obtain $K(t) \leq 0$, $i = \overline{1, m-1}$ and $\max_{t \in [0, 1]} |K(t)| = \frac{h^2}{8}$.

We have

$$\begin{aligned} \int_0^1 K(t) dt &= \int_0^h \left(\frac{t^2}{2} - \frac{h}{2}t \right) dt \\ &+ \sum_{i=2}^{m-1} \int_{(i-1)h}^{ih} \left[\frac{t^2}{2} - (2i-1)\frac{h}{2}t + \frac{h^2}{2}i(i-1) \right] dt \\ &= \frac{h^3}{12}(1-m) = -\frac{1}{12(m-1)^2}. \end{aligned}$$

□

Theorem 6 If $f \in H^2[0, 1]$ and there exist real numbers γ, Γ such that $\gamma \leq f''(t) \leq \Gamma$, $t \in [0, 1]$, then

$$|\mathcal{R}_m[f]| \leq \frac{1}{24(m-1)^2} [\Gamma - \gamma + |\Gamma + \gamma|].$$

Proof: We can write

$$\begin{aligned} \mathcal{R}_m[f] &= \int_0^1 K(t) f''(t) dt = \int_0^1 K(t) \left[f''(t) - \frac{\gamma + \Gamma}{2} \right] dt \\ &+ \frac{\gamma + \Gamma}{2} \int_0^1 K(t) dt = \int_0^1 K(t) \left[f''(t) - \frac{\gamma + \Gamma}{2} \right] dt \\ &- \frac{1}{24(m-1)^2} (\gamma + \Gamma) \end{aligned}$$

and we obtain

$$\begin{aligned} |\mathcal{R}_m[f]| &\leq \max_{t \in [0,1]} \left| f''(t) - \frac{\gamma + \Gamma}{2} \right| \cdot \int_0^1 |K(t)| dt \\ &+ |\gamma + \Gamma| \cdot \frac{1}{24(m-1)^2} = \frac{1}{12(m-1)^2} \\ &\cdot \left\{ \max_{t \in [0,1]} \left| f''(t) - \frac{\gamma + \Gamma}{2} \right| + \left| \frac{\gamma + \Gamma}{2} \right| \right\} \\ &= \frac{1}{12(m-1)^2} \left\{ \frac{\Gamma - \gamma}{2} + \frac{|\Gamma + \gamma|}{2} \right\} \\ &= \frac{1}{24(m-1)^2} \{ \Gamma - \gamma + |\Gamma + \gamma| \}. \end{aligned}$$

□

Theorem 7 If $f \in H^2[0, 1]$ and there exist real numbers γ, Γ such that $\gamma \leq f''(t) \leq \Gamma$, $t \in [0, 1]$, then we have

$$\frac{1}{24(m-1)^2} (3T - 5\Gamma) \leq \mathcal{R}_m[f] \leq \frac{1}{24(m-1)^2} (3T - 5\gamma),$$

where $T = f'(1) - f'(0)$.

Proof: We have

$$\int_0^1 K(t) [f''(t) - \gamma] dt = \mathcal{R}_m[f] + \frac{\gamma}{12(m-1)^2}.$$

Since

$$\begin{aligned} \int_0^1 K(t) [f''(t) - \gamma] dt &\leq \max_{t \in [0,1]} |K(t)| \int_0^1 (f''(t) - \gamma) dt \\ &= \frac{h^2}{8} [f'(1) - f'(0) - \gamma] = \frac{h^2}{8} (T - \gamma) \\ &= \frac{1}{8(m-1)^2} (T - \gamma), \end{aligned}$$

we obtain

$$\mathcal{R}_m[f] \leq \frac{1}{24(m-1)^2} (3T - 5\gamma).$$

On the other hand we have

$$\begin{aligned} \int_0^1 K(t) [\Gamma - f''(t)] dt &= -\frac{\Gamma}{12(m-1)^2} - \mathcal{R}_m[f], \\ \int_0^1 K(t) [\Gamma - f''(t)] dt &\leq \max_{t \in [0,1]} |K(t)| \int_0^1 (\Gamma - f''(t)) dt \\ &= \frac{h^2}{8} (\Gamma - f'(1) + f'(0)) = \frac{1}{8(m-1)^2} (\Gamma - T). \end{aligned}$$

From above relations we obtain

$$\mathcal{R}_m[f] \geq \frac{1}{24(m-1)^2} (3T - 5\Gamma).$$

□

3 The improvement of degree of exactness

In this section we want to improve the degree of exactness of quadrature formulae (7).

For simplicity of calculations we choose $a = 0$, $b = 1$. If denote $h = \frac{1}{m-1}$, we have $x_i = (i-2)h$, $i = \overline{3, m}$.

