been investigated by Karimi and Yazdanpanah [13]. Recently, a dynamical model for two-time scale systems is presented such that a portion of the dynamics may be treated as a norm-bounded dynamic uncertainty. Clearly, it means that the proposed approach deals with only those two-time scale systems where the fast subsystem is norm-bounded. Although, this might be considered as a restriction on systems under consideration, it covers many control systems, for instance mechanical systems having two types, i.e., slow and fast, behaviors. In this view, the synthesis is performed only for certain dynamics of the system.

This paper presents novel results on control synthesis for stabilization and disturbance attenuation of a class of time-delayed singularly perturbed systems with norm-bounded nonlinear uncertainties. The system under consideration consists of systems in state-space form with linear nominal parts, norm-bounded nonlinear uncertainties and time delays. Robust stabilization and disturbance attenuation of such systems is investigated using the Hamiltonian approach. The state feedback gain matrices can be constructed from the positive definite solutions to a certain Riccati inequality. Another advantage to this approach is that we can preserve the characteristic of the composite controller, i.e., the whole-dimensional process can be separated into two subsystems ([1]-[4]). Moreover, the presented stabilization design insures the stability for all $\varepsilon \in (0, \infty)$ and independently of the time delay.

Notation: The notations used throughout the paper are fairly standard. I and 0 represent identity matrix and zero matrix; the superscript $'T'$ stands for matrix transposition. $\|.\|$ refers to the Euclidean vector norm or the induced matrix 2-norm. The notation $P > 0$ means that P is real symmetric and positive definite.

2. PROBLEM FORMULATION

Consider a linear time-invariant state-delayed singularly perturbed system with norm-bounded nonlinear uncertainties in the form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \varepsilon \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} x_1(t-h) + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} x_2(t-h) + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} w(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t) + \begin{bmatrix} \Delta_1(x_1(t)) \\ \Delta_2(x_1(t)) \end{bmatrix} \quad (1)$$

$$x_1(t) = \varphi(t) \quad t \in [-h, 0] \quad (2)$$

$$z(t) = c_1 x_1(t) + c_2 x_2(t) + Du(t) \quad (3)$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, $n(= n_1 + n_2)$ is the order of the whole system, $u \in R^m$, $w \in R^k$, $z \in R^l$ are control vector, disturbance and controlled output, respectively, $\Delta_i(x_1) \quad (i = 1, 2)$ are nonlinear terms of the uncertainty space. The certain matrices $a_{11} \in R^{n_1 \times n_1}$, $a_{12} \in R^{n_1 \times n_2}$, $a_{21} \in R^{n_2 \times n_1}$, $a_{22} \in R^{n_2 \times n_2}$, $b_1 \in R^{n_1 \times m}$, $b_2 \in R^{n_2 \times m}$, $d_1 \in R^{n_1 \times k}$, $d_2 \in R^{n_2 \times k}$, $r_1 \in R^{n_1 \times n_1}$, $f_1 \in R^{n_1 \times n_2}$, $f_2 \in R^{n_2 \times n_2}$ and $r_2 \in R^{n_2 \times n_1}$ are constant and $\varepsilon \geq 0$ is scalar and real. For a vector ν, v^T is its transpose, and $\|v\|$ is its Euclidean norm and L^2 is the Lebesque space of square integrable functions.

Assumption 1. There exist the known real constant matrixes G_1, G_2 such that the known nonlinear uncertainties $\Delta_i(x_1(t)) \; (i = 1, 2)$ satisfy the following bounded condition,

$$\|\Delta_i(x_1(t))\| \leq \|G_i x_1(t)\| \quad \forall x_1(t) \in R^{n_1} \quad (4)$$

Denote the corresponding uncertainty set by

$$\Xi_i(x_1) = \{\Delta_i(x_1(t)) : \|\Delta_i(x_1(t))\| \leq \|G_i x_1(t)\|\} \quad (i = 1, 2) \quad (5)$$

Definition 1.

1) A state feedback

$$u = -k_1 x_1 - k_2 x_2,$$

$k_1 \in R^{m \times n_1}$, $k_2 \in R^{m \times n_2}$ is said to achieve robust global asymptotic stability if for $w = 0$ and any $\Delta_i(x_1) \in \Xi_i(x_1) \quad (i = 1, 2)$ the closed-loop system

$$\begin{bmatrix} \dot{x}_1 \\ \varepsilon \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} - b_1 k_1 & a_{12} - b_1 k_2 \\ a_{21} - b_2 k_1 & a_{22} - b_2 k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} x_1(t-h) + \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} x_1(t-h) + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} x_2(t-h) + \begin{bmatrix} \Delta_1(x_1) \\ \Delta_2(x_1) \end{bmatrix} \quad (6)$$

is globally asymptotically stable in the Lyapunov sense for all $\varepsilon \in (0,\infty)$ and independently of the time delay (h).

