Fuzzy Tracking Control Based on H_∞ Performance for Nonlinear Systems

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Abstract: - This paper presents a fuzzy tracking control design based on H_∞ performance for nonlinear systems. The Takagi-Sugeno fuzzy model is employed to approximate a nonlinear system. Based on the H∞ tracking performance, the nonlinear system output is controlled to track a reference signal, and at the same time the tracking performance is attenuated to a prescribed level. Linear matrix inequalities (LMI) techniques are used to solve the fuzzy tracking control problem. The proposed method has been applied to control of a laboratory pendulum-cart system. Hence, the performance has been evaluated in simulations as well as in real-time control.

Key-Words: - Fuzzy tracking control, H_∞ performance, LMI, Pendulum-cart system

1 Introduction

The Takagi-Sugeno (T-S) fuzzy model has become popular because of its efficiency in controlling nonlinear systems. The T-S fuzzy model has been proved to be a good representation for a class of nonlinear systems [1]. The main property of T-S fuzzy model is to describe the local dynamics by linear models. The overall model of the nonlinear system is obtained by fuzzy blending of these linear models through nonlinear fuzzy membership functions.

The tracking control design based on the Takagi-Sugeno fuzzy model has been treated by several researchers, for instance, Ma [2], Zhang [3], Tseng [4], and Uang [5]. The most important issue for fuzzy tracking control systems is that the output of the nonlinear system tracks a reference signal. In Ma [2], the tracking problem of nonlinear systems is solved using a synthesis of the fuzzy control theory and the linear multivariable control theory. Simulation results show that the proposed tracking control system can make the output of the system to asymptotically track the reference signal. The robust fuzzy tracking controller based on internal model principle is introduced to track a reference signal in [3]. Simulation results on the inverted pendulum system show that the stepwise signal for uncertain nonlinear system can be tracked via the proposed method. The nonlinear H_∞ control schemes have been introduces to deal with the robust performance design problem of nonlinear systems. In general, conventional H_∞ control scheme are not suitable for practical control system design [6]. In the work of Tseng [4], a fuzzy tracking control design method with a guaranteed H_∞ model reference tracking control scheme is proposed to systematically design for continuous-time systems. The proposed design is applied for the multi input multi output (MIMO) systems. However, the application of the design for single input multi output systems, we have to make adjustment on the structure of reference model. Despite the fact that much progress has been made in studying the tracking problem of nonlinear systems, it is still a challenge to apply the control system to the real plant.

For practical control design, a simple fuzzy tracking control design with guaranteed control performance is more appealing for nonlinear systems. In this work, the T-S fuzzy model is used to describe the dynamics of the nonlinear system. Then, a fuzzy tracking controller is introduced to track a reference signal, and the H_∞ tracking performance is guaranteed for a prescribed level for all external disturbance and reference signal. The LMI convex programming technique is used to solve this problem. The proposed technique is validated by means of a laboratory experiment; a pendulum-cart system.

The paper is organized as follows. Section 2 addresses the problem formulation. The H_∞ tracking control design is discussed in Section 3. In Section 4, the application of the proposed method in pendulum-cart system is derived. Also, the simulation and experiment results are provided in this section. Finally, Section 5 concludes this paper.

2 Problem Formulation

The main feature of Takagi–Sugeno fuzzy models is to represent the nonlinear dynamics by linear model according to the so-called fuzzy rules and then to blend all the linear models into an overall single model through nonlinear fuzzy membership functions [7]. The ith rule of the fuzzy model is of the following form:

Plant Rule i:

If $z_1(t)$ is F_{i1} and ··· and $z_g(t)$ is F_{ig}

Then $\dot{x}(t) = A_i x(t) + B_i u(t) + w(t)$

$$y(t) = C_i x(t) \quad \text{for } i = 1, 2, \cdots, L \tag{1}$$

where $x(t) \in R^{n\times 1}$ denotes the state vector, $u(t) \in R^{m\times 1}$ denotes the control input, $w(t) \in R^{n\times 1}$ denotes the bounded external disturbance, $y(t) \in R^q$ denotes the system output, $A_i \in R^{n\times n}$, $B_i \in R^{n\times m}$, and $C_i \in R^{q\times n}$, F_{ij} is the fuzzy set, L is the number of If-Then rules, and $z_1(t), z_2(t), \ldots, z_g(t)$ are the premise variables.

The final output of fuzzy model is inferred as follows [4],[5]:

$$\dot{x}(t) = \frac{\sum_{i=1}^{L} \mu_i(z(t))[A_i x(t) + B_i u(t) + w(t)]}{\sum_{i=1}^{L} \mu_i(z(t))}$$

$$= \sum_{i=1}^{L} h_i(z(t))[A_i x(t) + B_i u(t) + w(t)] \tag{2}$$

$$y(t) = \frac{\sum_{i=1}^{L} \mu_i(z(t)) C_i x(t)}{\sum_{i=1}^{r} \mu_i(z(t))}$$

$$= \sum_{i=1}^{L} h_i(z(t)) C_i x(t) \tag{3}$$

where

$$\mu_i(z(t)) = \prod_{j=1}^{g} F_{ij}(z_j(t))$$

$$h_i(z(t)) = \frac{\mu_i(z(t))}{\sum_{i=1}^{L} \mu_i(z(t))}$$

$$z(t) = [z_1(t), z_2(t), \cdots, z_g(t)]$$

and where $F_{ij}(z_j(t))$ is the grade of membership function of $z_j(t)$ in F_{ij}.

It assumed that

$$\mu_i(z(t)) \geq 0 \text{, and } \sum_{i=1}^{L} \mu_i(z(t)) > 0$$

for all t.