Now, we study the quadrature formula obtained for $m = 3$. In this case we have the following quadrature formula

$$\begin{aligned} \int_0^1 f(x) dx &= \frac{1}{4(2\alpha-1)} \left\{ (2\alpha-1)f(0) \right. \\ &+ 2 \left[\alpha f\left(\frac{\alpha}{2}\right) + (\alpha-1)f\left(\frac{1-\alpha}{2}\right) \right] + \alpha f\left(\frac{\alpha+1}{2}\right) \\ &\left. + (\alpha-1)f\left(\frac{2-\alpha}{2}\right) \right\} + \mathcal{R}_3[f]. \quad (18) \end{aligned}$$

This quadrature formula has degree of exactness equal 1. If we claim that $\mathcal{R}_3[e_2] = 0$, where $e_2(x) = x^2$, we obtain

$$18\alpha^3 - 27\alpha^2 + 13\alpha - 2 = 0, \text{ namely } \alpha \in \left\{ \frac{1}{3}, \frac{2}{3} \right\}.$$

We consider $\alpha \in \left\{ \frac{1}{3}, \frac{2}{3} \right\}$ and we observe that

$$\mathcal{R}_3[e_3] = -\frac{18\alpha^3 - 27\alpha^2 + 13\alpha - 2}{32(2\alpha - 1)} = 0,$$

where $e_3(x) = x^3$. Therefore, for $\alpha \in \left\{ \frac{1}{3}, \frac{2}{3} \right\}$, the quadrature formula (23) has degree of exactness equal 3 and can be written

$$\begin{aligned} \int_0^1 f(x)dx &= \frac{1}{4} \left\{ f(0) - 2f\left(\frac{1}{6}\right) + 4f\left(\frac{1}{3}\right) \right. \\ &\quad \left. - f\left(\frac{2}{3}\right) + 2f\left(\frac{5}{6}\right) \right\} + \mathcal{R}_3[f]. \end{aligned} \quad (19)$$

For $m \geq 4$ the quadrature formula (7) can be written

$$\begin{aligned} \int_0^1 f(x)dx &= \frac{h}{2\alpha - 1} \left\{ \frac{2\alpha - 1}{2} f(0) \right. \\ &\quad + \sum_{j=2}^{m-1} [\alpha f((j+\alpha-2)h) - (1-\alpha)f((j-1-\alpha)h)] \\ &\quad + \frac{1}{2} [\alpha f((m+\alpha-2)h) - (1-\alpha)f((m-1-\alpha)h)] \\ &\quad \left. + \mathcal{R}_m[f]. \right\} \end{aligned} \quad (20)$$

The quadrature formula (20) has degree of exactness equal 1 and we want to obtain the values of parameter α such that to improve degree of exactness of this quadrature formula.

We claim that $\mathcal{R}_m(e_2) = 0$, where $e_2(x) = x^2$ and we obtain

$$\begin{aligned} \int_0^1 x^2 dx &= \frac{h^3}{2\alpha - 1} \left\{ \sum_{j=2}^{m-1} [\alpha(j+\alpha-2)^2 \right. \\ &\quad - (1-\alpha)(j-1-\alpha)^2] + \frac{1}{2} [\alpha(m+\alpha-2)^2 \\ &\quad \left. - (1-\alpha)(m-1-\alpha)^2] \right\}, \end{aligned}$$

namely

$$\begin{aligned} h^3 \left\{ \sum_{j=2}^{m-1} [(j-1)^2 + \alpha(\alpha-1)] \right. \\ \left. + \frac{1}{2} [(m-1)^2 + \alpha(\alpha-1)] \right\} = \frac{1}{3}. \end{aligned} \quad (21)$$

From relation (21) we obtain

$$3(2m-3)\alpha^2 - 3(2m-3)\alpha + m - 1 = 0, \quad (22)$$

and degree of exactness of quadrature formula (20) is equal 2 for

$$\alpha \in \left\{ \frac{1}{2} \left(1 + \sqrt{\frac{2m-5}{6m-9}} \right), \frac{1}{2} \left(1 - \sqrt{\frac{2m-5}{6m-9}} \right) \right\}.$$

We observe that $\alpha \in [0, 1] \setminus \left\{ \frac{1}{2} \right\}$ for $m \geq 1$.

Now, we want to give the values of m such that degree of exactness of quadrature formula (20) is equal 3. From condition $\mathcal{R}_m(e_3) = 0$, where $e_3(x) = x^3$ we have

$$\begin{aligned} \frac{1}{(m-1)^4} \left\{ \sum_{j=1}^{m-2} [j^3 + 3\alpha(\alpha-1)j - \alpha(\alpha-1)] \right. \\ \left. + \frac{1}{2} [(m-1)^3 + 3\alpha(\alpha-1)(m-1) - \alpha(\alpha-1)] \right\} = \frac{1}{4} \end{aligned}$$

and using relation (22) we obtain $m = 3$. Therefore, there is not value of $m \in \mathbb{N}$, $m \geq 4$, such that the quadrature formula (20) to have degree of exactness equal 3.

