Optimal Allocation of Time in Radiation Detection Labs

AHMAD A. MOREB
HANI M. ABURAS
Department of Industrial Engineering,
King Abdul-Aziz University,
P.O. Box 80204, Jeddah 21589
SAUDI ARABIA
Email: dr.moreb@gmail.com ; haburas@kau.edu.sa

Abstract: - Nuclear radiation detection laboratories are confronted with huge number of products to be analyzed for dangerous radiation levels. Since the precision in determining the right level of radioactivity for a given product is directly proportional to the time allocated for that particular product, then the limited working hours for these laboratories become the scarce resource. In this paper, a nonlinear constrained model is developed to find the optimal allocation of time for each product in order to obtain the best possible estimate for the level of radiation for each given product. A validation procedure for this model was carried out using special cases.

Key-Words: - Nuclear Radiation, Minimization, Standard Deviation.

1 Mathematical Background

Nuclear radiation detection laboratories are overwhelmed with huge number of products to be analyzed for dangerous radiation levels constrained by limited working hours available at these laboratories. Such a situation was encountered during the Chernobyl accident when many nuclear laboratories were flooded by foodstuffs and other products to be analyzed [1-5]. The most recent work related to finding the optimal allocation of nuclear detector's time for a given number of products, under time constraints was done by Aljohani [1]. His methodology was based on minimizing the sum of associated standard deviations of the net counting rate of products. The assumption in Aljohani's [1] was based on having a constant radiation background for all products which may not be exactly true if products are measured by a spectrometer or if products are measured in different locations.

In nuclear radiation counting detection, only small fraction of nuclei are picked up and recorded by the detector. This is depicted as a sampling process done by the detector. The more samples picks up by the detector, the better will be the accuracy of the detector's readings. Therefore, the accuracy in measuring radioactivity of a product is directly proportional to the time allocated for measurement. The case of counting nuclear radiation events can best be modeled by a binomial distribution. Knoll [5] states that in the case of a trial that consists of observing a given radioactive nucleus for a period of time t_i, the number of trials is equal to the number of nuclei in the product under observation, and the measurement consists of counting those nuclei that undergo decay. The probability of success is identified as the proportion of nuclei that undergo decay, which is:

$$p = 1 - e^{-\lambda t} \qquad (1)$$

where λ is the decay constant of the radioactive product.

It is well known that the mean μ in a binomial distribution is equal to np and the variance σ^2 is equal to $n\,p\,(1-p)$, where n is the number of trials and p is the probability of success.

Binary processes with low probability of success for each individual trial can best be estimated by a Poisson distribution. In nuclear counting experiments large numbers of nuclei (in the order of Avogadro's number 10^{23}) make up the number of trials, whereas a relatively small fraction of these give rise to recorded counts. This small fraction of nuclei to be recorded by the detector is the primary focus in this study. Under these conditions, where $p \ll 1$ and x is the reading from a radiation counter for a time interval, the binomial distribution can be mathematically simplified by a Poisson distribution, as follows:

$$p(x) = \frac{(pn)^x e^{-pn}}{x!} \qquad (2)$$

Since the relationship $np = \bar{x}$ holds, then equation (2) becomes:

$$p(x) = \frac{(\bar{x})^x e^{-\bar{x}}}{x!} \qquad (3)$$

where $\bar{x}$ is the average of successive readings from a radiation counter for repeated time intervals of equal length. If $\bar{x}$ is large (traditionally > 20) then more simplification can be achieved as follows:

$$p(x) = \frac{1}{\sqrt{2\bar{x}\pi}} \exp\left[\frac{(x-\bar{x})^2}{2\bar{x}}\right] \qquad (4)$$

Under the aforementioned conditions this distribution is characterized by a single parameter $\bar{x}$, which is equal to np which is also equal to the predicted variance σ^2. From now on the main focus will be this single parameter σ^2 or $\sigma = \sqrt{\bar{x}}$.

2 Problem Formulation

The large number of radioactive products coming for measurement is constrained by time limits. This time is usually dictated by the supplier of these products and/or by the laboratory conditions. Since radioactivity usually differs from one product to another, the measurement time needed for each product has to be different. Thus, the challenge is to optimize the allocation of measurement time among these products. The longer the time allocated for measurement the higher the accuracy of results. However, with scarcity of time one cannot measure indefinitely.

In this paper, assuming that the number of products is (n) the mathematical manipulation and all related data is adopted from the authors' previous work. Since a smaller value for the standard deviation is used as a measure of accuracy, the problem translates into minimizing the standard deviations under time constraint. The objective is to minimize the sum of associated standard deviation of counting rates for (n) products. This objective function is nonlinear one, and thus, requiring a nonlinear programming algorithm to solve it. The solution found using the proposed nonlinear model was verified by analytically solving the system of equations under the assumption that the background radiation is negligible. Two examples are introduced; the results found in the examples for the verification stage are identical to the results found using the nonlinear model.

3 Definition of Variables

M_i = Counts due to both radioactive product i and background.

