Robust stability of linear fractional systems with non linear uncertainties

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Abstract :-Using the differentiation and integration of fractional order or non-integer order in systems control is gaining more and more interest from the systems control community.
In the present document, we study the stability of multivariable fractional systems using robust fractional controller. Some important mathematical inequalities and particularly the Gronwall inequality are employed to investigate the robust stability conditions which are based to non linear bounded uncertainties.

Key-words: Fractional system, robust stability, controller, Gronwall Bellman, inequality, multivariable system

1 Introduction

During the last decades, researches have shown that new fractional-order models are much more appropriate than the previously used integer-order ones; the given fundamental physical considerations are more in favour of using models that rely on derivations of no integer order. Many systems can be described with the help of fractional derivatives. These systems are known to display fractional-order dynamics: electrochemistry, electromagnetism and electrical machines, thermal systems and heat conduction, transmission and acoustics, viscoelastics materials and robotics. The fractional systems exhibit hereditarily properties and long memory transients, which can be represented more accurately by fractional order models.

In the particular domain of control theory, several authors have been interested by this aspect since the 1960s. The first contributions [2], [24], [30] provided generalizations of classical analysis methods for fractional order systems (transfer function, frequency response, pole and zero analysis, etc...).

In this field, there are many challenges and unsolved problems.

The stability analysis and stability proof of fractional order control system [19], [29] are still consider yet as open problem this is due to the fact that existing theory developed so for stability proof manly exists for integer order and generally is not applicable to fractional order control system [33].

In this paper, we generalize the results presented in [3] about robust linear integer controller design, for fractional order controller.

The robust fractional controller is introduced for the multivariable dynamic fractional system with parametrical as well as structural linear or non linear time varying model uncertainties. The parameters of a dynamic controller are selected to satisfy the requirement of robust stability under plant uncertainties. The Gronwall lemma is used to investigate the robust stability conditions which are based on the upper norm bounds of the uncertainties. T he proof of standard Gronwall theorem and some useful lemma have been presented.

The paper is organized as follow: in the section 2, we recall some concepts of the fractional differential system's theory; we present also some important mathematical inequalities and particularly the theorem of Gronwall-Bellman. The section 3, presents the problem of multivariable system bounded uncertainties. A theorem of robust stability is then proposed in section 4 and its demonstration is performed in section 5. Section 6 presents the conclusion of the paper.

2. Preliminary on Fractional Differential Systems

In this section, we recall the main definitions and results concerning fractional systems. We also present the theorem of Gronwall-Bellman.

2-1 Fractional derivative definition

The fractional calculus is a generalization of integration and derivation to non-integer order operators. The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695.

At first, we generalize the differential and integral operators into one fundamental operator ${}_aD_t^\alpha$ which is known as fractional calculus:

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & Re(\alpha)>0, \\ 1 & Re(\alpha)=0, \\ \int_a^t (d\tau)^{-\alpha} & Re(\alpha)<0. \end{cases} \quad (1) $$

Formulations of noninteger-order derivatives fall in two main classes: Grünwald definition and the Riemann-Liouville(RL) definition on one hand. The Grünwald definition is given here:

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh) \quad (2) $$

where $[x]$ means the integer part of x. the (RL) definition is given as:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \quad (3) $$

For $(n-1<\alpha<n)$ and $\Gamma(x)$ is the well known Euler's gamma function.

Or the Caputo derivative on the other, defined as:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{\frac{d^n f(\tau)}{dt^n}}{(t-\tau)^{\alpha-n+1}} d\tau \quad (4) $$

for $(n-1<\alpha<n)$

The physical interpretation of the fractional derivatives and the solutions of fractional differential equation was given in [20],[30]. Here and throughout, only the caputo definition is used since its intial conditions take on the same form as for the integer order differential equation.

The Laplace transform method is used for solving engineering problems. The formula for the Laplace transform of the caputo fractional derivative (4) has the form:

$$ \int_0^\infty e^{-st} {}_0D_t^\alpha f(t)dt = = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} {}_0D_t^k f(t)\Big|_{t=0} \quad (5) $$

for $(n-1<\alpha\le n)$.