4 A quadrature formula with degree of exactness equal to 3

Let $S_2(x_4, \dots, x_m)$ be the space of spline functions of degree 2 with knots at x_4, \dots, x_m . Let f be a function defined on $[x_3, x_{m+1}]$. The spline quasi-interpolant operator is

$$P_2 f = \sum_{j=1}^m c_j N_{j,2}.$$

To determine B -spline coefficients, c_j , by choosing the local interval $I = [x_{j+1}, x_{j+2}]$ and the local approximation method

$$P_2^I f(x) = \sum_{i=j-1}^{j+1} b_i N_{i,2}(x).$$

We consider that local approximation method is the polynomial interpolation knots

$$x_j^{(1)} = (1-\alpha)x_{j+1} + \alpha x_{j+2},$$

$$x_j^{(2)} = \frac{x_{j+1} + x_{j+2}}{2},$$

$$x_j^{(3)} = \alpha x_{j+1} + (1-\alpha)x_{j+2},$$

where $\alpha \in [0, 1] \setminus \left\{ \frac{1}{2} \right\}$. We have

$$\begin{cases} P_2^I f(x_j^{(1)}) = f(x_j^{(1)}), \\ P_2^I f(x_j^{(2)}) = f(x_j^{(2)}), \\ P_2^I f(x_j^{(3)}) = f(x_j^{(3)}), \end{cases}$$

namely

$$\left\{ \begin{array}{l} b_{j-1}N_{j-1,2}(x_j^{(1)}) + b_jN_{j,2}(x_j^{(1)}) + \\ b_{j+1}N_{j+1,2}(x_j^{(1)}) = f(x_j^{(1)}), \\ b_{j-1}N_{j-1,2}(x_j^{(2)}) + b_jN_{j,2}(x_j^{(2)}) + \\ b_{j+1}N_{j+1,2}(x_j^{(2)}) = f(x_j^{(2)}), \\ b_{j-1}N_{j-1,2}(x_j^{(3)}) + b_jN_{j,2}(x_j^{(3)}) + \\ b_{j+1}N_{j+1,2}(x_j^{(3)}) = f(x_j^{(3)}). \end{array} \right. \quad (23)$$

Using relations (23) and (2) we obtain

$$\begin{aligned} & (1-\alpha)^2 \frac{x_{j+2}-x_{j+1}}{x_{j+2}-x_j} b_{j-1} + \left[1 + \alpha^2 \frac{x_{j+1}-x_{j+2}}{x_{j+3}-x_{j+1}} + \right. \\ & \left. (1-\alpha)^2 \frac{x_{j+1}-x_{j+2}}{x_{j+2}-x_j} \right] b_j + \alpha^2 \frac{x_{j+2}-x_{j+1}}{x_{j+3}-x_{j+1}} b_{j+1} = f(x_j^{(1)}), \\ & \frac{x_{j+2}-x_{j+1}}{4(x_{j+2}-x_j)} b_{j-1} + \left[\frac{1}{2} + \frac{1}{4} \left(\frac{x_{j+3}-x_{j+2}}{x_{j+3}-x_{j+1}} + \frac{x_{j+1}-x_j}{x_{j+2}-x_j} \right) \right] b_j \\ & + \frac{x_{j+2}-x_{j+1}}{4(x_{j+3}-x_{j+1})} b_{j+1} = f(x_j^{(2)}), \\ & \alpha^2 \frac{x_{j+2}-x_{j+1}}{x_{j+2}-x_j} b_{j-1} + \left[1 + (1-\alpha)^2 \frac{x_{j+1}-x_{j+2}}{x_{j+3}-x_{j+1}} - \right. \\ & \left. \alpha^2 \frac{x_{j+2}-x_{j+1}}{x_{j+2}-x_j} \right] b_j + (1-\alpha)^2 \frac{x_{j+2}-x_{j+1}}{x_{j+3}-x_{j+1}} b_{j+1} = f(x_j^{(3)}). \end{aligned}$$

From the above relations we obtain

$$\begin{aligned} b_j &= -\frac{1}{2(2\alpha-1)^2} \left[f(x_j^{(1)}) \right. \\ &\quad \left. - 4(\alpha^2 + (1-\alpha)^2) f(x_j^{(2)}) + f(x_j^{(3)}) \right], \end{aligned}$$

for $1 < j < m$.

The expression for b_j is valid whenever $x_{j+1} < x_{j+2}$ which is not the case for $j = 1$ and $j = m$, since $x_1 = x_2 = x_3$ and $x_{m+1} = x_{m+2} = x_{m+3}$. The first value of j for which the general procedure works is $j = 2$. To obtain the value of b_1 by solving the above system with $j = 2$. We have

$$\left\{ \begin{array}{l} (1-\alpha)^2 b_1 + \left[1 - \alpha^2 \frac{x_4-x_1}{x_5-x_1} - (1-\alpha)^2 \right] b_2 + \\ \alpha^2 \frac{x_4-x_1}{x_5-x_1} b_3 = f(x_2^{(1)}), \\ \frac{1}{4} b_1 + \left[\frac{1}{2} + \frac{1}{4} \cdot \frac{x_5-x_4}{x_5-x_1} \right] b_2 + \frac{1}{4} \cdot \frac{x_4-x_1}{x_5-x_1} b_3 = f(x_2^{(2)}), \\ \alpha^2 b_1 + \left[1 - (1-\alpha)^2 \frac{x_4-x_1}{x_5-x_1} - \alpha^2 \right] b_2 + \\ +(1-\alpha)^2 \frac{x_4-x_1}{x_5-x_1} b_3 = f(x_2^{(3)}), \end{array} \right.$$