Therefore, we get [7], [8]

$$h_i(z(t)) \geq 0, \sum_{i=1}^{L} h_i(z(t)) = 1 \text{ for } i = 1, 2, \cdots, L\,. \tag{4}$$

The T-S fuzzy model in (2) is a general nonlinear time-varying equation and has been used to model the behaviours of nonlinear dynamic systems [7].

Consider the following reference model [9]:

$$\dot{x}_r(t) = A_r x_r(t) + B_r r(t) \tag{5}$$

where $x_r(t)$ denotes the reference state, A_r and B_r are the known linear system and input matrices, respectively; $r(t)$ is the bounded reference input (signal).

The H_∞ performance related to tracking error is denoted as follow [4], [5]:

$$\frac{\int_0^{t_f} \{[x(t) - x_r(t)]^T Q [x(t) - x_r(t)]\} dt}{\int_0^{t_f} \tilde{w}(t)^T \tilde{w}(t) dt} \leq \rho^2$$

or

$$\int_0^{t_f} \{[x(t) - x_r(t)]^T Q [x(t) - x_r(t)]\} dt$$

$$\leq \rho^2 \int_0^{t_f} \tilde{w}(t)^T \tilde{w}(t) dt \tag{6}$$

where $\tilde{w}(t) = [w(t), r(t)]^T$ for all reference input $r(t)$, and external disturbance $w(t)$; t_f is terminal time of control, Q is a positive definite weighting matrix, ρ is a prescribed attenuation level. The physical meaning of (6) is that the effect of any $\tilde{w}(t)$ on tracking error $x(t) - x_r(t)$ must be attenuated below a desired level ρ from the viewpoint of energy, i.e. the L_2 gain from $\tilde{w}(t)$ to $x(t) - x_r(t)$ must equal to or less than a prescribed value ρ^2.

By using the PDC scheme, the following fuzzy controller is employed to deal with the proposed control system design:

Control Rule j:

If $z_1(t)$ is F_{i1} and ⋯ and $z_g(t)$ is F_{ig}
Then $u(t) = K_j(x(t) - x_r(t))$, $j = 1, 2, \cdots, L$ (7)

where K_j is the controller gain for the jth controller rule.
Hence, the overall fuzzy controller is given by

$$u(t) = \sum_{j=1}^{L} h_j(z(t))(K_j(x(t) - x_r(t))) \tag{8}$$

where the weight $h_j(z(t))$ is the same as the weight of ith rule of the fuzzy system (2). Substituting (8) into (2) yields the closed-loop control system as follows:

$$\dot{x}(t) = \sum_{i=1}^{L} h_i(z(t))[(A_i + B_i K_j)x(t) - B_i K_j x_r(t) + w(t)] \tag{9}$$

Combining the controlled local linear model (9) and the reference model (5), we obtain the following augmented fuzzy system:

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t))[\tilde{A}_{ij}\tilde{x}(t) + \tilde{E}\tilde{w}(t)] \tag{10}$$

where

$$\tilde{A}_{ij} = \begin{bmatrix} A_i + B_i K_j & -B_i K_j \\ 0 & A_r \end{bmatrix}, \ \tilde{E} = \begin{bmatrix} I & 0 \\ 0 & B_r \end{bmatrix}$$

$$\tilde{x}(t) = \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix}, \ \tilde{w}(t) = \begin{bmatrix} w(t) \\ r(t) \end{bmatrix}.$$

If the initial condition is also considered, the H_∞ tracking performance in (6) can be modified as follows:

$$\int_0^{t_f} \{[x(t) - x_r(t)]^T Q[x(t) - x_r(t)\} dt = \int_0^{t_f} \tilde{x}(t)^T \tilde{Q}\tilde{x}(t)dt$$

$$\leq \tilde{x}(0)^T \tilde{P}\tilde{x}(0) + \rho^2 \int_0^{t_f} \tilde{w}(t)^T \tilde{w}(t)dt \tag{11}$$

where $\tilde{P}$ is a symmetric positive definite weighting matrix and

$$\tilde{Q} = \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix}.$$

3 H_∞ Tracking Control Design

The design purpose of this study is how to specify a fuzzy controller in (8) for the augmented system (10) with the guaranteed H_∞ tracking performance in (11) for all $w(t)$, and the output of system can follow the reference signal $r(t)$. Furthermore, the closed-loop system

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t))\tilde{A}_{ij}\tilde{x}(t) \tag{12}$$

is quadratically stable.

Let us choose a Lyapunov function for the system (10) as

$$V(t) = \tilde{x}(t)^T \tilde{P}\tilde{x}(t) \tag{13}$$

where the weighting matrix $\tilde{P} = \tilde{P}^T > 0$.
The time derivative of $V(t)$ is

$$\dot{V}(t) = \dot{\tilde{x}}(t)^T \tilde{P}\tilde{x}(t) + \tilde{x}(t)^T \tilde{P}\dot{\tilde{x}}(t) \tag{14}$$

By substituting (10) into (14), we get

$$\dot{V}(t) = \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t))(\tilde{A}_{ij}\tilde{x}(t) + \tilde{E}\tilde{w}(t))^T \tilde{P}x(t) + \tilde{x}^T \tilde{P}(\tilde{A}_{ij}\tilde{x}(t) + \tilde{E}\tilde{w}(t))$$

$$= \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t))\{\tilde{x}^T(\tilde{A}_{ij}^T\tilde{P} + \tilde{P}\tilde{A}_{ij})\tilde{x}(t)\} + \tilde{w}(t)^T \tilde{E}^T \tilde{P}x(t) + \tilde{x}^T \tilde{P}\tilde{E}\tilde{w}(t) . \tag{15}$$

Then, we get the following result.