For simplicity, we will note D^α for ${}_aD_t^\alpha$.

For numerical calculation of fractional-order derivation we can use the relation (5) derived from the Grünwald definition (2). This relation has the following form:

$$ {}_{(t-L)}D_t^\alpha f(t) \approx h^{-\alpha} \sum_{j=0}^{N(t)} b_j f(t-jh) \quad (6) $$

where L is the "memory length", h is the step size of the calculation,

$$ N(t) = min\left\{\left[\frac{t}{h}\right], \left[\frac{L}{h}\right]\right\} $$

$[x]$ is the integer part of x and b_j is the binomial coefficient:

$$ b_0 = 1, \quad b_j = \left(1 - \frac{1+\alpha}{j}\right) b_{j-1} $$

For the solution of the fractional- order differential equations (FODE) most effective and easy analytic methods were deployed based on formula of the Laplace transform method of the Mittag-Leffler function in two parameters. A two parameter function of Mittag-Leffler type is defined by the series expansion:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad \alpha>0,\beta>0 \qquad (7)$$

Where

$$E_{\alpha,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}\equiv E_{\alpha}(z)$$

is the one-parameter Mittag-Leffler function.

The Mittag-Leffler function is a generalization of exponential function e^z and the exponential function is particular case of the Mittag-Leffler function:

$$E_{1,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z$$

The Laplace transform of the two parameter Mittag-Leffler function is:

$$\int_0^{\infty} e^{-st}t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(at^{\alpha})dt=\frac{s^{\alpha-\beta}k!}{(s^{\alpha}-a)^{k+1}} \qquad (8)$$

where

$$E_{\alpha,\beta}^{(k)}=\frac{d^k}{dt^k}E_{\alpha,\beta}.$$

2-2 Some important mathematical inequalities:

2-2-1Lemma 1 [4]:

for $\alpha<1, z\in C$:

$$L_{\frac{1}{\alpha}}(z)=Re(z^{\frac{1}{\alpha}}) \text{ if} \left|Arg\ z\right|\le\frac{\pi}{2}\alpha \qquad (9)$$

$$L_{\frac{1}{\alpha}}(z)=0 \text{ if } \frac{\pi}{2}\alpha<\left|Arg\ z\right|\le\pi$$

In particular, $L_{\frac{1}{\alpha}}\ge 0$. Moreover, for every $\varepsilon>0$, there is a constant $C_{\varepsilon}>0$ such that

$$\left\|E_{\alpha,1}(z)\right\|\le C_{\varepsilon}\exp(L_{\frac{1}{\alpha}}(z)+\varepsilon\left|z\right|^{\frac{1}{\alpha}}) \qquad (10)$$

Where $L_{\frac{1}{\alpha}}$ means the regularized Lindel of indicator function of $E_{\alpha,1}$ with respect to the order $\frac{1}{\alpha}$.

2-2-2 Lemma 2 [25]:

There exist finit real constant $K_{\alpha,\alpha}>1$ such that for any $0<\alpha<1$ and A a constant matrix:

$$E_{\alpha,\alpha}(At^{\alpha})\le K_{\alpha,\alpha}\left\|exp(At)\right\| \qquad (11)$$

2-3 Gronwall inequality [6]:

Integral inequalities play an important role in the qualitative analysis of the solutions to differential and integral equations. The celebrate known now as Gronwall – Bellman- Raid inequality provided explicit bounds on solutions of a class of linear integral inequalities. On the basis of various motivations, this inequality has been extended a used in various context like the fractional system. This inequality plays a useful role such as of the problem of control and stability.

Firstly, let's present the classical Gronwall Bellman.