namely

$$\begin{aligned} b_1 &= \frac{1}{(1-2\alpha)^2} \left\{ (1-\alpha) f((1-\alpha)x_1 + \alpha x_4) \right. \\ &\quad \left. - 4\alpha(1-\alpha) f\left(\frac{x_1+x_4}{2}\right) + \alpha f(\alpha x_1 + (1-\alpha)x_4) \right\}. \end{aligned}$$

This procedure can obviously be used to determine value of b_m . For $j = m - 1$ we have

$$\left\{ \begin{array}{l} \alpha^2 b_m + \left[1 - \alpha^2 - (1-\alpha)^2 \frac{x_{m+1}-x_m}{x_{m+1}-x_{m-1}} \right] b_{m-1} + \\ (1-\alpha)^2 \frac{x_{m+1}-x_m}{x_{m+1}-x_{m-1}} b_{m-2} = f(x_{m-1}^{(1)}), \\ \frac{1}{4} b_m + \left[\frac{1}{2} + \frac{1}{4} \cdot \frac{x_m-x_{m-1}}{x_{m+1}-x_{m-1}} \right] b_{m-1} + \\ \frac{1}{4} \cdot \frac{x_{m+1}-x_m}{x_{m+1}-x_{m-1}} b_{m-2} = f(x_{m-1}^{(2)}), \\ (1-\alpha)^2 b_m + \left[1 - (1-\alpha)^2 - \alpha^2 \frac{x_{m+1}-x_m}{x_{m+1}-x_{m-1}} \right] b_{m-1} + \\ \alpha^2 \frac{x_{m+1}-x_m}{x_{m+1}-x_{m-1}} b_{m-2} = f(x_{m-1}^{(3)}), \end{array} \right.$$

and we obtain

$$\begin{aligned} b_m &= \frac{1}{(1-2\alpha)^2} \left\{ \alpha f((1-\alpha)x_m + \alpha x_{m+1}) \right. \\ &\quad \left. - 4\alpha(1-\alpha) f\left(\frac{x_m+x_{m+1}}{2}\right) \right. \\ &\quad \left. + (1-\alpha) f(\alpha x_m + (1-\alpha)x_{m+1}) \right\}. \end{aligned}$$

For $\alpha = 0$ or $\alpha = 1$ we obtain the following spline quasi-interpolant operator

$$P_2 f = \sum_{j=1}^m c_j N_{j,2},$$

where

$$c_j = \begin{cases} f(x_1), & \text{for } j = 1, \\ \frac{1}{2} \left[-f(x_{j+1}) + 4f\left(\frac{x_{j+1}+x_{j+2}}{2}\right) \right. \\ \left. - f(x_{j+2}) \right], & \text{for } 1 < j < m, \\ f(x_{m+1}), & \text{for } j = m. \end{cases}$$

Let $x_1 = x_2 = x_3 = a$, $x_{m+1} = x_{m+2} = x_{m+3} = b$ and $(x_i)_{i=3}^m$ are the equidistant nodes from the interval $[a, b]$. If we integrate the approximation formula of function f

$$f(x) = \sum_{j=1}^m c_j N_{j,2}(x) + r_m[f]$$

to obtain following quadrature formula

$$\int_a^b f(x)dx = \sum_{i=1}^m \frac{x_{i+3} - x_i}{3} \left[-\frac{1}{2}f(x_{i+1}) + 2f\left(\frac{x_{i+1}+x_{i+2}}{2}\right) - \frac{1}{2}f(x_{i+2}) \right] + \mathcal{R}_m[f]. \quad (24)$$

The quadrature formula (24) was studied in paper [1].

For $m = 3$ we have the Simpson's quadrature formula. We shall choose the equidistant nodes $(x_i)_{i=4}^m$ from the interval $[a, b]$.

For $m = 4$ we obtain the following quadrature formula

$$\begin{aligned} \int_a^b f(x)dx &= \frac{b-a}{3} \left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) \right. \\ &\quad \left. + 2f\left(\frac{a+3b}{4}\right) \right] + \mathcal{R}_4[f], \end{aligned}$$

and for $m = 5$ we have

$$\begin{aligned} \int_a^b f(x)dx &= \frac{b-a}{3} \left[\frac{4}{3}f\left(\frac{5a+b}{6}\right) - \frac{5}{6}f\left(\frac{2a+b}{3}\right) + \right. \\ &\quad \left. 2f\left(\frac{a+b}{2}\right) - \frac{5}{6}f\left(\frac{a+2b}{3}\right) + \frac{4}{3}f\left(\frac{a+5b}{6}\right) \right] + \mathcal{R}_5[f]. \end{aligned}$$