2) A state feedback

$$u = -k_1 x_1 - k_2 x_2$$

is said to achieve robust disturbance attenuation if under zero initial condition there exists $0 \le \gamma < \infty$ for which the performance bound is such that:

$$\| z(t) \| < \gamma \, \| w(t) \| \quad \forall w \in L^2, \Delta_i(x_1) \in \Xi_i(x_1) \quad \text{for } i = 1,2 \quad (7)$$

The main objective of the paper is to design $k_1 \in R^{m\times n_1}$, $k_2 \in R^{m\times n_2}$ such that the state feedback $u = -k_1 x_1 - k_2 x_2$ achieves simultaneously robust global asymptotic stability and robust disturbance attenuation for all $\varepsilon \in (0,\infty)$ and independently of the time delay (h). The main approach employed here is the standard HJI method. Hence, we define a quadratic energy function in the form:

$$E(x_1, x_2) = x_1^T P_1 x_1 + \varepsilon\, x_2^T P_2 x_2 + \int_{t-h}^{t} x_1^T(\sigma) Q x_1(\sigma)\, d\sigma + \int_{t-h}^{t} x_2^T(\sigma) Z x_2(\sigma)\, d\sigma \quad (8)$$

where $P_1 > 0, P_2 > 0, Q > 0$ and $Z > 0$ are to be determined. Define the Hamiltonian function

$$H[u, w, \Delta_1(x_1), \Delta_2(x_1)] = z^T z - \gamma^2 w^T w + \frac{dE}{dt} \quad (9)$$

where derivative of $E(t)$ is evaluated along the trajectory of the closed-loop system. It is well known that a sufficient condition for achieving robust disturbance attenuation is that the inequality

$$H[u, w, \Delta_1(x_1), \Delta_2(x_1)] < 0, \quad \forall w \in L^2, \Delta_i(x_1) \in \Xi_i(x_1),\ i = 1,2 \quad (10)$$

results in an $E(x)$ which is strictly radially unbounded ([5]-[6]), $E(x)$ may be regulated as a Lyapunov function for the closed-loop systems, and hence, robust stability is guaranteed for all $\varepsilon \in (0,\infty)$ and independently of the time delay (h).

In this paper we will establish conditions under which

$$\text{Inf}_u \, \text{Sup}_{\Delta_i \in \Xi_i} \, \text{Sup}_{w \in L^2} H[u, w, \Delta_1, \Delta_2] < 0 \quad \text{for } i = 1,2$$
$$\text{such that } \Delta_i := \Delta_i(x_1),\ \Xi_i := \Xi_i(x_1). \quad (11)$$

3. MAIN RESULTS

Before deriving the main results, some preliminary lemmas are reviewed.

Lemma 1 [32]. For any matrices *X* and *Y* with appropriate dimensions and for any constant $\eta > 0$, we have:

$$X^T Y + Y^T X \le \eta X^T X + \frac{1}{\eta} Y^T Y. \quad (12)$$

Lemma 2. For an arbitrary positive scalar $\varepsilon_i > 0$ and a positive definite $P_i > 0$, we have:

$$\Delta_i^T(x_i(t))\, P_i\, x_i(t) + x_i^T(t)\, P_i\, \Delta_i(x_i(t)) \le x_i^T(t)\, (\varepsilon_i P_i^2 + \frac{1}{\varepsilon_i} G_i^T G_i)\, x_i(t) \quad (13)$$

Proof. By using assumption 1 and lemma1, we can conclude (13).

One of the key technical contributions of this paper is utilization of Lemma 2, which establishes a representation of the nonlinear uncertainty set by the certain terms. This observation leads to the following Theorem, which is the main result of this paper. The approach employed here is the standard method of Riccati inequalities, which have been used, extensively in linear control for state-space systems [32]-[34].

