Stability condition for discrete systems with multiple state delays

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Abstract: - many natural systems have intrinsic delays, caused by input, output or state variables. In an open-loop system, input delay doesn't affect the stability but if feedback of the state is utilized for controlling the systems, the delay is transferred to closed-loop state and would affect the stability of the system. It is important to evaluate/examine the stability of continuous and discrete delayed-systems. In continuous systems existence of delay increases system dimension to infinity (number of poles is infinite). In discrete systems, number of poles increases but it is finite. Due to the lack of proper and powerful tools in discrete systems, the evaluations of stability in these systems are very important. In this paper, an approach has been proposed based on Lyapunov equation and frequency domain stability analysis for systems which have delay in their states. In this approach, stability conditions have been obtained for discrete systems with delay in a specific interval. All previous works on discrete systems are often for single-delay in state. The purpose of this paper is to extend the existence approaches for systems that have multiple delays in their states. Performance of the proposed method has been studied in several examples.

Key-Words: - Discrete-time systems with delay, stability, Lyapunov equation, state delay

1 Introduction

The existence of time delay in different industrial systems such as the turbojet engines of aircrafts, electricity networks, microwaves, vibrometers, nuclear reactors, roller machines, chemical processes, and power transmission lines is often the cause of instability in these systems. This delay can appear in the input, output, or the variables of state. In this article, the subject of delay in state variables is investigated. The issue of delay of state has been encountered many times in the control problems and physical systems. In recent years, the continuous systems with delays in state have attracted the attention of numerous investigators, and a large volume of research data has been prepared regarding the stability of these systems. The works related to the subject of the present study have also been presented in [1] and [2]. In [3-7], the stability conditions associated with delay have been considered; although, these conditions have been expressed conservatively, and only for a state in which the delay occurs in a specific interval.
On the other hand, less attention has been paid to discrete systems with delay in state, which have been expressed in the following equation.

$$x(k+1) = A_0 x(k) + \sum_{j=1}^{M} A_j x(k-N_j) \quad , \quad 0 < N_1 < \cdots N_M \tag{1}$$

In the single-delay state, we have:

$$x(k+1) = A_0 x(k) + A_1 x(k-N) \tag{2}$$

Where $x \in R^n$ is a variable of state. It is not surprising to have less attention; because by defining a new variable of state according to the following equation, system (2) can be converted to an equivalent non-delayed system:

$$z(k) \equiv \begin{bmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-N) \end{bmatrix} \in R^{n(N+1)\times 1} \tag{3}$$

Thus, there will be an equivalent non-delayed system as follows:

$$z(k+1) = A_N z(k) \tag{4}$$

$$A_N \equiv \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & A_1 \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I & 0 \end{bmatrix} \tag{5}$$

Now the stability of (2) can be determined by examining the stability of an ordinary non-delayed system (4). However, there are two important issues that indicate that the examination of the stability of the equivalent non-delayed system (4) is not sufficient for the fulfillment of the stability of system (2). The first issue is when the delay of state (N) is very large. The second issue is when the amount of delay of state (N) is not known exactly, and we only know that it is in a certain interval.

If N is very large, then matrix A_N in equation (4) will be very large as well $A_N \in R^{n(N+1)\times n(N+1)}$; thus, the numerical evaluation of the stability of A_N will be exhaustive, and sometimes, numerical calculations will be difficult to perform.

If we know that the delay of state (N) is in a specific interval, but its value is not known (for example, $N \in [0, N_{Max}]$), therefore, the stability for N should be checked for the interval $N \in [0, N_{Max}]$, which itself is a difficult and exhaustive numerical task, especially for the case where $N_{\max}$ is large.

In order to deal with the above two cases (especially for large N), we will show that for equation (4), the solving of the Lyapunov's equation can be turned into a simple linear equation whose only N-dependent component is the N^{th} power of a constant matrix. Then, this constant matrix is combined with the expression of the frequency domain, and while considering the existence of delay of state in a specific interval, a stability condition is proposed for equation (2).

In this paper, the suggested method in [1] and [2] (which has been evaluated only on single-delay systems) has been extended to two-delay, three-delay, and multi-delay systems, and the simulation results totally agree with those obtained from the proposed method. For example, a solution has been presented in [2] for determining the stability range for the single-delay systems with state delay N. In that article, the amount of delay has been considered as indefinite, and the maximum delay for which the system remains stable has been calculated. The authors of that paper have completed in their next article in [1], the necessary and sufficient conditions for the establishment of stability of a discrete-time system with delay. However, no specific method has been presented so far for discrete systems with multiple delays-in-state.