B_i = Counts due to background only while testing product i.

r_{M_i} = Counting rate due to both radioactive product i and background.

r_i = Counting rate due to the radioactive product i without background.

b_i = Counting rate due to background while testing product i.

t_i = Measurement time of radioactive product i with background.

t_{bi} = Measurement time for background only while testing product i.

T = Total time given to test all (n) products.

σ_i = Associated standard deviation of net counting rate for product i.

4 Theory

Consider the measurement of the net counting rate from a long-lived radioactive product in the presence of background. The net counting rate due to the radioactive product must be corrected by subtracting the background counting rate[2] as:

$$r_i = r_{M_i} - b_i \quad \text{, Or} \quad r_i = \frac{M_i}{t_i} - \frac{B_i}{t_{bi}} \qquad (5)$$

Applying error propagation formula[1] results in:

$$\sigma_i^{\,2} = \sigma_{M_i}^{\;2}\left(\frac{\partial r_i}{\partial M_i}\right)^2 + \sigma_{B_i}^{\;2}\left(\frac{\partial r_i}{\partial B_i}\right)^2 \qquad (6)$$

Or

$$\sigma_i = \left[\left(\frac{\sigma_{M_i}}{t_i}\right)^2 + \left(\frac{\sigma_{B_i}}{t_{bi}}\right)^2\right]^{\frac{1}{2}} \qquad (7)$$

From the mathematical background above, it is known that $\sigma_{M_i} = \sqrt{M_i}$ and $\sigma_{B_i} = \sqrt{B_i}$. Substituting into equation (7) yields:

$$\sigma_i = \left[\frac{M_i}{t_i^{\,2}} + \frac{B_i}{t_{bi}^{\;2}}\right]^{\frac{1}{2}} \qquad (8)$$

By definition, counting rates are $r_i + b_i = \frac{M_i}{t_i}$,

and $b_i = \frac{B_i}{t_{bi}}$, then equation (8) becomes :

$$\sigma_i = \left[\frac{r_i + b_i}{t_i} + \frac{b_i}{t_{bi}}\right]^{\frac{1}{2}} \qquad (9)$$

Assuming that (n) products are available for measurement and products are independent of each others, the total associated standard deviation of counting rates for all products is:

$$\sigma_T = \sigma_1 + \sigma_2 + \sigma_3 + \dots\dots\dots + \sigma_n \qquad (10)$$

Substituting (9) into (10) one gets:

$$\sigma_T = \left[\frac{r_1 + b_1}{t_1} + \frac{b_1}{t_{b1}}\right]^{\frac{1}{2}} + \left[\frac{r_2 + b_2}{t_2} + \frac{b_2}{t_{b2}}\right]^{\frac{1}{2}} + \dots\dots + \left[\frac{r_n + b_n}{t_n} + \frac{b_n}{t_{bn}}\right]^{\frac{1}{2}} \qquad (11)$$

Equation (11) represents the objective function that needs to be minimized with the following constraint:

$$T = t_1 + t_2 + t_3 + \dots\dots\dots + t_n \qquad (12)$$

5 Validation

The nonlinear objective function in (11) under the constraint in (12) is to be solved using any nonlinear programming package. Assuming that background radiation is negligible (i.e. $b_i = 0$), equation (11) may be simplified; and σ_T becomes:

$$\sigma_T = \left[\frac{r_1}{t_1}\right]^{\frac{1}{2}} + \left[\frac{r_2}{t_2}\right]^{\frac{1}{2}} + \dots\dots + \left[\frac{r_n}{t_n}\right]^{\frac{1}{2}} \qquad (13)$$

By differentiating σ_T with respect to $t_1, t_2, \dots\dots\dots, t_n$, setting all the derivatives to zero and rearranging terms, the following set of equations is obtained:

$$\frac{t_2}{t_1} = \left[\frac{r_2}{r_1}\right]^{\frac{1}{3}}, \frac{t_3}{t_1} = \left[\frac{r_3}{r_1}\right]^{\frac{1}{3}}, \dots, \frac{t_n}{t_1} = \left[\frac{r_n}{r_1}\right]^{\frac{1}{3}} \qquad (14)$$

This set of equations (14) ((n-1) equations & (n) unknowns), along with equation (12) can now be solved analytically. The results can then be compared with the results found using the proposed numerical nonlinear model.

Without loss of generality, equation (14) above can be rewritten as:

$$t_i = t_1\left[\frac{r_i}{r_1}\right]^{\frac{1}{3}} \quad (\textit{for } i = 2, 3, \dots, n) \qquad (15)$$

Substituting for t_i in the constraint equation (12) yields:

$$T = t_1 \left\{ 1 + \left[\frac{r_2}{r_1} \right]^{\frac{1}{3}} + \left[\frac{r_3}{r_1} \right]^{\frac{1}{3}} + \cdots + \left[\frac{r_n}{r_1} \right]^{\frac{1}{3}} \right\} \quad (16)$$

Since t_1 is the only unknown variable in equation (16) one can solve for t_1. The values for all t_i $(i = 2, 3, \cdots n)$ can then be found by substituting the value of t_1 in equations (15).

To illustrate the steps discussed above, two examples are presented. The first one will be solved using the proposed nonlinear model. The second example is solved using both, the proposed nonlinear model and the analytical method formulated specially for verification purposes.

