

Geometrical Model of Plant Presented on a State Surface of a Complex Error.

Oleg N. Agamalov
Electrical Department
Tashlyk Pump-Storage Power Plant
st. Drugby Narodov 33/152, Yushnoukrainsk, Mykolayv region, 55002
UKRAINE
aon@ukrsat.mk.ua, o_agamalov@sunpp.atom.gov.ua

Abstract: - in paper considered a geometric approach for presentation the mathematical model of a control plant, based on calculation a complex error, allowing define an optimal control law of a plant as an equation of the geodesic curve on a state surface of a complex error.

Key-Words: - control system, Hilbert transform, complex error of control, state surface of a complex error, geodesic curve.

1 Introduction

Analysis and synthesis of modern control systems are directed on controllers development, allowing to provide the required quality of the work for plant, which structure and parameters can be uncertain or are certain not precisely. Discrepancy of plant representation arises owing to impossibility of measurement its all state variables at various conditions and restrictions. For nonlinear plants, by development of their control systems, also following basic features [1] should be considered:

- the superposition principle is not carried out, and research of nonlinear system at several influences cannot be reduced to research as sum of single influence;
- transient stability depend at value of an initial deviation from equilibrium point;
- for the fixed external influences some positions of stability are possible.

The possible decision a problem at design of control system for nonlinear plants with unknown structure and parameters can be reached if to use only input and output signals of control system, having described the control law as the equation of a geodesic curve (further geodesic) on state surface S_e , which geometrical properties are defined by the control error, depending at plant structure and parameters. For one-dimensional plant or plant with single input and single output (*SISO*), without dependence from physical principles of its functioning, the state surface of a complex control error it is define in the Cartesian coordinates system.

The real axis R of this Cartesian coordinates system coincides with a vector of an input signal (the reference r), and imaginary axis I is displaced on an angle $\pi/2$ and physically defines delay in a plant [2]. Accordingly, for multi-dimension plant or plant with multi inputs and multi outputs (*MIMO*) the set of state surfaces of the complex errors defined between demanded pairs of vectors of input signals m and output signals n is formed. The general dimension of *MIMO* plant state space is defined by all surfaces of conditions for considered pairs "input-output". Any point \bar{e}_0 of complex error state surface S_e may be defined as position of radius-vector, which a difference between an input vector or reference \bar{r}_0 and output vector of plant \bar{y}_0 in the entered Cartesian coordinates system: $\bar{e}_0 = \bar{r}_0 - \bar{y}_0$. In offered work the geometrical mathematical model of the *SISO* plant presented on a state surface of a complex error. The given material includes sections with the following maintenance. In the second section are resulted definitions of a state surface (for *SISO* plant) and state space (for *MIMO* plant) of control system complex error and a measurement method of a complex error with use a Hilbert transform are presented. Opportunity of calculation the real and imaginary components of a complex error in the Cartesian coordinates system, and also instant phase and the module of a complex error in the polar coordinates system, which connected with entered Cartesian coordinates system is shown. Statement of problems the analysis, identification and optimum control of plant in parameters of internal geometry of a complex error state surface is

considered. In the third section the model of plant on a complex error state surface is presented. The stability conditions of control system and condition of optimal control, defining geodesic on a complex error state surface are formulated. In the fourth section the example of the controller, using only the real component of a complex error by plant of the first order with a transport delay are presented. Also, an adaptive and robust property of such controller is shown. In conclusion, the basic directions of the further researches for development the industrial controllers, using representation on a state surface and in a state space of a complex error are considered, and also the substantive provisions presented in given paper are reflected.

2 State Surface and State space of a Complex Error.

Description of control system depends from a choice the state variables and a coordinates system in which they are considered. Modern control system, for optimal control, their maintenance adaptive and robust properties should be based on the algorithms, which are not requiring the detailed aprioristic information about plant, and capable to execute identification, structural and parametrical optimization of a controller, being based on measurement only their inputs and outputs. Therefore important to note, that work of control system depends on exact knowledge not only that occurs in it, i.e. from values of input and output, but also from when it happens, i.e. from time (phase) correlation between changes of input and output [3]. Structure of controller is defined by an opportunity to installation the sensors, measuring the state variables of plant and allowing to generate the necessary feedbacks. Traditional representation of control system with feedback is shown on fig. 1,

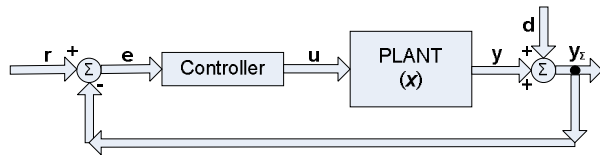


Fig.1. Control system with feedback.

where r - reference vector, $e = r - y_z$ - control error vector, u - input vector, y - output vector, d - disturbances vector, y_z - measured output vector.