In the present article, for the estimation of the maximum delay for which a system is stable, some methods have been proposed and evaluated. The notion that the solving of the Lyapunov's equation for equation (4) can be transformed into a simple linear equation whose only N-dependent component is the N^{th} power of a constant matrix, has been described in section 2. Then in section 3, this constant matrix has been combined with the expression of the frequency domain, and a stability condition has been proposed for equation (2) while considering the existence of delay of state in a specific interval. In section 4, the approach stated for single-delay systems has been extended to multi-delay systems. In section 5, numerical examples have been given to clarify the results obtained in this article, and in section 6, the conclusion has been presented.

2 Solving The Lyapunov Problem For Single-Delay Systems

The candidate function of Lyapunov for equation (4) is defined as follows:

$$V(z(k)) \equiv z(k)^T P z(k) \tag{6}$$

Where the symmetric matrix $P \in R^{n(N+1)\times n(N+1)}$ is adapted to dimension $z(k)$, and is expressed as follows:

$$P = \begin{bmatrix} P_{00} & P_{01}(N) & \cdots & P_{01}(1) \\ P_{01}(N) & P_{11}(N,N) & \cdots & P_{11}(N,1) \\ \vdots & \vdots & \ddots & \vdots \\ P_{10}(N) & P_{11}(1,N) & \cdots & P_{11}(1,1) \end{bmatrix} \quad (7)$$

The changes of the candidate function are given in the following relation.

$$V(z(k+1)) - V(z(k)) = z(k)^T (A_N^T P A_N - P) z(k) \quad (8)$$

System (4) is stable if, and only if, there is $P = P^T > 0$ which satisfies relation $A_N^T P A_N - P < 0$. By using the particular structure of $z(k)$ ($z(k)$ is stable if, and only if, $x(k)$ is stable), condition $A_N^T P A_N - P < 0$ can be modified into the following principle.

Principle 1: System (2) is stable if, and only if, there is $P = P^T > 0$ with $P_{00} > 0$, which is true for $Q = Q^T \in R^{n\times n} > 0$ in the following relation.

$$A_N^T P A_N - P + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} = 0 \quad (9)$$

Matrix P has three variables: P_{00}, $P_{01}(i) = P_{10}(i)^T$, and $P_{11}(i,j) = P_{11}(j,i)^T$.
The following principle has simplified the expression of P by using the variable $X(i)$.

Principle 2: The solution of P, which satisfies equation (9), has been given in the following form:

$$P_{00} = X(0), \quad P_{01}(i) = X(i)A_1,$$
$$P_{11}(i,j) = \begin{cases} A_1^T X(i-j)^T A_1, & 0 \le j \le i \le N, \\ A_1^T X(j-i) A_1, & 0 \le i \le j \le N, \end{cases} \quad (10)$$

In which, $X(k), 0 \le k \le N$ is expressed as follows:

$$\begin{aligned} & A_0^T X(0) A_0 + A_1 X(N)^T A_0 + A_0^T X(N) A_1 + \\ & \qquad A_1^T X(0) A_1 - X(0) + Q = 0, \\ & X(0) = X(0)^T, \\ & X(k+1) = A_0^T X(k) + A_1 X(N-k)^T \\ & \qquad , \quad 0 \le k \le N-1 \end{aligned} \quad (11)$$

Proof: The correct calculations of equation (9) result in equations (10) and (11) [1].
Since the matrix differential equation (the third equation) in equation (11) doesn't have a simple solution form, this matrix differential equation has been transformed into a kind of two-point boundary value problem in the next principle. Throughout this article, A_0 has been considered as a non-singular matrix. It should be mentioned that most of the discrete systems have a non-singular A_0 matrix.