Theorem 3.1 The augmented fuzzy system described by (10), if $\tilde{P} = \tilde{P}^T > 0$ is the common solution of the following matrix inequalities:

$$\tilde{A}_{ij}^T\tilde{P} + \tilde{P}\tilde{A}_{ij} + \frac{1}{\rho^2}\tilde{P}\tilde{E}\tilde{E}^T\tilde{P} + \tilde{Q} < 0 \tag{16}$$

for all $i, j = 1,2,\ldots,L$, then the H_∞ tracking performance in (11) is guaranteed for a prescribed ρ^2.

Proof: From (15), we get

$$\dot{V}(t) = \sum_{i=1}^{L}\sum_{j=1}^{L} h_i(z(t))h_j(z(t))\{\tilde{x}^T(\tilde{A}_{ij}^T\tilde{P} + \tilde{P}\tilde{A}_{ij})\tilde{x}(t)\} + [\tilde{x}^T \tilde{P}\tilde{E}\tilde{w}(t) + \tilde{w}^T \tilde{E}^T \tilde{P}\tilde{x}(t) \tag{4}$$

$$-\rho^2\tilde{w}(t)^T\tilde{w}(t)-\frac{1}{\rho^2}\tilde{x}^T\tilde{P}\tilde{E}\tilde{E}^T P\tilde{x}(t)]$$

$$+\frac{1}{\rho^2}\tilde{x}^T\tilde{P}\tilde{E}\tilde{E}^T\tilde{P}\tilde{x}(t)+\rho^2\tilde{w}(t)^T\tilde{w}(t)$$

$$\dot{V}(t)=\sum_{i=1}^{L}\sum_{j=1}^{L}h_i(z(t))h_j(z(t))\{\tilde{x}^T(\tilde{A}_{ij}^T\tilde{P}+\tilde{P}\tilde{A}_{ij})\tilde{x}(t)\}$$

$$-\left(\frac{1}{\rho}\tilde{E}^T\tilde{P}\tilde{x}(t)-\rho\tilde{w}(t)\right)^T\left(\frac{1}{\rho}\tilde{E}^T\tilde{P}\tilde{x}(t)-\rho\tilde{w}(t)\right)$$

$$+\frac{1}{\rho^2}\tilde{x}^T\tilde{P}\tilde{E}\tilde{E}^T P\tilde{x}(t)+\rho^2\tilde{w}(t)^T\tilde{w}(t)$$

$$\le\sum_{i=1}^{L}\sum_{j=1}^{L}h_i(z(t))h_j(z(t))\{\tilde{x}^T(\tilde{A}_{ij}^T\tilde{P}+\tilde{P}\tilde{A}_{ij}$$

$$+\frac{1}{\rho^2}\tilde{P}\tilde{E}\tilde{E}^T\tilde{P})\tilde{x}(t)\}+\rho^2\tilde{w}(t)^T\tilde{w}(t) \quad (17)$$

Note that the inequality (16) can be written as

$$\tilde{A}_{ij}^T\tilde{P}+\tilde{P}\tilde{A}_{ij}+\frac{1}{\rho^2}\tilde{P}\tilde{E}\tilde{E}^T\tilde{P}<-\tilde{Q}. \quad (18)$$

Therefore, from (17) and (18) we obtain

$$\dot{V}(t)\le\sum_{i=1}^{L}\sum_{j=1}^{L}h_i(z(t))h_j(z(t))\tilde{x}(t)^T(-\tilde{Q})\tilde{x}(t)$$

$$+\rho^2\tilde{w}^T\tilde{w}(t) \quad (19)$$

From the properties of $h_i(z(t))$ in (4), (19) can imply the following inequality:

$$\dot{V}(t)\le-\tilde{x}(t)^T\tilde{Q}\tilde{x}(t)+\rho^2\tilde{w}^T\tilde{w}(t) \quad (20)$$

Integrating (20) from $t=0$ to $t=t_f$ yields

$$V(t_f)-V(0)<-\int_0^{t_f}\tilde{x}(t)^T\tilde{Q}\tilde{x}(t)\,dt$$

$$+\rho^2\int_0^{t_f}\tilde{w}(t)^T\tilde{w}(t)\,dt \quad (21)$$

By substituting (13) into (21), we obtain

$$\int_0^{t_f}\tilde{x}(t)^T\tilde{Q}\tilde{x}(t)\,dt\le\tilde{x}(0)^T\tilde{P}\tilde{x}(0)$$

$$+\rho^2\int_0^{t_f}\tilde{w}(t)^T\tilde{w}(t)\,dt \quad (22)$$

This is (11) and the H_∞ tracking control performance is achieved with a prescribed ρ^2. This completes the proof.

The tracking control problem can be formulated as the following minimization problem to obtain better tracking performance:

$$\begin{aligned}&\mathit{min}\ \rho^2\\&\mathit{subject\ to}\ \tilde{P}>0\ \text{and (16)}.\end{aligned} \quad (23)$$

In the case $\tilde{w}(t)=0$, if the fuzzy controller (8) is employed in the closed-loop system (12) and there exists a positive definite matrix $\tilde{P}$ such that the matrix inequalities in (16) are satisfied, then the closed-loop system (12) is quadratically stable.

Proof: From (20) we obtain

$$\dot{V}(t)<-\tilde{x}(t)^T\tilde{Q}\tilde{x}(t)<0 \quad (24)$$

and from (15) we obtain

$$\dot{V}(t)=\sum_{i=1}^{L}\sum_{j=1}^{L}h_i(z(t))h_j(z(t))\{\tilde{x}^T(\tilde{A}_{ij}^T\tilde{P}+\tilde{P}\tilde{A}_{ij})\tilde{x}(t)\}$$

$$\ldots (25)$$

Therefore, the closed-loop system (12) is quadratically stable. This completes the proof.

To obtain the solution $\tilde{P}$ from the minimization problem in (23) is not easy. Fortunately, (23) can be transferred into the linear matrix inequalities problem (LMIP) [10]. The matrix inequalities in (16) are transformed to the equivalent LMIs by the following procedure.