Theorem 1. Let the matrix $D^T D$ be nonsingular. If there exist positive scalars $\varepsilon_1,\varepsilon_2,\varepsilon_3$ and positive definite solutions $P_1>0, P_2>0, Z>0$ and $Q>0$ to the Matrix inequality

$$\begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} & P_1 r_1 & P_2 f_1 & P_1 d_1 & \varepsilon_1 P_1 & \varepsilon_2 P_2 \\ \hat{R}_{12}^T & \hat{R}_{22} & P_2 r_2 & P_2 f_2 & P_2 d_2 & 0 & 0 \\ r_1^T P_1 & r_2^T P_2 & -Q & 0 & 0 & 0 & 0 \\ f_1^T P_1 & f_2^T P_2 & 0 & -Z & 0 & 0 & 0 \\ d_1^T P_1 & d_2^T P_2 & 0 & 0 & -\gamma^2 I & 0 & 0 \\ \varepsilon_1 P_1 & 0 & 0 & 0 & 0 & -\varepsilon_1 I & 0 \\ \varepsilon_2 P_2 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0 \tag{14}$$

such that

$$\hat{R}_{11} = P_1 a_{11} + a_{11}^T P_1 + Q + c_1^T c_1 - 2P_1 b_1 (D^T D)^{-1} D^T c_1 + \varepsilon_1^{-1} G_1^T G - c_1^T D (D^T D)^{-1} D^T c_1 + \overline{\chi}_1 - 2P_1$$

$$\hat{R}_{22} = P_2 a_{22} + a_{22}^T P_2 + c_2^T c_2 + Z - 2P_2 b_2 (D^T D)^{-1} D^T c_2 + \varepsilon_2^{-1} G_2^T G_2 - c_2^T D (D^T D)^{-1} D^T c_2 + \overline{\chi}_2 - 2P_2$$

$$\hat{R}_{12} = P_1 a_{12} + a_{21}^T P_2 + c_1^T c_2 + \varepsilon_1 (\overline{\chi}_1 - 2P_1 - 2P_1 b_1 (D^T D)^{-1} D^T c_1 - c_1^T D (D^T D)^{-1} D^T c_1) + \varepsilon_1^{-1} (\overline{\chi}_2 - 2P_2 - 2P_2 b_2 (D^T D)^{-1} D^T c_2 - c_2^T D (D^T D)^{-1} D^T c_2)$$

with $\overline{\chi}_1 := \chi_1^{-1}$ and $\overline{\chi}_2 := \chi_2^{-1}$ and $\chi_1 = b_1 (D^T D)^{-1} b_1^T > 0$ and $\chi_2 = b_2 (D^T D)^{-1} b_2^T > 0$. Then, the control law

$$u(t) = -(D^T D)^{-1} \left((b_1^T P_1 + D^T c_1) x_1 + (b_2^T P_2 + D^T c_2) x_2\right) \tag{15}$$

achieves robust global asymptotic stability and robust disturbance attenuation in the sense of (6) and (7), respectively and independently of the time delay (h).

Proof. We will prove the Theorem by showing that the control law (15) will guarantee the inequality of (10).

Noting to the expression (8) and according to (9), we have:

$$\begin{aligned} H(u,w,\Delta_1,\Delta_2) &= x_1^T (a_{11}^T P_1 + P_1 a_{11} + c_1^T c_1 + Q) x_1 \\ &+ x_2^T (a_{22}^T P_2 + P_2 a_{22} + c_2^T c_2 + Z) x_2 \\ &+ x_1^T (P_1 a_{12} + a_{21}^T P_2 + c_1^T c_2) x_2 \\ &+ x_2^T (a_{12}^T P_1 + P_2 a_{21} + c_2^T c_1) x_1 \\ &+ u^T (b_1^T P_1 + D^T c_1) x_1 \\ &+ u^T (b_2^T P_2 + D^T c_2) x_2 \\ &+ x_1^T (c_1^T D + P_1 b_1) u \\ &+ x_2^T (c_2^T D + P_2 b_2) u + u^T D^T D u \\ &- \gamma^2 w^T w + w^T (d_1^T P_1 x_1 + d_2^T P_2 x_2) \\ &+ (x_1^T P_1 d_1 + x_2^T P_2 d_2) w - x_{1h}^T Q x_{1h} \\ &+ x_{1h}^T (r_2^T P_2 x_2 + r_1^T P_1 x_1) - x_{2h}^T Z x_{2h} \\ &+ x_{2h}^T (f_2^T P_2 x_2 + f_1^T P_1 x_1) \\ &+ (r_2^T P_2 x_2 + r_1^T P_1 x_1)^T x_h + \Delta_1^T P_1 x_1 \\ &+ x_1^T P_1 \Delta_1 + \Delta_2^T P_2 x_2 + x_2^T P_2 \Delta_2 \end{aligned} \tag{16}$$

such that $x_{1h} := x_1(t-h), x_{2h} \triangleq x_2(t-h)$.