Principle 3: The matrix differential equation in (11) is equivalent to the following relations.
Matrix $M \in C^{n\times n}$ can be represented as:

$$M = \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix} \quad (12)$$

csM is defined as:

$$csM \equiv \left[m_{11} \cdots m_{n1} \middle| \cdots \middle| m_{1n} \cdots m_{nn} \right]^T \in C^{n^2\times 1} \quad (13)$$

$$\begin{bmatrix} csX(k+1) \\ csX(N-k-1) \end{bmatrix} = \begin{bmatrix} A & B \\ -A^{-1}BA & A^{-1}(I - BB) \end{bmatrix} \begin{bmatrix} csX(k) \\ csX(N-k) \end{bmatrix} \quad (14)$$

where $A \equiv (I \otimes A_0^T)$, $B \equiv (I \otimes A_1^T)T$, and the symbol $\otimes$ denotes the Kronecker product.

$$T \equiv \left[|T_1| \, |T_2| \ldots |T_{n^2-1}| \, |T_{n^2}| \right], T_l \in R^{n^2\times 1} \quad (15)$$

And the row vectors $T_l, 1 \le l \le n^2$ have been defined as follows:

$$T_{(i-1)n+j} \equiv e_{(j-1)n+i}, \quad 1 \le i, j \le n, \quad (16)$$

Where $e_l \in R^{n^2 \times 1}, 1 \le l \le n^2$ is a row vector whose l[th] element is 1 and the rest of its elements are zero (0).

Proof: The following properties are mentioned.

$$\begin{aligned} cs(ABC) &= (C^T \otimes A)csB \\ csA^T &= TcsA \end{aligned} \tag{17}$$

In these relations, $A, B, C \in R^{n \times n}$, and by using these properties, we obtain from the forward difference equation in (11) :

$$\begin{aligned} &csX(k+1) = (I \otimes A_0^T)csX(k) + \\ &(I \otimes A_1^T)TcsX(N-k) = AcsX(k) + BcsX(N-k) \end{aligned}$$

So, we have:

$$csX(k+1) = AcsX(k) + BcsX(N-k) \tag{18}$$

Thus, the equation below can be obtained from the above equations (it should be mentioned that A is non-singular if, and only if, A_0^T is non-singular).

$$csX(k) = A^{-1}csX(k+1) - A^{-1}BcsX(N-k) \tag{19}$$

By inserting $k \equiv N-k-1$ into the above equation, we will have:

$$\begin{aligned} csX(N-k-1) = -A^{-1}BAcsX(k) + \\ A^{-1}(I-BB)csX(N-k) \end{aligned} \tag{20}$$

By combining equations (18) and (20), equation (14) is obtained.

The two matrices of H (equation (14)) and J are defined as follows

$$H = \begin{bmatrix} A & B \\ -A^{-1}BA & A^{-1}(I-BB) \end{bmatrix}, J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{21}$$

Now, we get back to the solving of equation (11) to obtain the Q. The Lyapunov's matrix equation (the first equation) and the matrix differential equation (the third equation) have been combined in equation (11).

To solve the Lyapunov's equation, it is necessary to obtain a pair of $X(0)$ and $X(N)$ which satisfy the matrix differential equation, or equivalently, equation (14). The application of constraint on every pair of $X(0)$ and $X(N)$ that satisfy equation (14) can be expressed by using the following boundary conditions.

$$\begin{bmatrix} csX(N) \\ csX(0) \end{bmatrix} = H^N \begin{bmatrix} csX(0) \\ csX(N) \end{bmatrix} \tag{22}$$

And by using J (equation (21)), the above equation is written as:

$$(I - JH^N)\begin{bmatrix} csX(0) \\ csX(N) \end{bmatrix} = 0 \tag{23}$$

Principle 4: The following equation is established [2].

$$\dim Null(I - JH^N) = n^2 \tag{24}$$

Principle 5: If z is an eigenvalue of H, then, z^{-1} will also be an eigenvalue of H [1].

3 Review Of Stability Conditions For Single-Delay Systems

In this section, it will be demonstrated that the eigenvalues and eigenvectors of H are almost associated with the stability of the frequency domain of equation (2).

Based on this observation, a new stability condition is proposed, which provides the stability of equation (2) for all the $N \in [0, N_{\max}]$.