2-3-1 Theorem 1:

If

$$x(t)\le h(t)+\int_{t_0}^{t}k(s)x(s)ds, \qquad t\in\left[t_0,T\right) \qquad (12)$$

where all the functions involved are continuous on $\left[t_0,T\right)$, $t\le\infty$, and $k(t)\ge 0$, then $x(t)$ satisfies:

$$x(t)\le h(t)+\int_{t_0}^{t}h(s)k(s)exp\left[\int_{s}^{t}k(u)du\right]ds, \quad t\in\left[t_0,T\right) \qquad (13)$$

If, in addition, $h(t)$ is nondecreasing, then

$$x(t)\le h(t)exp\left(\int_{t_0}^{t}k(s)ds\right), t\in\left[t_0,T\right). \qquad (14)$$

Secondly, the generalized Gronwall Bellman with Riemann-Liouville fractional derivatives is presented as follows.

2-3-2 Theorem 2*:*

Suppose $\alpha > 0$, $a(t)$ is a nonnegative function and locally integrable on $0 \le t < T$ (some $T \le +\infty$) and $g(t)$ is a nonnegative, nondecreasing, continuous function defined on $0 \le t < T, g(t) \le M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \le t < T$ with

$$u(t) \le a(t) + g(t)\int_0^t (t-s)^{\alpha-1} u(s)ds \qquad (15)$$

on the interval. Then :

$$u(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds \qquad (16)$$

$$0 \le t < T$$

This inequality has a close connection to the Riemann-Liouville derivative. It also can be used to estimate the bound of the Lyapunov exponents for both the Riemann-Liouville fractional differential systems and the Caputo ones.

Proof of theorem 2:

Let

$$B\varphi(t) = g(t)\int_1^t (ln\frac{t}{s})^{n\alpha-1} \varphi(s)\frac{ds}{s}, t \ge 1,$$

for locally integrable functions φ. Then

$$u(t) \le a(t) + Bu(t)$$

Iterating the inequality, one has:

$$u(t) \le \sum_{k=0}^{n-1} B^k a(t) + B^n u(t).$$

In the following, we should prove that

$$B^n u(t) \le \int_1^t \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (ln\frac{t}{s})^{n\alpha-1} u(s)\frac{ds}{s} \qquad (17)$$

holds, and $B^n u(t) - +\infty$ for each t in $1 \le t < T$. Obviously, the relation (17) holds as $n = 1$; suppose it holds for some $n = k$, if $n = k+1$ then one has:

$$B^{k+1}u(t) = B(B^k u(t))$$

$$\le g(t)\int_1^t (ln\frac{t}{s})^{\alpha-1}$$

$$\left[\int_1^s \frac{((g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (ln\frac{s}{\tau})^{k\alpha-1} u(\tau)\frac{d\tau}{\tau} \right] \frac{ds}{s}$$

Under the condition that $g(t)$ is nondecreasing, one obtain:

$$B^{k+1}u(t)$$

$$\le (g(t))^{k+1}\int_1^t (ln\frac{t}{s})^{\alpha-1} \left[\int_1^s \frac{\Gamma(\alpha)^k}{\Gamma(k\alpha)} (ln\frac{s}{\tau})^{k\alpha-1} u(\tau)\frac{d\tau}{\tau} \right] \frac{ds}{s}$$

By interchanging the order of integration, one has:

$$B^{k+1}u(t)$$

$$\le (g(t))^{k+1}\int_1^t \left[\int_\tau^t \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} (ln\frac{t}{s})^{\alpha-1} (ln\frac{s}{\tau})^{k\alpha-1} \frac{ds}{s} \right] u(\tau)\frac{d\tau}{\tau}$$

$$= \int_1^t \frac{(g(t)\Gamma(\alpha))^{k+1}}{\Gamma((k+1)\alpha)} (ln\frac{t}{s})^{(k+1)\alpha-1} u(s)\frac{ds}{s}$$

where the integral

$$\int_\tau^t (ln\frac{t}{s})^{\alpha-1} (ln\frac{s}{\tau})^{k\alpha-1} \frac{ds}{s}$$