For the convenience of design, we assume

$$\tilde{P}=\begin{bmatrix}\tilde{P}_{11} & 0\\ 0 & \tilde{P}_{22}\end{bmatrix}. \quad (26)$$

Substituting (26) into (16), we obtain

$$\begin{bmatrix}A_i+B_iK_j & -B_iK_j\\ 0 & A_r\end{bmatrix}^T\begin{bmatrix}\tilde{P}_{11} & 0\\ 0 & \tilde{P}_{22}\end{bmatrix}$$

$$+\begin{bmatrix}\tilde{P}_{11} & 0\\ 0 & \tilde{P}_{22}\end{bmatrix}\begin{bmatrix}A_i+B_iK_j & -B_iK_j\\ 0 & A_r\end{bmatrix}$$

$$+\frac{1}{\rho^2}\begin{bmatrix}\tilde{P}_{11} & 0\\ 0 & \tilde{P}_{22}\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & B_r\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & B_r\end{bmatrix}^T\begin{bmatrix}\tilde{P}_{11} & 0\\ 0 & \tilde{P}_{22}\end{bmatrix}$$

$$+\begin{bmatrix}Q & -Q\\ -Q & Q\end{bmatrix}<0 \quad (27)$$

This inequality can be written as

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} < 0 \tag{28}$$

where

$$S_{11} = (A_i + B_i K_j)^T \tilde{P}_{11} + \tilde{P}_{11}(A_i + B_i K_j) + \frac{1}{\rho^2}\tilde{P}_{11}\tilde{P}_{11} + Q$$

$$S_{12} = S_{21}^T = -\tilde{P}_{11} B_i K_j - Q$$

$$S_{22} = A_r^T \tilde{P}_{22} + \tilde{P}_{22} A_r + \frac{1}{\rho^2}\tilde{P}_{22} B_r B_r^T \tilde{P}_{22} + Q$$

By the Schur complements, (28) is equivalent to the following LMIs:

$$\begin{bmatrix} H_{11} & H_{12} & 0 \\ H_{21} & H_{22} & \tilde{P}_{22} B_r \\ 0 & B_r^T \tilde{P}_{22} & -\rho^2 I \end{bmatrix} < 0 \tag{29}$$

where

$$H_{11} = (A_i + B_i K_j)^T \tilde{P}_{11} + \tilde{P}_{11}(A_i + B_i K_j) + \frac{1}{\rho^2}\tilde{P}_{11}\tilde{P}_{11} + Q$$

$$H_{12} = H_{21}^T = -\tilde{P}_{11} B_i K_j - Q$$

$$H_{22} = A_r^T \tilde{P}_{22} + \tilde{P}_{22} A_r + Q$$

We can solve $\tilde{P}_{11}$, $\tilde{P}_{22}$, and K_j using the following two-step procedures [4]. First, find the solution of $H_{11} < 0$, we obtain K_j and $\tilde{P}_{11}$, then substituting them into (29) to obtain $\tilde{P}_{22}$.

In the first step, the solution of

$$(A_i + B_i K_j)^T \tilde{P}_{11} + \tilde{P}_{11}(A_i + B_i K_j) + \frac{1}{\rho^2}\tilde{P}_{11}\tilde{P}_{11} + Q < 0 \quad \ldots (30)$$

can be obtain by change of variables $Y_{11} = \tilde{P}_{11}^{-1}$ and $X_j = K_j Y_{11}$, then (30) is equivalent to the following inequality

$$Y_{11} A_i^T + A_i Y_{11} + B_i X_j + (B_i X_j)^T + \frac{1}{\rho^2} I + Y_{11} Q Y_{11} < 0 \quad \ldots (31)$$

By Schur complement, (31) is equivalent to the following LMIs:

$$\begin{bmatrix} Y_{11} A_i^T + A_i Y_{11} + B_i X_j + (B_i X_j)^T + \frac{1}{\rho^2} I & Y_{11} \\ Y_{11} & -Q^{-1} \end{bmatrix} < 0 \quad \ldots (32)$$

The parameters Y_{11} and X_j can be obtained by solving the LMIP in (32).

In the second step, by substituting $\tilde{P}_{11}$ and K_j into (29), then (29) become standard LMIs. Similarly, we can obtain $\tilde{P}_{22}$ by solving the LMIP in (29). If there exits positive definite solutions for $\tilde{P}$, then the closed-loop system is stable and the H∞ optimization design for fuzzy control system of (1) is formulated as the following optimization problem:

$$\min_{\{\tilde{P}_{11}, \tilde{P}_{22}\}} \quad \rho^2$$

$$\text{subject to } \tilde{P}_{11} = \tilde{P}_{11}^T > 0,\ \tilde{P}_{22} = \tilde{P}_{22}^T > 0 \text{ and (29).} \quad \ldots (33)$$

According to the analysis above, the fuzzy tracking control based on H∞ performance for nonlinear systems is summarized as follows.

Design Procedures:

1) Select membership functions and construct fuzzy plant rules in (1).
2) Given an initial attenuation level ρ^2.
3) Solve the LMIP in (32) to obtain Y_{11} and X_j (thus $\tilde{P}_{11}$ and K_j can also be obtained).
4) Substitute $\tilde{P}_{11}$ and K_j into (29) and then solve the LMIP in (29) to obtain $\tilde{P}_{22}$.
5) Decrease ρ^2 and repeat Steps 3–5 until positive definite solutions $\tilde{P}_{11}$ and $\tilde{P}_{22}$ can not be found.
6) Construct the fuzzy controller (8).

This minimization problem can be solved very efficiently by means of the Matlab LMI Toolbox software.