Equation (2) is stable if, and only if, all the roots of the characteristic equation (25) are inside the unit circle.

$$\det(z^{N+1}I - A_0 z^N - A_1) = 0 \tag{25}$$

Since $\det M = \det M^T$, therefore equation (2) is stable if, and only if, all the roots of the following characteristic equation are inside the unit circle.

$$\det(z^{N+1}I - A_0^T z^N - A_1^T) = 0 \tag{26}$$

In this article, a simple method is presented for checking to see whether all the roots of equation (26) are inside the unit circle.

$$f(z,r) \equiv \det(z^{r+1} I - A_0^T z^r - A_1^T),$$
$$W(r) \equiv \{z \in C \mid f(z,r) = 0\}, \tag{27}$$
$$W_B(r,\varepsilon) \equiv \{z \in C \mid \|z - \hat{z}\| < \varepsilon, \hat{z} \in W(r)\}.$$

Principle 6: If equation (2) is stable for $N = 0$, and for every real number $r \in [0, N_{\max}]$, $W(r)$ has no element on the unit circle, then, equation (2) will be stable for every $N \in [0, N_{\max}]$ [1].

The next principle shows that the unit circle element of $W(r)$ can be evaluated from the eigenvalues of H .

Principle 7: If $W(r)$ has an element on the unit circle, then, that root will be an eigenvalue of H as well [2].

By using principle 7, $N_{\max}$ can be so calculated as to make equation (2) stable for every $N \in [0, N_{\max}]$.

Assume that $e^{jw_i}, w_i \in R \geq 0$ is an eigenvalue of the unit circle H , and v_i is the eigenvector associated with it, and $r_i \in R \geq 0$ is defined as follows:

$$r_i \equiv \begin{cases} \dfrac{\left|\lim\left(Ln\left(\dfrac{\beta_k}{\gamma_k}\right)\right)\right|}{\omega_i} & \omega_i \neq 0 \\ 0 & \omega_i = 0 \end{cases} \tag{28}$$

where β_k is the k th element v_i and γ_k is the (n^2+k) th element from vector v_i . $k \leq n^2$, which can be arbitrarily chosen to the length of the k th non-zero element of v_i .

(i) If equation (2) is stable for $N = 0$ and $N_{\max}$ is the largest whole number not bigger than $\min r_i$, then, equation (2) will be stable for all the $N \in [0, N_{\max}]$.

(ii) If equation (2) is stable for $N = 0$ and H has no eigenvalue on the unit circle, then, equation (2) will be stable for $N \geq 0$.

Proof of (i): Let's assume that in equation (28), $k = 1$. From equation $v = [u \ \bar{u}]^T$, $v \in C^{2n^2}$, we obtain:

$$\frac{\beta_1}{\gamma_1} = e^{jw_i r_i}$$

where $r = r_i$ is a root on the unit circle from $f(z,r) = 0$. From principle 7, it is concluded that $W(r)$ has no element on $r < \min r_i$. Therefore, equation (26) doesn't have a root on the unit circle for $N < \min r_i$. So, according to principle 6, equation (2) is stable for all the $N \in [0, N_{\max}]$.

4 Extending The Expressed Method To Multi-Delay Systems

In this section, the intention is to extend the method expressed for single-delay systems to multi-delay systems. Consider the system given in equation (1). To extend the method cited in the previous section to multi-delay systems, we first find a realization that transforms system (1) into system (2); in other words, we find a realization that transforms the multi-delay system into an equivalent single-delay system. Then, the stability condition obtained in the previous section can also be applied to these systems. The equivalent single-delay system is defined as follows:

$$y(k+1) = A_0' y(k) + A_1' y(k - N') \tag{29}$$

In equation (29), the delay of N' has been specified as $N' = N_M - N_{M-1}$. It should be mentioned that in this state, the goal is to obtain the interval of delay of N_M for which the system is stable. Variables $y(k), y(k+1)$ have been defined as:

$$y(k)=\begin{bmatrix} x(k) \\ \vdots \\ x(k-N_1) \\ \vdots \\ x(k-N_j) \\ \vdots \\ x(k-N_{M-1}) \end{bmatrix}, \quad y(k+1)=\begin{bmatrix} x(k+1) \\ \vdots \\ x(k-N_1+1) \\ \vdots \\ x(k-N_j+1) \\ \vdots \\ x(k-N_{M-1}+1) \end{bmatrix} \tag{30}$$