$$= (ln\frac{t}{\tau})^{k\alpha+\alpha-1} \int_0^1 (1-z)^{\alpha-1} z^{k\alpha-1} dz$$

$$= (ln\frac{t}{\tau})^{(k+1)\alpha-1} B(k\alpha, \alpha)$$

$$= \frac{\Gamma(\alpha)\Gamma(k\alpha)}{\Gamma((k+1)\alpha)} (ln\frac{t}{\tau})^{(k+1)\alpha-1}$$

is obtained in terms of the $ln\, s = ln\, \tau + z\, ln\frac{t}{\tau}$. Therefore, the relation (17) holds. Moreover, since:

$$B^n u(t) \le \int_1^t \frac{(M\Gamma(\alpha))^n}{\Gamma(n\alpha)}(ln\frac{t}{s})^{n\alpha-1}u(s)\frac{ds}{s} \to 0$$

as $n \to +\infty$, for $t \in [1,T)$.

This completes the proof.

2-3-3 Corollary 1
If $a(t)$ be a nondecreasing function on $[0,T]$, then

$$u(t) \le a(t)E_\beta(g(t)\Gamma(\beta)t^\beta) \quad (18)$$

where E_β is the Mittag Leffler function defined by:

$$E_\beta(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\beta+1)}$$

Proof:
We have:

$$u(t) \le a(t) + \int_0^t\left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\right]ds$$

and $a(t)$ is a nondecreasing function on $[0,T]$, then

$$u(t) \le a(t)\left[1+\int_0^t\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}ds\right]$$

So

$$u(t) \le a(t)\sum_{n=0}^{\infty}\frac{(g(t)\Gamma(\beta)t^\beta)^n}{\Gamma(n\beta+1)}$$

Finally

$$u(t) \le a(t)E_\beta(g(t)\Gamma(\beta)t^\beta)$$

3 Problem statements

Consider the following multivariable dynamic system with parametrical uncertainties:

$$\begin{cases} D^\alpha x = Ax+Bu+\Delta A(x)+\Delta B(u) \\ y = Cx+Du+\Delta C(x)+\Delta D(u), \quad 0<\alpha<1 \\ x(0)=x_0 \end{cases} \quad (19)$$

where $x(t) \in IR^n$ is the state vector, $u(t) \in IR^m$ is the input vector, $y(t) \in IR^p$ is the output vector, and A, B, C and D are constant matrices. $\Delta A(x), \Delta B(u), \Delta C(x)$ and $\Delta D(u)$ are nonlinear time-varying parametrical uncertainties with the following known upper norm-bounds.

$$\begin{matrix} \|\Delta A(x)\| \le \delta_1\|x\|, & \|\Delta B(u)\| \le \delta_2\|u\|, \\ \|\Delta C(x)\| \le \delta_3\|x\|, & \|\Delta D(u)\| \le \delta_4\|u\| \end{matrix} \quad (20)$$

where $\delta_1, \delta_2, \delta_3$ and δ_4 are constant positives.

The norm of real vector $x \in R^n$, denoted by $\|x\|$ is defined as :

$$\|x\| = \sum_{i=1}^{n}|x_i|$$

And the induced matrix norm corresponding to the vector norm is given as:

$$\|A\| = \max_j \sum_{i=1}^{n}|A_{ij}|$$

where $x_i, i = 1,2,...,n$ denotes the element of the vector x and $A_{ij}, i,j = 1,2,....,n$ denotes the entries of matrix A.