4 Application in Pendulum-Cart System

Consider the familiar pendulum-cart system experiment, found in many undergraduate control laboratories. The design objective of the application of the proposed fuzzy tracking control method that are the cart can track a sinusoidal reference signal, and H∞ performance is achieved for a prescribed ρ^2. The state equations of the pendulum-cart system including external disturbances are given by [11]

$$\dot{x}_1 = x_3 + w_1(t)$$
$$\dot{x}_2 = x_4 + w_2(t)$$
$$\dot{x}_3 = \frac{a(u - T_c - \mu x_4^2 \sin x_2) + l\cos x_2(\mu g \sin x_2 - f_p x_4)}{J + \mu l \sin^2 x_2}$$
$$\dot{x}_4 = \frac{l\cos x_2(u - T_c - \mu x_4^2 \sin x_2) + \mu g \sin x_2 - f_p x_4}{J + \mu l \sin^2 x_2} \quad \ldots (34)$$

where $a = l^2 + (J/(m_c + m_p))$, $\mu = l(m_c + m_p)$, x_1 denotes the cart position (m), x_2 denotes the angle of the pendulum from the vertical (rad), x_3 is the cart velocity (m/s), and x_4 is the pendulum angular velocity (rad/s), g = 9.8 m/s^2 is the gravity constant, m_p is the mass of the pendulum (kg), m_c is the mass of the cart (kg), l is the distance from the axis of rotation to the centre of mass of the pendulum-cart system, J is the moment of inertia of the pendulum-cart system with respect to the centre of mass, F is the force applied to the cart (N), T_c is the friction force, and f_p is the pendulum friction constant (kg·m^2/s), $w_1(t)$ and $w_2(t)$ are the bounded external disturbances.

The pendulum-cart system parameters used for simulation and experiment are m_c=1.12 kg, m_p= 0.12 kg, J= 0.0135735 kg·m^2, l = 0.01679 m, and f_p=0.000107 kg·m/s [11]. All pendulum frictions are considered to be negligible.

The T-S fuzzy model for the nonlinear system in (34) is given by the following three-rule fuzzy model:

R$_1$: If $x_2(t)$ is F_1 (about 0 rad)
Then $\dot{x}(t) = A_1 x(t) + B_1 u(t) + w(t)$
$y(t) = Cx(t)$

R$_2$: If $x_2(t)$ is F_2 ($\pm\pi$/15 rad)
Then $\dot{x}(t) = A_2 x(t) + B_2 u(t) + w(t)$
$y(t) = Cx(t)$

R$_3$: If $x_2(t)$ is F_3 ($\pm\pi$/7.5 rad)
Then $\dot{x}(t) = A_3 x(t) + B_3 u(t) + w(t)$
$y(t) = Cx(t)$

where

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.2524 & 0 & -0.0001 \\ 0 & 15.0319 & 0 & -0.0079 \end{bmatrix};\ B_1 = \begin{bmatrix} 0 \\ 0 \\ 0.8272 \\ 1.2370 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.2298 & 0 & -0.0001 \\ 0 & 14.6544 & 0 & -0.0079 \end{bmatrix};\ B_2 = \begin{bmatrix} 0 \\ 0 \\ 0.8263 \\ 1.2086 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.1664 & 0 & -0.0001 \\ 0 & 13.5581 & 0 & -0.0079 \end{bmatrix};\ B_3 = \begin{bmatrix} 0 \\ 0 \\ 0.8237 \\ 1.1253 \end{bmatrix}$$

$C = \mathrm{diag}\,(1,1,1,1)$

and $w(t) = [w_1(t), w_2(t), w_3(t), w_4(t)]^T$.

Membership functions for Plant Rules 1–3 are

$$F_1(x_2(t)) = \left\{1.0 - \frac{1.0}{1.0 + e^{-80[x_2(t) - \pi/30]}}\right\} \cdot \frac{1.0}{1.0 + e^{-80[x_2(t) + \pi/30]}}$$

$$F_2(x_2(t)) = \left\{1.0 - \frac{1.0}{1.0 + e^{-80[(x_2(t) - (\pi/15)) - \pi/30]}}\right\} \cdot \frac{1.0}{1.0 + e^{-80[(x_2(t) - (\pi/15)) + \pi/30]}}$$

$$F_3(x_2(t)) = \frac{1.0}{1.0 + e^{-80[(x_2(t) - (\pi/7.5)) + \pi/30]}}\ .$$

The external disturbances in (35) are $w_1(t) = w_2(t) =$ 0.01 sin(0.4πt), and $w_3(t) = w_4(t) = 0$.

The reference model is given as

$$\dot{x}_r(t) = A_r x_r(t) + B_r r(t)$$

where

$$A_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{bmatrix};\ B_r = \begin{bmatrix} 0 \\ -3.1 \\ 0 \\ 4.2 \end{bmatrix}$$

and the reference signal $r(t)$ = 0.1 sin(0.2πt).

Select $Q = 10^{-5}$ diag (50, 50, 50, 50). The optimal ρ^2 = 0.75 is found after several iterations using the LMI optimization toolbox in Matlab [12]. In this case, we obtain the solution for (33) as follows:

$$\tilde{P}_{11} = \begin{bmatrix} 0.0041 & -0.0131 & 0.0036 & -0.0033 \\ -0.0131 & 0.0621 & -0.0156 & 0.0155 \\ 0.0036 & -0.0156 & 0.0042 & -0.0040 \\ -0.0033 & 0.0155 & -0.0040 & 0.0040 \end{bmatrix}$$

$$\tilde{P}_{22} = \begin{bmatrix} 120.7562 & 8.7187 & 89.6012 & 6.4900 \\ 8.7187 & 14.4529 & 6.4987 & 10.6842 \\ 89.6012 & 6.4987 & 67.8523 & 4.9439 \\ 6.4900 & 10.6842 & 4.9439 & 8.1367 \end{bmatrix}$$

and

$$K_1 = [55.2760 \quad -310.4392 \quad 69.6191 \quad -79.1678]$$

$$K_2 = [55.1650 \quad -310.7216 \quad 69.5844 \quad -79.3040]$$

$$K_3 = [54.2518 \quad -309.2168 \quad 68.8104 \quad -79.1144].$$

Therefore, we obtain the control law

$$u(t) = \sum_{j=1}^{3} h_j(x_2(t)) K_j (x(t) - x_r(t)).$$

For comparison, the simulations are made with the fuzzy tracking controller using stabilizing compensator structure (FTC) [13]. We apply the fuzzy tracking control systems to the original system (34). The simulation program is realized by Matlab/Simulink.