Next to study the quadrature formula for $m \geq 6$. For simplicity of calculations we choose $a = 0, b = 1$. If denote $h = \frac{1}{m-2}$, we have $x_i = (i-3)h$, $i = \overline{4, m}$ and the quadrature formula (24) can be written

$$\int_0^1 f(x)dx = h \left\{ \frac{4}{3}f\left(\frac{h}{2}\right) - \frac{5}{6}f(h) + 2 \sum_{k=2}^{m-3} f\left(\frac{2k-1}{2}h\right) \right.$$

$$\left. - \sum_{k=2}^{m-4} f(kh) - \frac{5}{6}f((m-3)h) + \frac{4}{3}f\left(\frac{2m-5}{2}h\right) \right\} + \mathcal{R}_m[f]. \quad (25)$$

The exactness degree of quadrature formula (25) is equal to 3 and from Theorem 1 the remainder term has the form

$$\begin{aligned} \mathcal{R}_m[f] &= \frac{1}{6} \int_0^1 K(t)f^{(4)}(t)dt, \text{ where } f \in H^4[0, 1] \\ K(t) &= \mathcal{R}_m[(\cdot - t)_+^3] = \\ &\quad \frac{(1-t)^4}{4} - h \left\{ \frac{4}{3} \left(\frac{h}{2} - t \right)_+^3 - \frac{5}{6}(h-t)_+^3 \right. \\ &\quad \left. + 2 \sum_{k=2}^{m-3} \left(\frac{2k-1}{2}h - t \right)_+^3 - \sum_{k=2}^{m-4} (kh - t)_+^3 \right. \\ &\quad \left. - \frac{5}{6} ((m-3)h - t)_+^3 + \frac{4}{3} \left(\frac{2m-5}{2}h - t \right)_+^3 \right\}. \end{aligned}$$

In paper [1] we obtain the following properties of function K :

Lemma 8 [1] The function K verifies

$$K(t) = K(1-t) \text{ any } t \in [0, 1] \quad (26)$$

$$K(t) \geq 0 \text{ any } t \in [0, 1] \quad (27)$$

$$\max_{t \in [0, 1]} K(t) = \frac{h^4}{12} \quad (28)$$

$$\int_0^1 K(t)dt = \frac{1}{480} \cdot \frac{29m-88}{(m-2)^5}. \quad (29)$$

In paper [1] we give some estimates of the quadrature error (24) for smooth functions by using the associated Peano kernel.

Theorem 9 [1] If $f \in H^4[0, 1]$ and there exist real numbers γ, Γ such that $\gamma \leq f^{(4)}(t) \leq \Gamma$, $t \in [0, 1]$, then

$$|\mathcal{R}_m[f]| \leq \frac{29m-88}{2880(m-2)^5} \left\{ \frac{\Gamma - \gamma}{2} + \left| \frac{\Gamma + \gamma}{2} \right| \right\}. \quad (30)$$

Theorem 10 [1] Let $f \in H^4[0, 1]$. If there exist a real number γ such that $\gamma \leq f^{(4)}(t)$, $t \in [0, 1]$, then

$$|\mathcal{R}_m[f]| \leq \frac{1}{72(m-2)^4} \cdot \left[T - \gamma + \frac{29m-88}{40(m-2)} |\gamma| \right] \quad (31)$$

where $T = f^{(3)}(1) - f^{(3)}(0)$.

If there exist a real number Γ such that $f^{(4)}(t) \leq \Gamma$, $t \in [0, 1]$, then

$$|\mathcal{R}_m[f]| \leq \frac{1}{72(m-2)^4} \cdot \left[\Gamma - T + \frac{29m-88}{40(m-2)} |\Gamma| \right]. \quad (32)$$

In this paper we give the following double inequality.

Theorem 11 If $f \in H^4[0, 1]$ and there exist real numbers γ, Γ such that $\gamma \leq f^{(4)}(t) \leq \Gamma$, $t \in [0, 1]$, then we have

$$\begin{aligned} & \frac{1}{2880(m-2)^5} \{40T(m-2)-\Gamma(11m+8)\} \leq \mathcal{R}_m[f] \\ & \leq \frac{1}{2880(m-2)^5} \{40T(m-2)-\gamma(11m+8)\} \end{aligned}$$

where $T = f^{(3)}(1) - f^{(3)}(0)$.