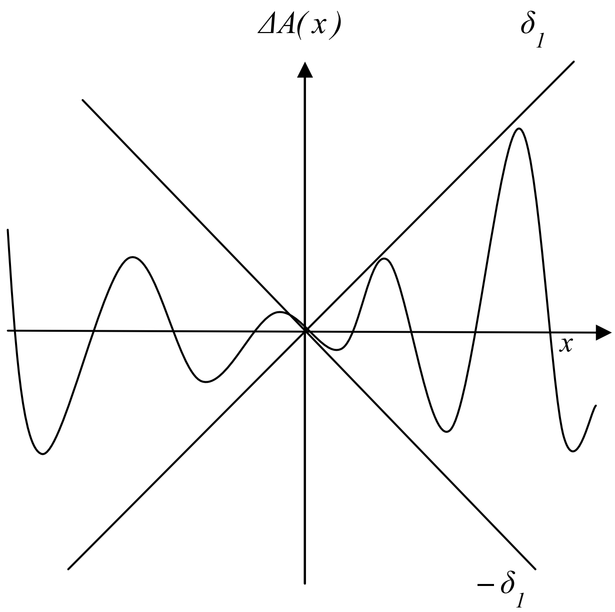

Fig.1

This system (19) yields an approximate dynamic system as follows:

$$\begin{cases} D^{\alpha} x = Ax + Bu \\ y = Cx + Du \qquad\qquad 0 < \alpha < 1 \\ x(0) = x_0 \end{cases} \tag{21}$$

Thus, system (21) represents the "approximate model" of the plant. Without loss of generality, we will assume that (A, B) is controllable and (C, A) is observable.

If the plant uncertainties are unknown and cannot be expressed as parametrical uncertainties in (19) but can be described by the input-output form as a structural model error with an upper bound, then:

$$\begin{cases} D^{\alpha} x = Ax + Bu \\ y = Cx + Du + \Delta h(u), 0 < \alpha < 1 \\ x(0) = x_0 \end{cases} \tag{22}$$

where the structural model uncertainty Δh is nonlinear and bounded

$$\|\Delta h(u(t))\| \leq r\|u(t)\|$$

where r is positive constant.

In this note the dynamic controller has the following structure:

$$\begin{cases} D^{\alpha} x_{\ell} = A_{\ell} x_{\ell} + B_{\ell} y \\ u(t) = L_{\ell} x_{\ell} \end{cases} \qquad 0 < \alpha < 1 \tag{23}$$

where $x_{\ell} \in IR$ and A_{ℓ}, B_{ℓ} and L_{ℓ} are constant matrices with appropriate dimensions.

Thus our problems are formulated as follow:

Problem 1: **Robust stabilization of fractional system (19).** The first design problem is to choose the parameters A_{ℓ}, B_{ℓ} and L_{ℓ} in the dynamic controller (23) such that the closed-loop system with uncertainties of (19) and (23) is asymptotically stable, the uncertainties can be tolerated in our design and the controller is a robust controller.

Problem 2: **Robust stabilization of fractional system (22).** The second problem is to choose the dynamic controller (23) such that the closed-loop system with uncertainties of (22) and (23) is asymptotically stable.

4 The main results:

4-1 Problem 1: Robust stabilization of fractional system (19)

The closed-loop system with nonlinear parametrical uncertainties is described by (19) and (23). Combining (19) and (23) we get:

$$\begin{bmatrix} D^{\alpha} x \\ D^{\alpha} x_{\ell} \end{bmatrix} = \begin{bmatrix} A & BL_{\ell} \\ B_{\ell} C & A_{\ell} + B_{\ell} DL_{\ell} \end{bmatrix} \begin{bmatrix} x \\ x_{\ell} \end{bmatrix} + \begin{bmatrix} \Delta A(x) + \Delta B(u) \\ B_{\ell}(\Delta C(x) + \Delta D(u)) \end{bmatrix}$$

$$y = \begin{bmatrix} C & DL_{\ell} \end{bmatrix} \begin{bmatrix} x \\ x_{\ell} \end{bmatrix} + \Delta C(x) + \Delta D(u) \tag{24}$$

where $0 < \alpha < 1$.