Simulations results are depicted in Figs. 1 and 2. The results indicate that the output (cart position) of the tracking control system can follow the reference signal $r(t)=0.1\ \sin(0.2\pi t)$. The system can also stabilize the pendulum in the upright position. The time response of the cart required to follow the reference signal for the system with the proposed fuzzy tracking controller (H-infinity) is shorter than that of the FTC. The settling time of the response of pendulum angle converging back to the equilibrium for the system with the proposed fuzzy tracking controller is faster than that of the FTC. It can be concluded that the proposed fuzzy tracking controller has better tracking performance than that of the FTC.

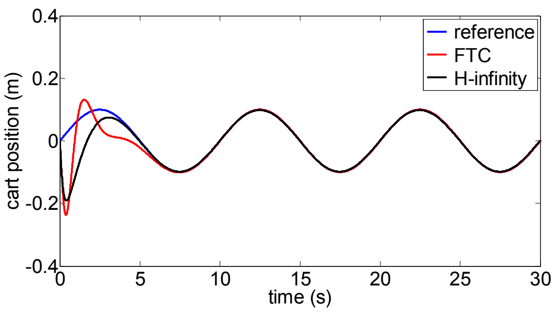

Fig.1. Simulation responses of the cart position of the pendulum-cart system.

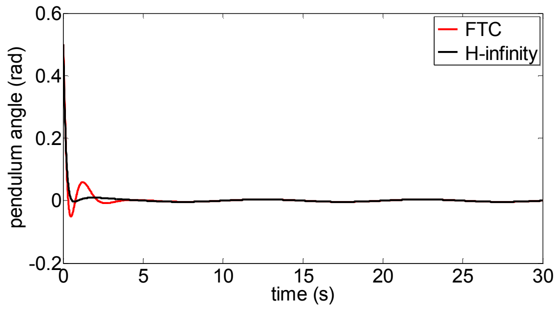

Fig.2. Simulation responses of the pendulum angle of the pendulum-cart system.

A real-time experiment on the pendulum-cart system from The Feedback Instrument Ltd. is conducted to verify the performance of the proposed fuzzy tracking control system using the experimental setup depicted in Fig. 3. The control system is performed using Matlab/Simulink with Real-Time Workshop on a personal computer with the 16-bit AD/DA converter. Two differentiators with 100 s^{-1} cutoff frequency are used for the cart velocity and the pendulum angular velocity calculations. All external disturbances in (34) are removed.

Figs. 4 and 5 show the experiment results using the initial condition $x(0)$ = (0, about 0.3 rad, 0, 0). These results are almost a replica of the simulation results. It can be concluded that the responses of the proposed control system met the designed criteria, i.e. the cart can follow the sinusoidal reference signal and the pendulum is stable in upright position, and H_∞ performance is achieved for a prescribed ρ^2. However, the control signal of the proposed tracking controller is larger than that of the FTC (Fig. 5). This is an issue of the implementation of the proposed method that should be considered.

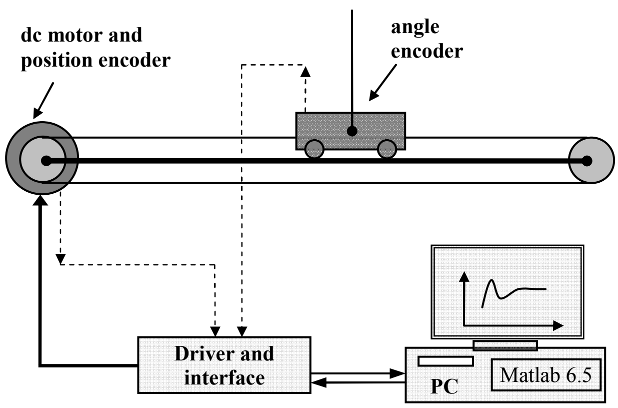

Fig.3. Experimental setup for the pendulum-cart system.

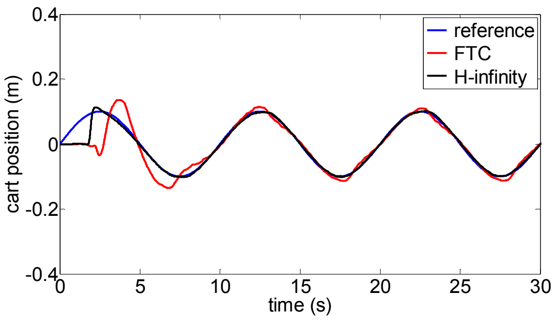

Fig.4. Experiment responses of the cart position of the pendulum-cart system.

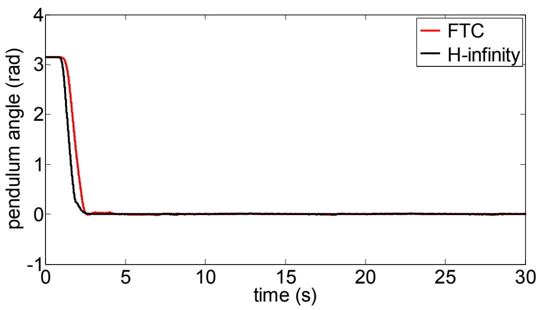

Fig.5. Experiment responses of the pendulum angle of the pendulum-cart system.