It is easy to show that the worst case disturbance occurs when

$$w^* = \gamma^{-2} (d_1^T P_1 x_1 + d_2^T P_2 x_2). \tag{17}$$

It follows that

$$\begin{aligned} H_1(u,\Delta_1,\Delta_2) &= \sup_{w \in L^2} H(u,w,\Delta_1,\Delta_2) \\ &= x_1^T R_{11} x_1 + x_2^T R_{22} x_2 + x_1^T R_{12} x_2 \\ &+ x_2^T R_{12}^T x_1 + u^T G_1^T (x_1,x_2) \\ &+ G_1(x_1,x_2) u + u^T D^T D u \\ &- x_{1h}^T Q x_{1h} - x_{2h}^T Z x_{2h} \\ &+ 2x_{1h}^T G_2(x_1,x_2) + 2x_{2h}^T G_3(x_1,x_2) \\ &+ x_1^T P_1 \Delta_1 + \Delta_1^T P_1 x_1 + x_2^T P_2 \Delta_2 + \Delta_2^T P_2 x_2 \end{aligned} \tag{18}$$

where

$$R_{11} = a_{11}^T P_1 + P_1 a_{11} + c_1^T c_1 + \gamma^{-2} P_1 d_1 d_1^T P_1 + Q$$

$$R_{22} = a_{22}^T P_2 + P_2 a_{22} + c_2^T c_2 + \gamma^{-2} P_2 d_2 d_2^T P_2 + Z$$

$$R_{12} = a_{21}^T P_2 + P_1 a_{12} + c_1^T c_2 + \gamma^{-2} P_1 d_1 d_2^T P_2$$

$$G_1(x_1,x_2) = x_1^T(P_1 b_1 + c_1^T D) + x_2^T(P_2 b_2 + c_2^T D)$$

$$G_2(x_1,x_2) = r_1^T P_1 x_1 + r_2^T P_2 x_2,$$

$$G_3(x_1,x_2) = f_1^T P_1 x_1 + f_2^T P_2 x_2.$$

According to Lemma 2, we have

$$\begin{aligned}\operatorname*{Sup}_{\Delta_i\in\Xi_i} H_1(u,\Delta_1,\Delta_2) \le\ & x_1^T(R_{11} + \varepsilon_1 P_1^2 + \frac{1}{\varepsilon_1} G_1^T G_1) x_1 \\ & + x_2^T(R_{22} + \varepsilon_2 P_2^2 + \frac{1}{\varepsilon_2} G_2^T G_2) x_2 \\ & + x_1^T R_{12} x_2 + x_2^T R_{12}^T x_1 \\ & + u^T G_1^T(x_1,x_2) + G_1(x_1,x_2) u \\ & + u^T D^T D u - x_{1h}^T Q x_{1h} \\ & - x_{2h}^T Z x_{2h} + 2x_{1h}^T G_2(x_1,x_2) \\ & + 2x_{2h}^T G_2(x_1,x_2)\end{aligned} \tag{19}$$

The optimal control law, which minimizes the right-hand side of (19), is given by

$$u(t) = -(D^T D)^{-1}((b_1^T P_1 + D^T c_1) x_1 + (b_2^T P_2 + D^T c_2) x_2). \tag{20}$$

As a result, we have:

$$\operatorname*{Inf}_{u}\operatorname*{Sup}_{\Delta_i\in\Xi_i} H_1(u,\Delta_1,\Delta_2) \le F(x_1,x_2,x_h) \tag{21}$$

where

$$\begin{aligned}F(x_1,x_2,x_h) :=& \begin{bmatrix} x_1 \\ x_2 \\ x_{1h} \\ x_{2h}\end{bmatrix}^T M \begin{bmatrix} x_1 \\ x_2 \\ x_{1h} \\ x_{2h}\end{bmatrix} \\ =& x_1^T \overline{R}_{11} x_1 + x_2^T \overline{R}_{22} x_2 + x_1^T \overline{R}_{12} x_2 \\ & + x_2^T \overline{R}_{12}^T x_1 - x_{1h}^T Q x_{1h} - x_{2h}^T Q x_{2h} \\ & + 2x_{1h}^T G_2(x_1,x_2) + 2x_{2h}^T G_2(x_1,x_2)\end{aligned} \tag{22}$$