By considering (29) and (30), we will have:

$$y(k+1)= \underbrace{\begin{bmatrix} A_0 & 0 \cdots 0 & A_1 & 0 \cdots 0 & A_j & 0 \cdots 0 & A_{M-1} \\ I & 0 \cdots 0 & \cdots & 0 \cdots 0 & \cdots & 0 \cdots 0 & 0 \\ 0 & I \ddots 0 & \ddots & 0 \ddots 0 & \ddots & 0 \ddots 0 & 0 \\ 0 & 0 \; I \ddots & \ddots & \ddots & \ddots & \ddots 0 & 0 \\ \vdots & \vdots \; 0 \ddots & \ddots & \ddots & \ddots & \ddots 0 & \vdots \\ \vdots & \vdots \; \vdots \ddots & \ddots & \ddots & \ddots & \ddots \vdots & \vdots \\ 0 & 0 \; 0 \ddots & \ddots & \ddots & \ddots & \ddots 0 & 0 \\ 0 & 0 \; 0 \; 0 & \cdots & \cdots & \cdots & I \; 0 \; 0 & 0 \\ 0 & 0 \; 0 \; 0 & \cdots & \cdots & \cdots & 0 \; I \; 0 & 0 \\ 0 & 0 \; 0 \; 0 & \cdots & \cdots & \cdots & 0 \; 0 \; I & 0 \end{bmatrix}}_{A_0'} \underbrace{\begin{bmatrix} x(k) \\ \vdots \\ x(k-N_1) \\ \vdots \\ x(k-N_j) \\ \vdots \\ x(k-N_{M-1}) \end{bmatrix}}_{y(k)} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & A_M \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}}_{A_1'} \underbrace{\begin{bmatrix} x(k-N_M+N_{M-1}) \\ \vdots \\ \vdots \\ x(k-N_M) \end{bmatrix}}_{y(k-N')} \tag{31}$$

With regards to (21), the two matrices of H and J are defined for multi-delay systems as follows:

$$H=\begin{bmatrix} A' & B' \\ -A'^{-1}B'A' & A'^{-1}(I-B'B') \end{bmatrix}, \quad J=\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{32}$$

where $A' \equiv (I \otimes A_0'^{T})$, $B' \equiv (I \otimes A_1'^{T})T$.

5 Examples

Now, the effectiveness of the proposed approach in this article is evaluated for systems with delay in state.

Example 1- Consider the following single-delay system:

$$x(k+1)=A_0x(k)+A_1x(k-N), \quad A_0=\begin{bmatrix} 0.3 & 0.15 \\ 0 & 0.7 \end{bmatrix}, A_1=\begin{bmatrix} 0.1 & -0.2 \\ 0.1 & -0.4 \end{bmatrix} \tag{33}$$

This system is stable for $N=0$, and the eigenvalues of H have been given as follows:

{0.0081 - 0.0070i, -0.0432 + 0.0749i, -0.0802 + 0.0322i, 0.5259 - 0.4565i, -0.0107, 0.0817 + 0.0283i, 0.0817 - 0.0283i, 0.6924}

First, the eigenvalues of matrix H are plotted. As is clear from Fig. 1, matrix H has two eigenvalues on the unit circle; one of these eigenvalues has $\omega \geq 0$ and is equal to $\omega = 0.2368$. Now, by considering equation (28) and (i), the value of N_{Max} can be calculated.

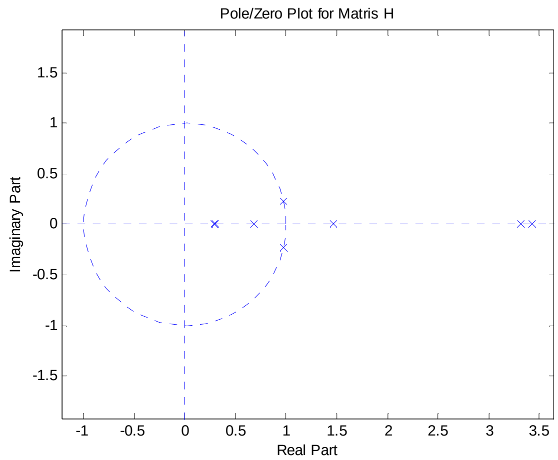

Fig. 1: Eigenvalues of matrix H (single-delay system)

$$Log\left((0.0081-0.0070i)/(-0.0107)\right)=0.0005+2.4289i$$

$$r=2.4289/0.238=10.2572 \Rightarrow N_{Max}=10$$

In this system, $N_{Max}=10$, and this means that the system is stable for the delay of $N\leq 10$, and unstable for $N=11$. Therefore, the system described in this example is stable for $N\in[0,10]$.