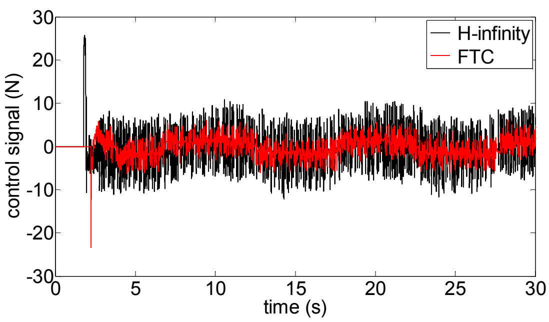

Fig.6. Experiment responses of the control signal of the pendulum-cart system.

In order to indicate the robustness of the control system designed, a disturbance, parameter variation, and measurement noise are applied to the system. To avoid the component failure of the control system, the magnitude of a disturbance on the control input, the standard deviation of the measurement noise, and the variation of system parameter must be applied with awareness.

A disturbance $d(t)$ (see Fig. 7) is used in the experiment. Figs. 8-9 show that the pendulum-cart system with control signal in Fig. 10 is robust to the disturbance with only a little deviation for the pendulum in a short period. It can be seen that the cart can not track the reference signal during the disturbance is applied.

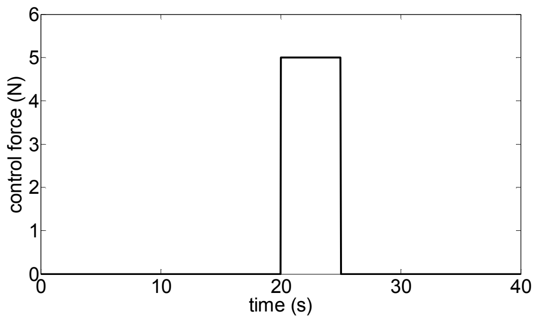

Fig.7. Disturbance $d(t)$.

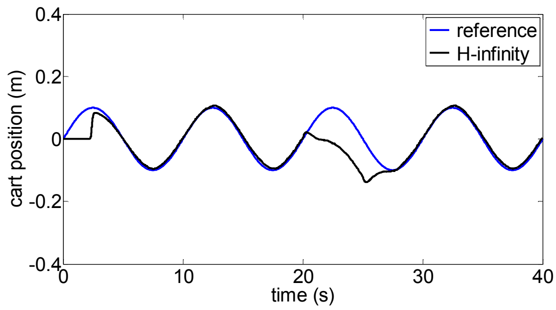

Fig.8. The response of the cart position of the system when $d(t)$ is included.

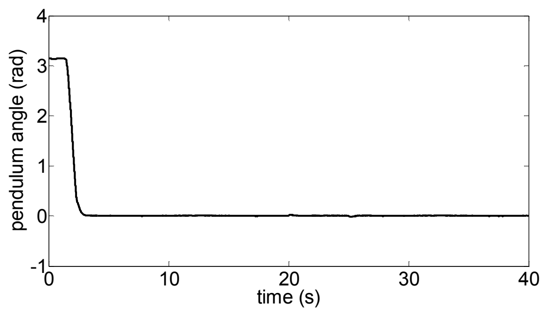

Fig.9. The response of the pendulum angle of the system when $d(t)$ is included.

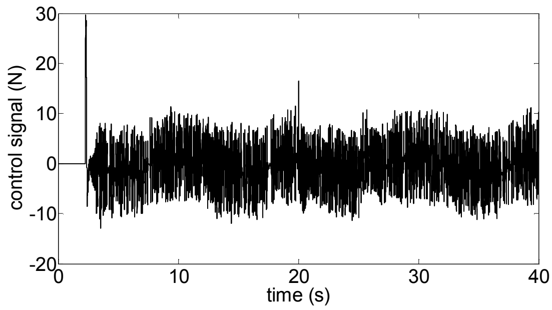

Fig.10. The control signal of the system when disturbance is included.

The cart mass is changed to be m_c = 1.34 kg, which is +20% variation of the nominal cart mass. Figs. 11-13 show the experiment results. The reference signal is tracked by the cart with only a little shift. It observed that the tracking controller can balance the pendulum in the upright position. The control signal is almost the same magnitude as the control signal of the system without additional mass (Fig. 6).

A noise on the cart position measurement, v_1, is assumed to be zero-mean white noise with standard deviation equals 0.1%. Figs. 14-16 show the response of the cart position, the pendulum angle, and the control signal. The cart can follow the reference signal with a slight shift, and a slight deviation on the steady state response of the pendulum angle is also observed. These good performances must be compensated with large control signal (Fig. 16).

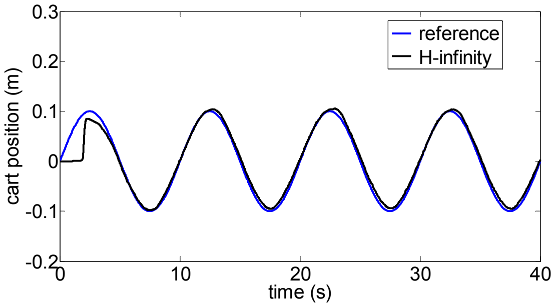

Fig.11. The cart position response when the cart mass varied from its nominal value.

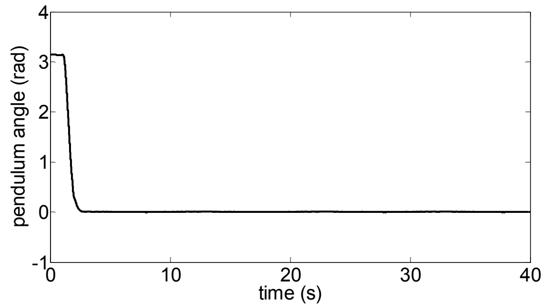

Fig.12. The pendulum angle response when the cart mass varied from its nominal value.