and

$$\begin{aligned}\overline{R}_{11} =& P_1 a_{11} + a_{11}^T P_1 + Q + c_1^T c_1 + \gamma^{-2} P_1 d_1 d_1^T P_1 + \varepsilon_1 P_1^2 \\ & + \frac{1}{\varepsilon_1} G_1^T G - (P_1 b_1 + c_1^T D)(D^T D)^{-1}(P_1 b_1 + c_1^T D)^T\end{aligned}$$

$$\begin{aligned}\overline{R}_{22} =& P_2 a_{22} + a_{22}^T P_2 + c_2^T c_2 + Z + \gamma^{-2} P_2 d_2 d_2^T P_2 \\ & + \varepsilon_2 P_2^2 + \frac{1}{\varepsilon_2} G_2^T G_2 - (P_2 b_2 + c_2^T D)(D^T D)^{-1}(P_2 b_2 + c_2^T D)^T\end{aligned}$$

$$\begin{aligned}\overline{R}_{12} =& P_1 a_{12} + a_{21}^T P_2 + c_1^T c_2 + \gamma^{-2} P_1 d_1 d_2^T P_2 \\ & - (P_1 b_1 + c_1^T D)(D^T D)^{-1}(P_2 b_2 + c_2^T D)^T \\ \le& P_1 a_{12} + a_{21}^T P_2 + c_1^T c_2 + \gamma^{-2} P_1 d_1 d_2^T P_2 \\ & - \varepsilon_3 (P_1 b_1 + c_1^T D)(D^T D)^{-1}(P_1 b_1 + c_1^T D)^T \\ & - \varepsilon_3^{-1}(P_2 b_2 + c_2^T D)(D^T D)^{-1}(P_2 b_2 + c_2^T D)^T\end{aligned}$$

$$M = \begin{bmatrix} \overline{R}_{11} & \overline{R}_{12} & P_1 r_1 & P_2 f_1 \\ \overline{R}_{12}^T & \overline{R}_{22} & P_2 r_2 & P_2 f_2 \\ r_1^T P_1 & r_2^T P_2 & -Q & 0 \\ f_1^T P_1 & f_2^T P_2 & 0 & -Z \end{bmatrix}. \tag{23}$$

Consequently, if there exist positive definite solutions

$$P_1 > 0, P_2 > 0, Q > 0 \text{ and } Z > 0$$

to the Matrix inequality

$$M < 0$$

then we have

$$\begin{aligned}& H[u,w,\Delta_1(x_1(t)),\Delta_2(x_1(t))] < 0, \\ & \quad \forall\ w \in L^2, \Delta_i(x_1(t)) \in \Xi_i(x_1(t)),\ i = 1,2\end{aligned} \tag{24}$$

It is clear that the inequality $M < 0$ is no longer a linear matrix inequality (LMI). By noticing that

$$\chi_1 = b_1(D^T D)^{-1} b_1^T > 0$$

and

$$\chi_2 = b_2(D^T D)^{-1} b_2^T > 0,$$

we have

$$(\overline{\chi}_1 - P_1)\chi_1(\overline{\chi}_1 - P_1) \ge 0$$

and

$$(\overline{\chi}_2 - P_2)\chi_2(\overline{\chi}_2 - P_2) \ge 0,$$

which are equivalent to

$$-P_1\chi_1P_1 \le \bar{\chi}_1 - 2P_1 \tag{25a}$$

$$-P_2\chi_2P_2 \le \bar{\chi}_2 - 2P_2 \tag{25b}$$

where $\bar{\chi}_1 := \chi_1^{-1}$ and $\bar{\chi}_2 := \chi_2^{-1}$.

Consequently, applying Schur complement to $M<0$ and considering (25a)-(25b), the LMI (14) holds. Thus the proof is completed.