We define

$$\bar{x} = \begin{bmatrix} x \\ x_{\ell} \end{bmatrix}, \quad D^{\alpha} \bar{x} = \begin{bmatrix} D^{\alpha} x \\ D^{\alpha} x_{\ell} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & BL_{\ell} \\ B_{\ell} C & A_{\ell} + B_{\ell} DL_{\ell} \end{bmatrix} \tag{25}$$

$$\Delta\bar{A}(\bar{x}) = \begin{bmatrix} \Delta A(x) + \Delta B(u) \\ B_{\ell}(\Delta C(x) + \Delta D(u)) \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & DL_{\ell} \end{bmatrix}, \quad \Delta\bar{C}(\bar{x}) = \begin{bmatrix} \Delta C(x) + \Delta D(u) \end{bmatrix} \tag{26}$$

Then, the closed-loop system with parametrical uncertainties can be represented as:

$$\begin{cases} D^{\alpha} \bar{x} = \bar{A}\bar{x} + \Delta\bar{A}(\bar{x}) \\ y = \bar{C}\bar{x} + \Delta\bar{C}(\bar{x}) \end{cases} \qquad \bar{x}(0) = \begin{bmatrix} x_0 \\ x_{\ell_0} \end{bmatrix}, \quad 0 < \alpha < 1 \tag{27}$$

And the approximate closed feedback system is given by:

$$\begin{cases} D^{\alpha}\bar{x} = \bar{A}\bar{x} \\ y = \bar{C}\,\bar{x} \end{cases}$$
$$\bar{x}(0) = \begin{bmatrix} x_0 \\ x_{\ell_0} \end{bmatrix} \quad , \quad 0 < \alpha < 1 \tag{28}$$

The following theorem gives a solution to the problem of the uncertain fractional system (19).

4-1-1 Theorem 3

In problem 1, we suppose the nonlinear parametrical uncertainties are bounded by (20) and if we choose the control parameters of (23) such the following inequality is satisfied:

$$(H\Gamma(\alpha))^{\frac{1}{\alpha}}(1+\varepsilon') + \bar{\varpi} < 0 \tag{29}$$

H, ε' and $\bar{\varpi}$ are defined after in the proof of theorem.

Then, the nonlinear parametrical perturbed closed-loop system (27) is asymptotically stable.

4-1-2 Proof of theorem 3

Using the Laplace transform to the system (27), we obtain:

$$\bar{X}(s) = \left(I_n s^{\alpha} - \bar{A}\right)^{-1}\left(s^{\alpha-1}\bar{x}_0 + L(\Delta\bar{A}(\bar{x}))\right) \tag{30}$$

Then, by applying the opposite transform of Laplace of equation (30), obtained on the one hand by the opposite transform of the function of Mittag-Liffler with two parameters, and on the other hand, by using the integral of convolution, we obtain the following equality:

$$\bar{x}(t) = E_{\alpha,1}(\bar{A}t^{\alpha})\bar{x}_0 + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\bar{A}(t-\tau)^{\alpha})\Delta\bar{A}(\bar{x}(\tau))d\tau \tag{31}$$

we put

$$\bar{x}_1(t) = E_{\alpha,1}(\bar{A}t^{\alpha})\bar{x}_0$$

and

$$\bar{x}_2(t) = \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\bar{A}(t-\tau))\Delta\bar{A}(\bar{x}(\tau))d\tau$$

So

$$\|\bar{x}(t)\| \le \|\bar{x}_1(t)\| + \|\bar{x}_2(t)\| \tag{32}$$

Using (20) we obtain:

$$\begin{aligned} &\|\Delta\bar{A}(\bar{x}(\tau)\| \\ &\le \delta_1\|x(\tau\| + \delta_2\|u(\tau)\| + \|B_\ell\|(\delta_3\|x(\tau)\| + \delta_4\|u(\tau)\|) \\ &\le \left(\delta_1 + \delta_3\|B_\ell\|\right)\|x(\tau\| + \left(\delta_2 + \delta_4\|B_\ell\|\right)\left\|\begin{bmatrix}0 & L_\ell\end{bmatrix}\right\|\|\bar{x}(\tau)\| \\ &\le \left(\delta_1 + \delta_3\|B_\ell\| + \left(\delta_2 + \delta_4\|B_\ell\|\right)\|L_\ell\|\right)\|\bar{x}(\tau)\| \end{aligned} \tag{33}$$

we put :

$$R = \left(\delta_1 + \delta_3\|B_\ell\| + (\delta_2 + \delta_4\|B_\ell\|)\|L_\ell\|\right)$$