6 Examples

The following examples are presented below along with their corresponding solutions:

6.1 Example 1

The total time allotted for testing the radioactivity of 8 products is T = 7224.9923 minutes. The counting rates in counts per minute due to both product and background and that due to background alone are given below:

$r_1 + b_1$	$r_2 + b_2$	$r_3 + b_3$	$r_4 + b_4$
611	1017	2022	1781

$r_5 + b_5$	$r_6 + b_6$	$r_7 + b_6$	$r_8 + b_8$
922	792	1415	921

And,

b_1	b_2	b_3	b_4
121	140	230	180

b_5	b_6	b_6	b_8
160	90	160	80

The above example was solved using the proposed nonlinear model presented in this paper. The minimum objective function is σ_T = 12.0944 and the optimal allocations for the 7224.9923 minutes allotted to the 8 products are:

t_1	t_2	t_3	t_4
532.36	642.101	814.1235	784.1783

t_5	t_6	t_7	t_8
614.696	595.68	722.9819	633.1941

And,

t_{b1}	t_{b2}	t_{b3}	t_{b4}
236.909	238.2416	274.5851	249.3172

t_{b5}	t_{b6}	t_{b7}	t_{b8}
256.0668	200.8176	243.1223	186.6179

6.2 Example 2

This example was solved using the proposed nonlinear model and validated using the analytical method. The background radiation is assumed negligible (i.e. b = 0).

The total time allotted for testing 6 products is T = 2117 minutes. Counting rates per minute are given below:

r_1	r_2	r_3	r_4	r_5	r_6
1015	921	102	201	333	621

Using the proposed nonlinear model, the minimum value for the objective function is σ_T = 6.730412; the optimal allocation of the 2117 minutes allotted to 6 products is:

t_1	t_2	t_3
464.8093	449.9932	216.0991

t_4	t_5	t_6
270.9265	320.5783	394.5935

The same example was solved using the analytical method, the minimum value for the objective function is σ_T = 6.7304; and the optimal allocations of the 2117 minutes allotted to 6 products are:

t_1	t_2	t_3
464.8093	449.9933	216.0991

t_4	t_5	t_6
270.9265	320.5784	394.5936

7 Discussion

It should be noted that the objective function σ_T in equation (11) along with the constraint equation (12) constitute a nonlinear model; both are functions of $t_1, t_2, t_3, ..., t_n, t_b$. Fortunately, the objective function σ_T is a monotonically decreasing function of $t_1, t_2, t_3, ..., t_n, t_b$ and the constraint equation (12) is purely linear. Therefore, the solution found is a global minimum. Comparison of results found from the nonlinear model with those from the analytical one show a high degree of compatibility between them. This degree of consistency between results establishes a proof of effectiveness of the proposed nonlinear model presented in this paper.

References:

[1] Aljohani, M. S. " Optimal Allocation of Nuclear Detector's Time for Radioactive Samples", *Journal of Nuclear And Related Technologies*, Volume 1, No,1, June (2004). pp. 56-62.

[2] Abdul-Fattah A., Abulfaraj W., Abdul-Majid S., (1988), Food Analysis and Radioactivity in Saudi Arabia, Final Report No.96-407,Faculty of Engineering, King AbdulAziz University, Jeddah , Saudi Arabia.

[3] Abdul-Majid S., Abulfaraj W., Aljohani M., Mamoon A., Abdul-fattah A. & Abubakar K. M. (1988) "Radiation Monitoring of Imported Food to Saudi Arabia after Chernobyl Accident". Radiation Protection Practice. 7th Int'l Congress of the Inte'l radiation Protection Association, Sydney, Australia, Vol. 2, pp.1090.

[4] Abulfaraj W. H., Abdul-Majid S. and Abdul-Fattah A. F.(1987) "Radiation Monitoring of Imported Food to Saudi Arabia after Chernobyl" *Transactions of the American Nuclear Society*, Volume 54, No. 39.

[5] Glen F. Knoll, "Radiation Detection and Measurement" John Wiley & Sons Inc. 1979

[6] Mamoon A., Abdul-Fattah A., Abulfaraj W., Abdul-Majid S., Al-Johani M., & Abubakar K. (1988),"Monitoring Radioactivity in Imported Foodstuffs Experience Gained & Recommenda-tions". *Radiation Protection Practice*. Vol. 2, 7th Int'l Congress of the Inte'l radiation Protection Association, Sydney, Australia, p.1094.

[7] Price, William J. , *Nuclear Radiation Detection*, McGraw-Hill Book Company,1994.

[8] Ballerini L. & Franzén, L. (2003), "Fractal Analysis of Microscopic Images of Breast Tissue", *WSEAS Transactions on Circuits*, Vol. 2, No. 1 pp. 270-275,.

[9] Filippopoulos G., Karabetsos E. (2005), "A Review of the Adverse Health Effects of ELF and RF Electromagnetic Fields Needed to Know in order to Assess Compliance with Safety Limits", *WSEAS Transactions on Circuits & Systems*, Vol.4, No.7, pp.719-727.