Proof: We have

$$\begin{aligned} & \frac{1}{6} \int_0^1 K(t) [f^{(4)}(t) - \gamma] dt = \\ & \mathcal{R}_m[f] - \frac{\gamma}{2880} \cdot \frac{29m-88}{(m-2)^5}. \end{aligned}$$

Since

$$\begin{aligned} & \int_0^1 K(t) [f^{(4)}(t) - \gamma] dt \\ & \leq \max_{t \in [0,1]} |K(t)| \int_0^1 [f^{(4)}(t) - \gamma] dt \\ & = \frac{h^4}{12} [f^{(3)}(1) - f^{(3)}(0) - \gamma] \\ & = \frac{h^4}{12} (T - \gamma) = \frac{1}{12(m-2)^4} (T - \gamma), \end{aligned}$$

we obtain

$$\mathcal{R}_m[f] \leq \frac{1}{2880(m-2)^5} \{40T(m-2)-\gamma(11m+8)\}.$$

On the other hand we have

$$\begin{aligned} & \frac{1}{6} \int_0^1 K(t) [\Gamma - f^{(4)}(t)] dt \\ & = \frac{\Gamma}{2880} \cdot \frac{29m-88}{(m-2)^5} - \mathcal{R}_m[f], \\ & \int_0^1 K(t) [\Gamma - f^{(4)}(t)] dt \\ & \leq \max_{t \in [0,1]} |K(t)| \int_0^1 (\Gamma - f^{(4)}(t)) dt \\ & = \frac{h^4}{12} (\Gamma - f^{(3)}(1) + f^{(3)}(0)) = \frac{1}{12(m-2)^4} (\Gamma - T). \end{aligned}$$

From above relations we obtain

$$\mathcal{R}_m[f] \geq \frac{1}{2880(m-2)^5} \{40T(m-2)-\Gamma(11m+8)\}.$$

□

5 An intermediate point property in the quadrature formulas

In this section we study a property of the intermediate point for the quadrature formulae of type (9) and (25).

Lemma 12 [9] If $-\infty < \alpha < \beta < +\infty$ and w is a weight on (α, β) and

$$\int_\alpha^\beta f(t)w(t)dt = \sum_{k=1}^n c_k f(z_k) + r_n[f], \quad f \in L_w^1(\alpha, \beta)$$

then

$$\begin{aligned} W(x) &= w \left(\alpha + (\beta - \alpha) \frac{x - a}{b - a} \right), \\ x \in (a, b), \quad -\infty &< a < b < +\infty \end{aligned}$$

is a weight on (a, b) and

$$\begin{aligned} \int_a^b F(x)W(x)dx &= \frac{b-a}{\beta-\alpha} \sum_{k=1}^n c_k F \left(\left(a + (b-a) \frac{z_k - \alpha}{\beta - \alpha} \right) \right. \\ &\quad \left. + \mathcal{R}_n[F] \right) \end{aligned}$$

where $F \in L_w^1(a, b)$ and

$$\mathcal{R}_n[F] = \frac{b-a}{\beta-\alpha} r_n[\tilde{F}], \quad \tilde{F}(t) = F \left(a + (b-a) \frac{t-\alpha}{\beta-\alpha} \right).$$

Let $f : [a, b] \rightarrow \mathbf{R}$, $f \in C^2[a, b]$. By using Lemma 5 and Lemma 12, the quadrature formula (9) can be written

$$\begin{aligned} \int_a^x f(t)dt &= \frac{x-a}{2(m-1)} \left\{ f(a) + 2 \sum_{j=1}^{m-2} f \left(a + j \frac{x-a}{m-1} \right) \right. \\ &\quad \left. + f(x) \right\} - \frac{(x-a)^3}{12(m-1)^2} f''(c_x), \quad c_x \in (a, x), \quad x \in (a, b]. \end{aligned} \quad (33)$$

Theorem 13 If $f \in C^4[a, b]$ and $f'''(a) \neq 0$, then for the intermediate point c_x that appears in formula (33), it follows

$$\lim_{x \rightarrow a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

Proof: Let $F, G : [a, b] \rightarrow \mathbf{R}$ defined as follows

$$\begin{aligned} F(x) &= \int_a^x f(t)dt - \frac{x-a}{2(m-1)}f(a) - \frac{x-a}{m-1} \\ &\quad \cdot \sum_{j=1}^{m-2} f\left(a+j\frac{x-a}{m-1}\right) - \frac{x-a}{2(m-1)}f(x) \\ &\quad + \frac{(x-a)^3}{12(m-1)^2}f''(a), \end{aligned}$$

$$G(x) = (x-a)^4.$$

We observe that $F(a) = 0$ and for $i = \overline{1,4}$ we have

$$\begin{aligned} F^{(i)}(x) &= f^{(i-1)}(x) - \left(\frac{x-a}{2(m-1)}\right)^{(i)} f(a) - \\ &\quad \sum_{k=0}^i \binom{i}{k} \left(\frac{x-a}{m-1}\right)^{(k)} \left(\sum_{j=1}^{m-2} f\left(a+j\frac{x-a}{m-1}\right)\right)^{(i-k)} - \\ &\quad \sum_{k=0}^i \binom{i}{k} \left(\frac{x-a}{2(m-1)}\right)^{(k)} f^{(i-k)}(x) + \left(\frac{(x-a)^3}{12(m-1)^2}\right)^{(i)} f''(a). \end{aligned} \quad (34)$$