So

$$\|\Delta\bar{A}\bar{x}(\tau)\| \le R\|\bar{x}(\tau)\| \tag{34}$$

Using the lemma 2, we obtain:

$$\begin{aligned} &\exists K_{\alpha,\alpha} > 1, \\ &\|E_{\alpha,\alpha}(\bar{A}(t-\tau)^{\alpha}\| \le K_{\alpha,\alpha}\|exp(\bar{A}(t-\tau))\| \end{aligned} \tag{35}$$

Then

$$\|\bar{x}_2(t)\| \le RK_{\alpha,\alpha}\int_0^t (t-\tau)^{\alpha-1}\,exp(\bar{A}(t-\tau))\bar{x}(\tau)d\tau$$

And we know that $\exists M > 1, \omega \in IR$

$$\|exp(\bar{A}(t-\tau))\| \le M\,exp(\omega(t-\tau)) \quad ,$$

So

$$\|\bar{x}_2(t)\| \le H\int_0^t (t-\tau)^{\alpha-1}exp(\omega(t-\tau))\|\bar{x}(\tau)\|d\tau, \tag{36}$$

$$H = MRK_{\alpha\alpha}$$

Using the lemma 1 we obtain

$$\|E_{\alpha,1}(\bar{A}t^{\alpha})\| < C_{\varepsilon}\,exp\left(L_{\alpha}\left(\|\bar{A}\|t^{\alpha}\right) + \varepsilon\left(\|\bar{A}\|t^{\alpha}\right)^{\frac{1}{\alpha}}\right) \tag{37}$$

$$\|E_{\alpha,1}(\bar{A}t^{\alpha})\| < C_{\varepsilon}\,exp(\omega' t), \qquad \omega' = \|\bar{A}\|^{\frac{1}{\alpha}}(1+\varepsilon)$$

So

$$\|\bar{x}_1(t)\| \le C_{\varepsilon}\|\bar{x}_0\|exp(\omega' t) \tag{38}$$

From (35) and (37) we obtain:

$$\begin{aligned} &\|\bar{x}(t)\| \le F\,exp(\bar{\omega}t) + H\int_0^t (t-\tau)^{\alpha-1}exp(\bar{\omega}(t-\tau))\|\bar{x}(\tau)\|d\tau, \\ &F = C_{\varepsilon}\|\bar{x}_0\|, \bar{\omega} = sup(\omega', \omega) \end{aligned}$$

By multiply the both side of inequality, we obtain

$$\|\bar{x}(t)\| exp(-\bar{\omega}t) \le F + + H\int_0^t (t-\tau)^{\alpha-1} exp(-\bar{\omega}t)\|\bar{x}(\tau)\| d\tau \quad (39)$$

By applying the the lemma 1 of Gronwal Bellman , we have:

$$\|\bar{x}(t)\| exp(-\bar{\omega}t) \le FE_{\alpha,1}(H\Gamma(\alpha))t^{\alpha})$$

Using the lemma 1, we find:

$$\|\bar{x}(t)\| exp(-\bar{\omega}t) \le FC_{\varepsilon'} \exp\left((H\Gamma(\alpha))^{\frac{1}{\alpha}}(1+\varepsilon')t\right)$$

$$\|\bar{x}(t)\| \le FC_{\varepsilon'} \exp\left((H\Gamma(\alpha))^{\frac{1}{\alpha}}(1+\varepsilon')+\bar{\omega}\right)t \quad (40)$$

So the system is asymptotically stable if:

$$(H\Gamma(\alpha))^{\frac{1}{\alpha}}(1+\varepsilon')+\bar{\omega} < 0$$

4-2 Problem 2: Robust stabilization of fractional system (22)