Consider Fig.2, the poles of the characteristic equation (33) according to formula (26) for $N=11$ is plotted.

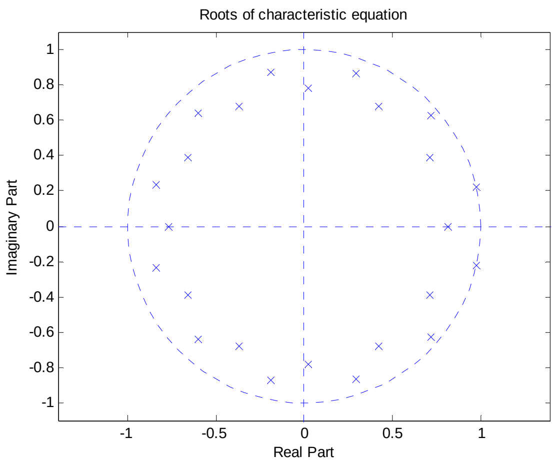

Fig. 2: The location of poles characteristic equation (33) for $N=11$

Conjugate poles of the characteristic equation of the system have been transferred outside the unit circle and have been lead to system instability. Unstable poles as follows:

$$\left.\begin{array}{l}P_1=0.9763+0.2222i\\ P_2=0.9763-0.2222i\end{array}\right\}\Rightarrow |P_1|=|P_2|=1.0012$$

Example 2- Consider the following two-delay system:

$$x(k+1)=A_0x(k)+A_1x(k-N_1)+A_2x(k-N_2) \quad , \quad N_1\leq N_2$$

$$A_0=\begin{bmatrix}0.3 & 0.15\\ -0.1 & 0.6\end{bmatrix},\quad A_1=\begin{bmatrix}0.2 & -0.2\\ 0.1 & 0.1\end{bmatrix},\quad A_2=\begin{bmatrix}0.1 & 0.4\\ -0.6 & -0.55\end{bmatrix} \tag{34}$$

This system is stable for $N_1=N_2=0$. In his example, by assuming $N_1=1$, the range of delay of N_2 is obtained. First, the eigenvalues of matrix H are plotted. It is observed in Fig. 3 that matrix H has four eigenvalues on the unit circle, which two of these eigenvalues have $\omega\geq 0$; so, by considering equation (31) and using equation (28) and (i), the value of N_2 can be estimated.

$$\left.\begin{array}{ll}\omega_1=0.1565 & r_1=10.8096\\ \omega_2=0.2087 & r_2=14.8654\end{array}\right\}\rightarrow$$

$$Min\ (r_1,r_2)=N'=10$$

$$N_1=1 \quad\rightarrow\quad N_2-N_1=N'=10 \quad\rightarrow$$

$$N_2=11 \quad\rightarrow\quad N_2\in[1,11]$$

It can be concluded that, for the delay of $N_1=1$, this system will be stable for $N_2\in[1,11]$

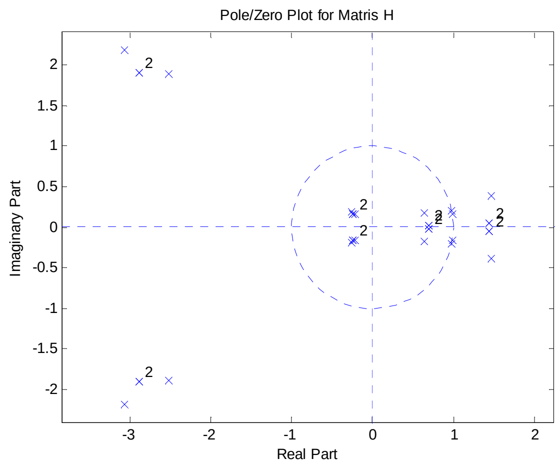

Fig. 3: Eigenvalues of matrix H (two-delay system)