Corollary 1. Let the matrix D^TD be nonsingular. If there exist positive scalars $\varepsilon_1,\varepsilon_2,\varepsilon_3$ and positive definite solutions $P_1>0, P_2>0$ to the linear Matrix inequality

$$\begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} & P_1 d_1 & \varepsilon_1 P_1 & \varepsilon_2 P_2 \\ \hat{R}_{12}^T & \hat{R}_{22} & P_2 d_2 & 0 & 0 \\ d_1^T P_1 & d_2^T P_2 & -\gamma^2 I & 0 & 0 \\ \varepsilon_1 P_1 & 0 & 0 & -\varepsilon_1 I & 0 \\ \varepsilon_2 P_2 & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0$$

such that

$$\hat{R}_{11} = P_1 a_{11} + a_{11}^T P_1 + Q + c_1^T c_1 - 2P_1 b_1 (D^TD)^{-1} D^T c_1 + \varepsilon_1^{-1} G_1^T G - c_1^T D (D^TD)^{-1} D^T c_1 + \bar{\chi}_1 - 2P_1$$

$$\hat{R}_{12} = P_1 a_{12} + a_{21}^T P_2 + c_1^T c_2 + \varepsilon_1(\bar{\chi}_1 - 2P_1 - 2P_1 b_1 (D^TD)^{-1} D^T c_1 - c_1^T D (D^TD)^{-1} D^T c_1) + \varepsilon_1^{-1}(\bar{\chi}_2 - 2P_2 - 2P_2 b_2 (D^TD)^{-1} D^T c_2 - c_2^T D (D^TD)^{-1} D^T c_2)$$

$$\hat{R}_{22} = P_2 a_{22} + a_{22}^T P_2 + c_2^T c_2 + Z - 2P_2 b_2 (D^TD)^{-1} D^T c_2 + \varepsilon_2^{-1} G_2^T G_2 - c_2^T D (D^TD)^{-1} D^T c_2 + \bar{\chi}_2 - 2P_2$$

then, the control law

$$u(t) = -(D^TD)^{-1} ((b_1^T P_1 + D^T c_1) x_1 + (b_2^T P_2 + D^T c_2) x_2)$$

achieves robust global asymptotic stability and robust disturbance attenuation in the sense of (6) and (7), respectively.

4. EXAMPLE

Consider a fourth-order singularly perturbed system with time delay in the slow state variable:

$$\begin{bmatrix} \dot{x}_{s_1}(t) \\ \dot{x}_{s_2}(t) \\ \varepsilon\dot{x}_{f_1}(t) \\ \varepsilon\dot{x}_{f_2}(t) \end{bmatrix} = \begin{bmatrix} -9 & 0 & 0 & 0.1 \\ 0.1 & -8 & 0.05 & 0.1 \\ 0 & 0 & -15 & 0 \\ 0.01 & 0.003 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_{s_1}(t) \\ x_{s_2}(t) \\ x_{f_1}(t) \\ x_{f_2}(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x_1(t-h) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0.5 \\ 0.5 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.5 \end{bmatrix} w(t) + \begin{bmatrix} \Delta_1(x_1(t)) \\ \Delta_2(x_1(t)) \end{bmatrix} \tag{26}$$

$$x_1(t) = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix}^T \quad \forall t \in [-h, 0]$$

$$z(t) = \begin{bmatrix} 0.4 & 0.15 \\ 0.1 & 0.6 \end{bmatrix} \begin{bmatrix} x_{s_1}(t) \\ x_{s_2}(t) \end{bmatrix} + \begin{bmatrix} 0.2 & -0.1 \\ -0.2 & 0.9 \end{bmatrix} \begin{bmatrix} x_{f_1}(t) \\ x_{f_2}(t) \end{bmatrix} + \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 0.1 \end{bmatrix} u(t)$$

where $x_1 = [x_{s_1} \quad x_{s_2}]^T$, $x_2 = [x_{f_1} \quad x_{f_2}]^T$ and the uncertainty terms $\Delta_i(x_1) \quad (i = 1, 2)$, are assumed to be norm-bounded such that the matrixes G_1, G_2 have been considered as follows:

$$G_1 = G_2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

Consider also $\gamma = 0.1$ as the performance bound, $\varepsilon = 0.1$ as the perturbed parameter and $h = 2$ second as the time delay parameter. From (14), we can choose the positive definite solutions $P_1 > 0, P_2 > 0$ and $Q > 0$ as follows:

$$P_1 = \begin{bmatrix} 2.1137 & -0.7639 \\ -0.7639 & 0.6814 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 2.0760 & -0.1406 \\ -0.1406 & 0.1390 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1.5 & 0 \\ -1500 & 0.015 \end{bmatrix}.$$