From relation (34) we obtain

$$\begin{aligned} F^{(i)}(a) &= 0, \text{ for } i = \overline{1,3}, \\ F^{(4)}(a) &= f'''(a) \left\{ 1 - \frac{4}{(m-1)^4} \sum_{j=1}^{m-2} j^3 - \frac{2}{m-1} \right\} \\ &= -\frac{f'''(a)}{(m-1)^2}. \end{aligned}$$

By using successive l' Hospital we obtain:

$$\lim_{x \rightarrow a} \frac{F(x)}{G(x)} = \lim_{x \rightarrow a} \frac{F^{(4)}(x)}{G^{(4)}(x)} = -\frac{f'''(a)}{4!(m-1)^2}, \quad (35)$$

$$\begin{aligned} \lim_{x \rightarrow a} \frac{F(x)}{G(x)} &= \lim_{x \rightarrow a} -\frac{(x-a)^3}{12(m-1)^2} \cdot \frac{f''(c_x) - f''(a)}{(x-a)^4} \\ &= \lim_{x \rightarrow a} -\frac{1}{12(m-1)^2} \cdot \frac{f''(c_x) - f''(a)}{c_x - a} \cdot \frac{c_x - a}{x - a} \\ &= -\frac{1}{12(m-1)^2} f'''(a) \cdot \lim_{x \rightarrow a} \frac{c_x - a}{x - a}. \end{aligned} \quad (36)$$

From (35) and (36) we obtain

$$\lim_{x \rightarrow a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

□

Let $f : [a, b] \rightarrow \mathbf{R}$, $f \in C^4[a, b]$. By using Lemma 8 and Lemma 12, the quadrature formula (25) can be written

$$\begin{aligned} \int_a^x f(t)dt &= \frac{x-a}{m-2} \left\{ \frac{4}{3} f\left(a + \frac{1}{2} \cdot \frac{x-a}{m-2}\right) - \frac{5}{6} f\left(a + \frac{x-a}{m-2}\right) \right. \\ &\quad \left. + 2 \sum_{k=2}^{m-3} f\left(a + \frac{2k-1}{2} \cdot \frac{x-a}{m-2}\right) - \sum_{k=2}^{m-4} f\left(a + k \frac{x-a}{m-2}\right) \right. \\ &\quad \left. - \frac{5}{6} f\left(a + (m-3) \frac{x-a}{m-2}\right) + \frac{4}{3} f\left(a + \frac{2m-5}{2} \cdot \frac{x-a}{m-2}\right) \right\} \\ &\quad + \frac{(x-a)^5}{(m-2)^5} \cdot \frac{29m-88}{2880} f^{(4)}(c_x), c_x \in (a, x), x \in (a, b]. \end{aligned} \quad (37)$$

Theorem 14 If $f \in C^6[a, b]$ and $f^{(5)}(a) \neq 0$, then for the intermediate point c_x that appears in formula (37), it follows

$$\lim_{x \rightarrow a} \frac{c_x - a}{x - a} = \frac{1}{2}$$

Proof: Let $F, G : [a, b] \rightarrow \mathbf{R}$ defined as follows

$$\begin{aligned} F(x) &= \int_a^x f(t)dt - \frac{4}{3} \frac{x-a}{m-2} f\left(a + \frac{1}{2} \cdot \frac{x-a}{m-2}\right) \\ &\quad + \frac{5}{6} \frac{x-a}{m-2} f\left(a + \frac{x-a}{m-2}\right) \\ &\quad - 2 \frac{x-a}{m-2} \sum_{k=2}^{m-3} f\left(a + \frac{2k-1}{2} \cdot \frac{x-a}{m-2}\right) \\ &\quad + \frac{x-a}{m-2} \sum_{k=2}^{m-4} f\left(a + k \frac{x-a}{m-2}\right) \\ &\quad + \frac{5}{6} \frac{x-a}{m-2} f\left(a + (m-3) \frac{x-a}{m-2}\right) \\ &\quad - \frac{4}{3} \frac{x-a}{m-2} f\left(a + \frac{2m-5}{2} \cdot \frac{x-a}{m-2}\right) \\ &\quad - \frac{(x-a)^5}{(m-2)^5} \cdot \frac{29m-88}{2880} f^{(4)}(a), \\ G(x) &= (x-a)^6. \end{aligned}$$

We have $F^{(i)}(a) = 0$, for $i = \overline{0,5}$ and

$$F^{(6)}(a) = \frac{f^{(5)}(a)}{(m-2)^5} \cdot \frac{29m-88}{8}.$$

By using successive l' Hospital we obtain:

$$\lim_{x \rightarrow a} \frac{F(x)}{G(x)} = \lim_{x \rightarrow a} \frac{F^{(6)}(x)}{G^{(6)}(x)} = \frac{29m-88}{5760} \frac{f^{(5)}(a)}{(m-2)^5}, \quad (38)$$

$$\begin{aligned} \lim_{x \rightarrow a} \frac{F(x)}{G(x)} &= \lim_{x \rightarrow a} \frac{(x-a)^5}{(m-2)^5} \frac{29m-88}{2880} \frac{f^{(4)}(c_x) - f^{(4)}(a)}{(x-a)^6} \\ &= \lim_{x \rightarrow a} \frac{29m-88}{2880(m-2)^5} \cdot \frac{f^{(4)}(c_x) - f^{(4)}(a)}{c_x - a} \cdot \frac{c_x - a}{x - a} \\ &= \frac{29m-88}{2880(m-2)^5} f^{(5)}(a) \cdot \lim_{x \rightarrow a} \frac{c_x - a}{x - a}. \quad (39) \end{aligned}$$