The fractional system (22) combined with (23) give the following system:

$$\begin{cases} D^{\alpha}\bar{x} = A\bar{x} + \begin{bmatrix} 0 \\ B_{\ell}\Delta h(u) \end{bmatrix} \\ \bar{x}(0) = \begin{bmatrix} x_0 \\ x_{\ell_0} \end{bmatrix} \qquad ,0<\alpha<1 \\ y = \bar{C}\bar{x} + \Delta h(u) \end{cases} \quad (41)$$

4-2-1 Theorem 4

In problem 2, we suppose that the structural model uncertainty Δh is nonlinear and bounded as:

$$\|\Delta h(u(t))\| \le r\|u(t)\|$$

where r is positive constant, and if we choose the control parameters of (23) such the following inequality is satisfied:

$$(H'\Gamma(\alpha))^{\frac{1}{\alpha}}(1+\varepsilon'')+\bar{\omega} < 0 \quad (42)$$

H', ε'' and $\bar{\omega}$ are defined after in the proof of theorem.

Then, the nonlinear parametrical perturbed closed-loop system (41) is asymptotically stable.

4-2-2 Proof of theorem 4

Using the Laplace transform to system (42), we obtain:

$$\bar{x}(s) = (I_1 s^{\alpha} - \bar{A})^{-1}(s^{\alpha-1}\bar{x}_0 + L\begin{bmatrix} 0 \\ B_{\ell}\Delta h(u(t)) \end{bmatrix} \quad (43)$$

Then,by applying the opposite transform of Laplace of equation (42), we obtain the following equality:

$$\bar{x}(t) = E_{\alpha,1}(\bar{A}t^{\alpha})\bar{x}_0 + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\bar{A}(t-\tau)^{\alpha})\begin{bmatrix} 0 \\ B_{\ell}\Delta h(u(\tau)) \end{bmatrix} d\tau$$

Then

$$\|\bar{x}(t)\| \le \|E_{\alpha,1}(\bar{A}t^{\alpha})\|\|\bar{x}_0\| + \int_0^t (t-\tau)^{\alpha-1}\|E_{\alpha,\alpha}(\bar{A}(t-\tau)^{\alpha}\|\|B_{\ell}\|\|\Delta h(u(t))\| d\tau$$

We have

$$\|\Delta h(u(\tau))\| \le r\|u(\tau)\| \le r\|[0 \quad L_{\ell}]\bar{x}(\tau)\| \le r\|L_{\ell}\|\|\bar{x}(\tau)\|$$

So

$$\|\bar{x}(t)\| \le \|E_{\alpha,1}(\bar{A}t^{\alpha})\|\|\bar{x}_0\| + \int_0^t r(t-\tau)^{\alpha-1}\|E_{\alpha,\alpha}(\bar{A}(t-\tau)^{\alpha}\|\|B_{\ell}\|\|L_{\ell}\|\bar{x}(\tau)$$

We put $R' = r\|B_{\ell}\|\|L_{\ell}\|$

Using the same demonstration as theorem1, we obtain:

$$\|\bar{x}(t)\| \le FC_{\varepsilon''} \exp\left((H'\Gamma(\alpha))^{\frac{1}{\alpha}}(1+\varepsilon'')+\bar{\omega})t\right)$$

With

$$H' = MR'K_{\alpha,\alpha}$$

So the system is asymptotically stable if:

$$(H'\Gamma(\alpha))^{\frac{1}{\alpha}}(1+\varepsilon'')+\bar{\omega} < 0$$

5 Conclusion

We presented a robust fractional controller for the multivariable fractional system with linear, nonlinear, or time varying model uncertainties. Our work is simplified to choose the dynamical control parameters to satisfy the conditions of robust stability which are based on the upper norm bounds of the uncertainties. Two important theorems were proposed and its demonstrations were performed using the Gronwall inequality.

More general multivariable fractional system should be considered in the future work with another form of uncertainties.

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