Also, positive numbers of $\varepsilon_1, \varepsilon_2$ are obtained as follows:

$\varepsilon_1 = 1.8, \quad \varepsilon_2 = 1$

The required state feedback control law is given by

$$u = -k_1 x_1 - k_2 x_2$$

with

$$k_1 = \begin{bmatrix} -13.5405 & 5.0117 \\ 11.2441 & -6.5131 \end{bmatrix},$$
$$k_2 = \begin{bmatrix} -6.1792 & -1.9220 \\ -0.1409 & 1.4103 \end{bmatrix}$$

Robust stability and disturbance attenuation of the slow and fast dynamics in the presence of disturbance (Gussian noise) have been depicted in Figures 1 and 2. Therefore, we conclude that system (26) can be stabilized by the control law (15) for all $\varepsilon \in (0, \infty)$ and independently of the time delay (h), which has been depicted in Figure 3 and the correctness of the attenuation level of the disturbance on the controlled output has been depicted in Figure 4.

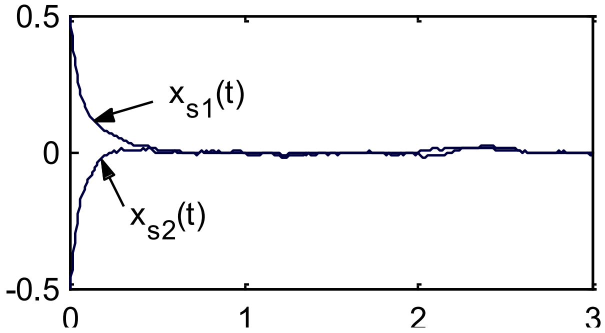

Fig. 1. Robust stability and disturbance attenuation of slow dynamics

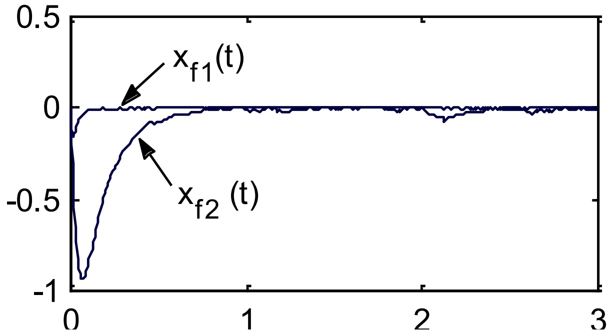

Fig. 2. Robust stability and disturbance attenuation of fast dynamics

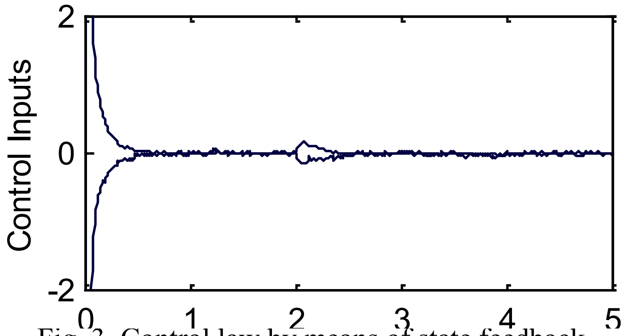

Fig. 3. Control law by means of state feedback

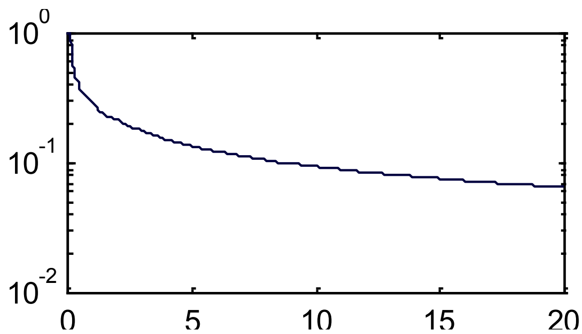

Fig. 4. Attenuation level of the disturbance on the controlled output

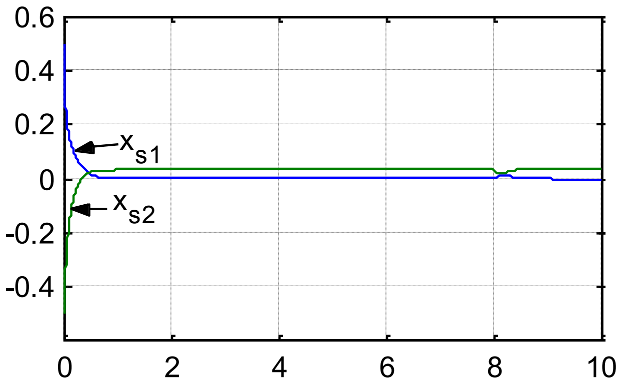

Fig. 5. Robust stability and disturbance attenuation of slow dynamics

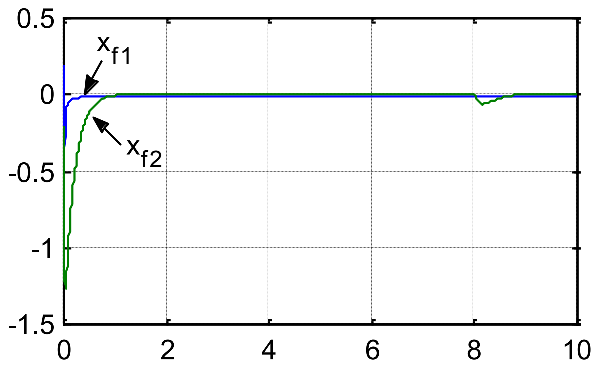

Fig. 6. Robust stability and disturbance attenuation of fast dynamics

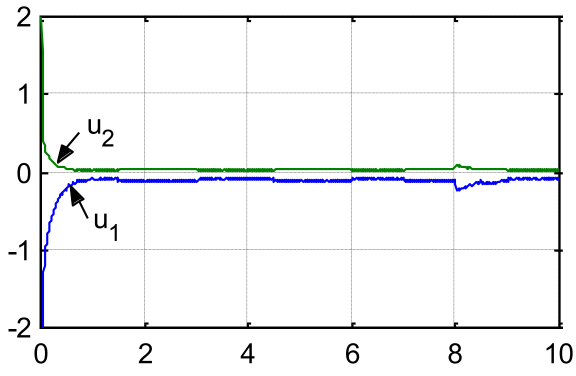

Fig. 7. Control law by means of state feedback

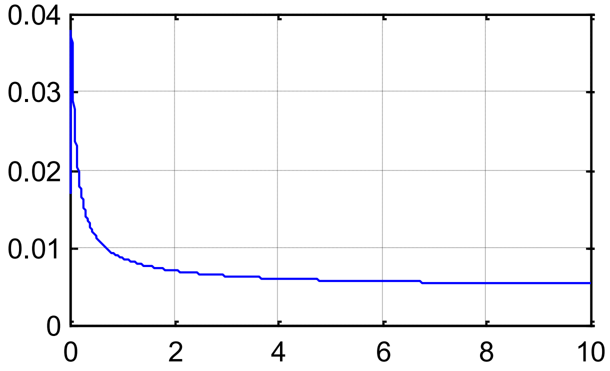

Fig. 8. Attenuation level of the disturbance on the controlled output