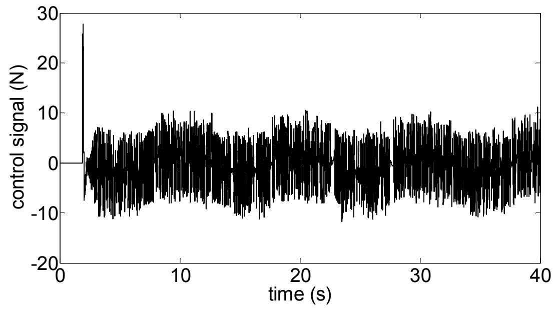

Fig.13. The control signal when the cart mass varied from its nominal value.

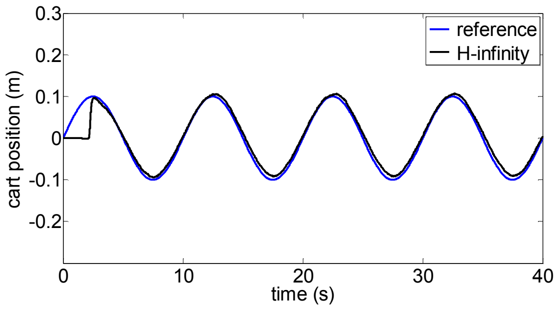

Fig.14. The cart position of the control system with measurement noise v_1.

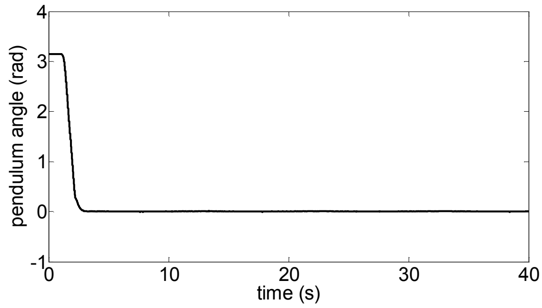

Fig.15. The pendulum angle of the control system with measurement noise v_1.

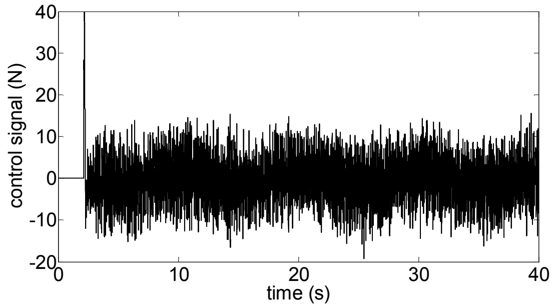

Fig.16. The control signal of the control system with measurement noise v_1.

Figs. 17-19 show the responses of the tracking control system when a white noise (zero-mean, 0.25% of standard deviation) v_2 is applied to the pendulum measurement. It can be observed that the performance of the control system meet the design objectives, i.e. the cart can track the reference signal and the pendulum stabilize in the upright position. However, the control signal is too large. It might damage the actuator in the system.

From the robustness test of the proposed fuzzy tracking control, we can conclude that the performance of the system still satisfy the design objective with slight performance degradation.

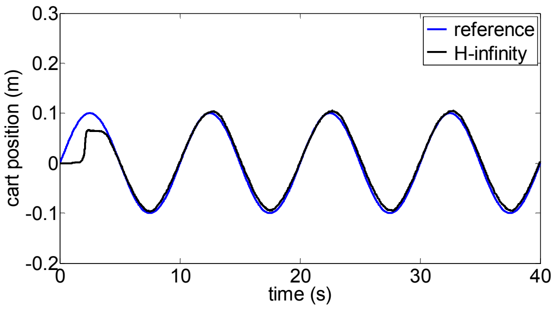

Fig.16. The cart position of the control system with measurement noise v_2.

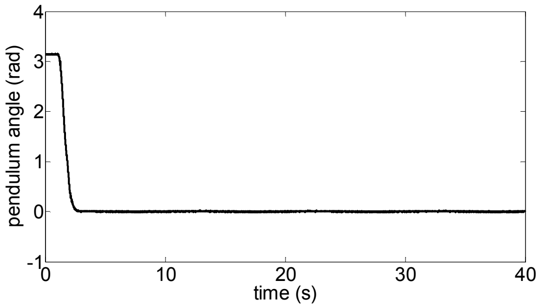

Fig.18. The cart position of the control system with measurement noise v_2.

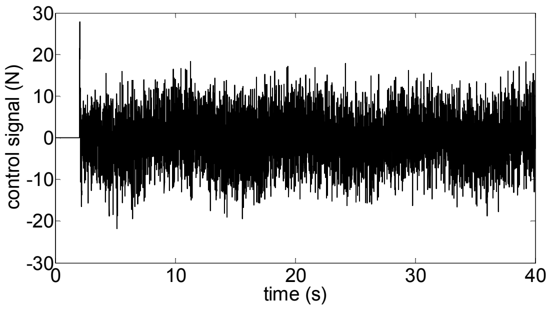

Fig.19. The control signal of the control system with measurement noise v_2.

5 Conclusion

In this paper we have presented a systematic design method of fuzzy tracking control system based on H_∞ tracking performance for nonlinear systems. Based on the T-S fuzzy model, the control system is developed to make the system output able to track a reference signal by minimizing the attenuation level ρ^2. The stability of the closed-loop nonlinear systems is also discussed in this paper. By employing the H_∞ attenuation technique, the performance of the fuzzy tracking control design for nonlinear systems can be improved.

The fuzzy tracking control problem is parameterized in terms of a LMIP. The LMIP can be solved efficiently by LMI optimization toolbox in Matlab. An application on the pendulum-cart system is given to illustrate the design procedures. Simulation and experiment results show that the desired performance for nonlinear systems can be achieved via the proposed tracking control method.

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