The poles of the characteristic equation (34) according to formula (35) for $N_1=1$ and $N_2=12$ are plotted in Fig. 4

$$\det(z^{N_1+N_2+1}I-A_0^Tz^{N_1+N_2}-A_1^Tz^{N_2}-A_2^Tz^{N_1})=0 \tag{35}$$

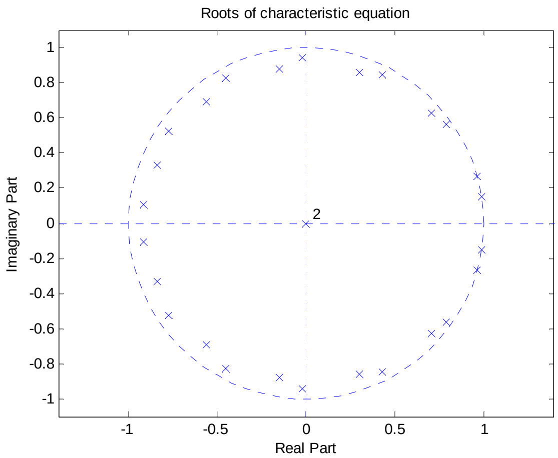

Fig. 4: The location of poles characteristic equation (34) for $N_1 = 1$ and $N_2 = 12$

According to Fig. 4 , Because the two poles of the unit circle is outside, the system is unstable, Unstable poles as follows:

$$\left.\begin{aligned} P_1 &= 0.9882 + 0.1538i \\ P_2 &= 0.9882 - 0.1538i \end{aligned}\right\} \Rightarrow |P_1| = |P_2| = 1.0001$$

Now, for a delay value of $N_1 = 2$, we want to determine the range of delay of N_2 for system stability. In this state also, matrix H has two eigenvalues on the unit circle, one of which has $\omega \geq 0$; so, in view of equation (28) and (i), the amount of delay of N_2 can be calculated.

$$\left.\begin{aligned} \omega_1 &= 0.2241 \qquad & r_1 &= 12.2249 \\ \omega_2 &= 0.0830 \qquad & r_2 &= 22.4386 \end{aligned}\right\} \rightarrow$$
$$Min\ (r_1, r_2) = N' = 12$$

$$N_1 = 2 \quad \rightarrow \quad N_2 - N_1 = N' = 12 \quad \rightarrow$$
$$N_2 = 14 \quad \rightarrow \quad N_2 \in [1,14]$$

As can be seen, for the delay of $N_1 = 2$, this system will be stable for $N_2 \in [1,14]$. By using the same approach, the range of delay of N_2 for system stability can be obtained for different delay values of N_1.

The poles of the characteristic equation (34) according to formula (35) for $N_1 = 2$ and $N_2 = 15$ are plotted in Fig. 5

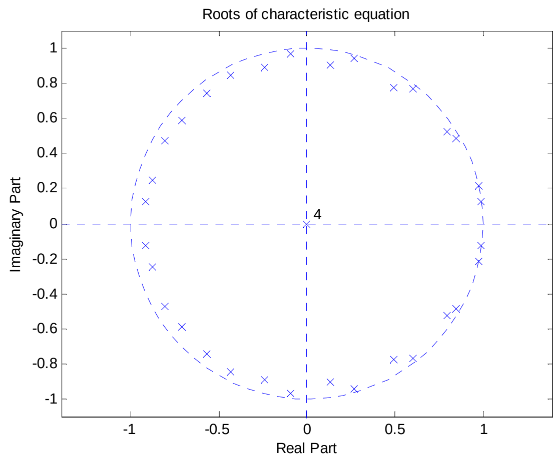

Fig. 5: The location of poles characteristic equation (34) for $N_1 = 2$ and $N_2 = 15$

Unstable poles as follows:

$$\left.\begin{aligned} P_3 &= 0.9779 + 0.2132i \\ P_4 &= 0.9779 - 0.2132i \end{aligned}\right\} \Rightarrow |P_3| = |P_4| = 1.0009$$

Example 3- The following three-delay discrete-time system is considered:

$$\begin{aligned} x(k+1) &= A_0 x(k) + A_1 x(k-1) + A_2 x(k-2) + \\ & A_3 x(k - N_3) \quad N_1 \leq N_2 \leq N_3 \end{aligned} \tag{36}$$

$$A_0 = \begin{bmatrix} 0.3 & 0.15 \\ 0 & 0.7 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0.1 & -0.3 \\ 0.2 & -0.3 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & 0.52 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.4 & -0.1 \\ -0.1 & -0.3 \end{bmatrix}$$

In the given three-delay system, the objective is the determination of the maximum amount of delay of N_3 for which the system remains stable. In equation (36), it has been assumed that $N_1 = 1$ and $N_2 = 2$. First, the eigenvalues of matrix H are plotted. It is observed in Fig. 6 that matrix H has several eigenvalues on the unit circle, and one of these eigenvalues has $\omega \geq 0$ which is equal to $\omega = 0.1656$. Now, by considering equation (31) and using equation (28) and (i), the value of N_3 can be calculated.