In the case $\varepsilon_1 = \varepsilon_2 = 10$ as the perturbed parameter and $h = 8$ second as the time delay parameter, from (14), we can find the positive definite solutions $P_1 > 0, P_2 > 0$ and $Q > 0$ as follows:

$$P_1 = \begin{bmatrix} 0.1845 & 0.0411 \\ 0.0411 & 0.0750 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.0377 & -0.0015 \\ -0.0015 & 0.0477 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.1000 & 0 \\ 0 & 0.1000 \end{bmatrix}$$

Figures 5 and 6 represent time behavior of slow and fast dynamics of the system. The control signal is depicted in Figure 7 and the correctness of the attenuation level of the disturbance on the controlled output has been depicted in Figure 8.

5. CONCLUSION

The problem of robust control design for a class of uncertain singularly perturbed system with discrete time-delay was investigated in this paper. A robust control design methodology is proposed to achieve the robust stabilization and disturbance attenuation for all $\varepsilon \in (0, \infty)$ and independently of time delay. Major contributions of the paper are threefold: One is that the type of norm-bounded nonlinear uncertainties considered in this class of systems coincides with the certain terms by utilization of Lemma 2. The other is that the state feedback gain matrices can be determined in terms of linear matrix inequalities (LMIs), and the last is that the closed-loop system is stable for all $\varepsilon \in (0, \infty)$. In this paper, the results are presented on the two-time-scale case, and the extension of results to multiple-time-scale and multiple time delays such as discrete and neutral and distributed delays is a topic currently under study.

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