From (38) and (39) we obtain

$$\lim_{x \rightarrow a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

□

References:

- [1] A.M. Acu, Spline quasi-interpolants and quadrature formulas, *Acta Universitatis Apulensis*, No. 13/2007, pp. 21–36.
- [2] B.D. Bojanov, H.A. Hakopian and A.A. Sahakian, *Spline Functions and Multivariate Interpolations*, Kluwer Academic Publishers, 1993.
- [3] D. Barrera, M.J. Ibáñez, P. Sablonnière, Near-best discrete quasi-interpolants on uniform and nonuniform partitions, *Curve and Surface Fitting*, Saint-Malo 2002, A. Cohen, J.L. Merrien and L.L. Schumaker (eds), Nashboro Press, Brentwood, 2003, pp. 31–40.
- [4] D. Barrera, M.J. Ibáñez, P. Sablonnière, D. Sbibih, Near-minimally normed univariate spline quasi-interpolants on uniform partitions, *J. Comput. Appl. Math.* 181, 2005, pp. 211-233.
- [5] D. Barrera, M.J. Ibáñez, P. Sablonnière, D. Sbibih, Near-best quasi-interpolants associated with H-splines on a three-direction mesh, *J. Comput. Appl. Math.* 181, 2005, pp. 133–152.
- [6] N.M. Breaz, *Modele de regresie bazate pe functii spline*, Presa Universitară Clujeană, 2007.
- [7] H.B. Curry, I.J. Schoenberg, On Pólya frequency functions IV: The fundamental spline functions and their limits, *J. Analyse Math.*, 17, 1966, pp. 71–107.
- [8] B.G. Lee, T. Lyche, L.L. Schumaker, Some examples of quasi-interpolants constructed from local spline projectors, *Mathematical methods for curves and surfaces*: Oslo 2000, T. Lyche, L.L. Schumaker (eds), Vanderbilt University Press, Nashville 2001, pp. 243–252.
- [9] A. Lupaş, C. Manole, *Capitole de Analiza Numerică*, Universitatea din Sibiu, Colectia Facultatii de Stiinte-Seria Matematica 3, Sibiu 1994.
- [10] T. Lyche, K. Mørken, *Spline Methods Draft*, <http://heim.ifi.uio.no/~in329/kap8-10.pdf>.
- [11] M. Podisuk, S. Phummark, Newton-Cotes Formulas in Runge-Kutta Method, *WSEAS Transactions on Mathematics*, Issue 1, Volume 4, January 2005, pp. 18-23.
- [12] P. Sablonnière, On some multivariate quadratic spline quasi-interpolants on bounded domains, *In Modern developments in multivariate approximation*, W. Haussmann, K. Jetter, M. Reimer, J. Stockler (eds), ISNM Vol.145, Birkhauser-Verlag, Basel 2003, pp. 263-278.
- [13] P. Sablonnière, Quadratic spline quasi-interpolants on bounded domains of \mathbb{R}^d , $d = 1, 2, 3$, *Spline and radial functions*, Rend. Sem. Univ. Pol. Torino, Vol. 61, 2003, pp. 61–78.
- [14] P. Sablonnière, Recent progress on univariate and multivariate polynomial or spline quasi-interpolants, *In Trends and applications in constructive approximation*, M.G. de Bruijn, D.H. Mache and J. Szabados (eds), ISNM Vol. 151 BV, 2005, pp. 229–245.
- [15] P. Sablonnière, Recent results on near-best spline quasi-interpolants, *Fifth International Meeting on Approximation Theory of the University of Jaen* (Ubeda, June 9-14, 2004), Prepublication IRMAR 04-50, Universite de Rennes, October 2004.
- [16] P. Sablonnière, A quadrature formula associated with a univariate quadratic spline quasi-interpolant, *Prepublication IRMAR*, Rennes, April 2005.
- [17] D. Simian, On some Hermite Bivariate Interpolation Schemes, *WSEAS Transactions on Mathematics*, Issue 12, Volume 5, December 2006, pp. 1322-1329.
- [18] F. Sofonea, Approximation Properties of a Sequence of Linear and Positive Operators, *Proc. of 11th WSEAS International Conference on Computers*, Agios Nikolaos, Crete, Greece, 2007, pp.328-332.
- [19] F. Sofonea, On a Class of Linear and Positive Operators, *WSEAS Transactions on Mathematics*, Issue 12, Vol. 5, December 2006, pp. 1263-1267.