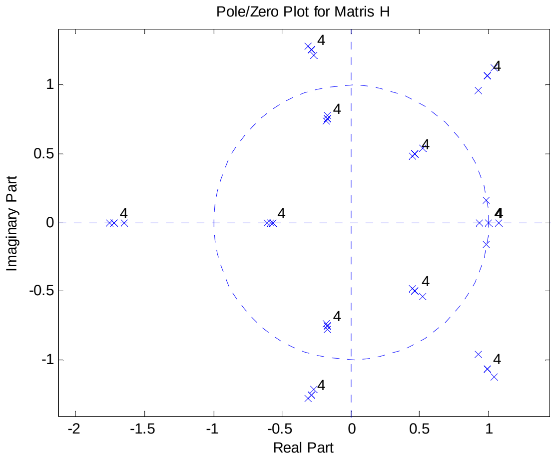

Fig. 6: Eigenvalues of matrix H (three-delay system)

$$\omega = 0.1656 \rightarrow \quad r = 8.0501 \rightarrow \quad N' = 8$$

$$N_2 = 2 \quad \rightarrow \quad N_3 - N_2 = N' = 8 \quad \rightarrow$$
$$N_3 = 10 \quad \rightarrow \quad N_3 \in [1,10]$$

As can be observed, for the delay values of $N_1 = 1$ and $N_2 = 2$, this system will be stable for $N_3 \in [1,10]$. Using the same approach, for different delay values of N_1 and N_2, the range of delay of N_3 for system stability can be obtained.

The poles of the characteristic equation (36) according to formula (37) for $N_1 = 1$, $N_2 = 2$ and $N_3 = 11$ are plotted in Fig. 7

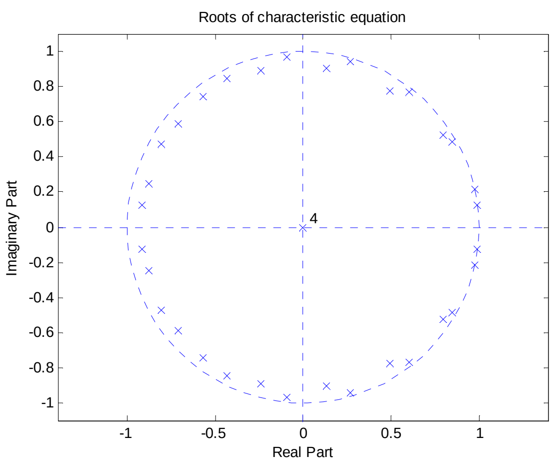

Fig. 7: The location of poles characteristic equation (34) for $N_1 = 1$, $N_2 = 2$ and $N_3 = 11$

Conjugate poles of the characteristic equation of the system have been transferred outside the unit circle and have been lead to system instability. Unstable poles as follows:

$$\left.\begin{array}{l} P_1 = 0.9941 + 0.1548i \\ P_2 = 0.9941 - 0.1548i \end{array}\right\} \Rightarrow |P_1| = |P_2| = 1.0061$$

6 Conclusion

In this article, the proposed method in [1], which concerns the necessary and sufficient conditions for the stability of single-delay systems, has been extended to multi-delay systems. For this purpose, first, a simple approach for solving the Lyapunov's equation for a system with delay in state has been presented. By using the proposed approach, the Lyapunov's equation can be easily solved, even for large values of N .Then, by presenting a realization that transforms a multi-delay system into a single-delay one, the stability conditions in [1] could be extended to multi-delay systems. It should be mentioned that this stability condition has been defined based on the relationship between the stability of the frequency domain and a constant matrix that appears in the Lyapunov's equation. The presented stability condition guarantees the stability of discrete systems with delay in state, whose delay of state is not specified exactly, and we only know that it falls in a particular interval.

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