So as the feedback of control system is defined by interaction of their separate elements, there is a phase delay between output and input (reference). For the control, based only on the information about

correlation between changes of values and phases of input and output of control system, let us enter the definition of a complex error:

Definition 1. *Complex error e* of a control system is the difference between an input (reference) vector r and an output vector y_z , presented on a complex plane $R - I$, where axis R is certain by a direction of an input (reference) vector r , and axis I is displaced on an angle $\pi/2$ in the Cartesian coordinates system, that allows to consider delay (displacement) of an output vector y_z of a plant concerning an input (axis R). The complex error can be presented as $e = r - y_z = e_R + je_I$, where $j = \sqrt{-1}$ - imaginary unit, e_R and e_I - the real and imaginary components of a complex error in the chosen Cartesian coordinates system. Phase delay $\Delta\phi$ of control system, real e_R and imaginary e_I components of a complex error e in the Cartesian coordinates system, and also the module e and argument δ of a complex error in the polar coordinates system, that displaced concerning chosen Cartesian coordinates system on an input (reference) vector r , are defined according to the geometrical correlation, fig. 2:

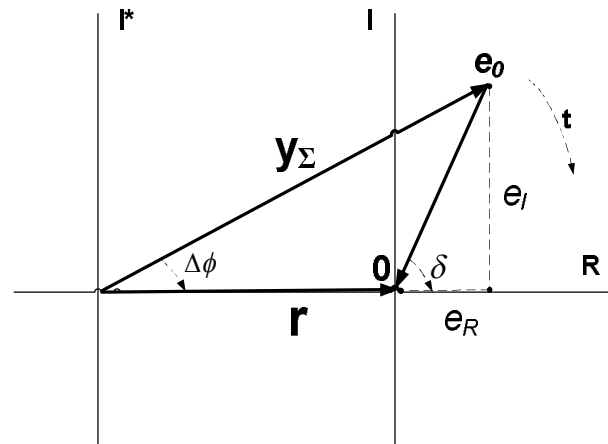


Fig.2. Complex error of a control system.

$$\begin{aligned} \Delta\phi &= \operatorname{tg}^{-1} \left(\frac{e_I}{r + e_R} \right) = \sin^{-1} \left(\frac{e_I}{y_z} \right); \\ \delta &= \sin^{-1} \left[\frac{y_z \cdot \sin(\Delta\phi)}{r^2 + y_z^2 - 2 \cdot r \cdot y_z \cdot \cos(\Delta\phi)} \right] \\ e^2 &= r^2 + y_z^2 - 2 \cdot r \cdot y_z \cdot \cos(\Delta\phi); \\ e_R &= y_z \cdot \cos(\Delta\phi) - r = e \cdot \cos \delta; \\ e_I &= y_z \cdot \sin(\Delta\phi) = e \cdot \sin \delta, \end{aligned} \quad (1)$$

where r - module of an input (reference) vector, y_z - module of an output vector of a plant, $\Delta\phi$ - the phase delay of control system, defined in the Cartesian coordinates system $R - I^*$, δ - argument of a complex error in the polar coordinates system.

Phase delay $\Delta\varphi$ of a control system can be defined with use of a Hilbert transform [4], that map input and output as the analytical signals. The analytical signal represents a sum of two orthogonal signals, which components are shifted on a phase on $\pi/2$ and for which the instant phase and frequency [5] can be defined. The imaginary part of an analytical signal $Z_s(t)$ is in a complex interfaced to its real part $\text{Re} Z_s(t) = s(t)$ and defined the Hilbert transform (HT): $\text{Im} Z_s(t) = \tilde{s}(t) = HT[s(t)]$. Accordingly the analytical signal is defined as:

$$\begin{aligned} Z_s(t) &= s(t) + j \cdot \tilde{s}(t) = S(t)e^{ja(t)}, \\ S(t) &= \sqrt{s^2(t) + \tilde{s}^2(t)}, a(t) = tg^{-1} \frac{\tilde{s}(t)}{s(t)} \end{aligned} \quad (2)$$

Hilbert transform implements the turn of initial phases for all frequency components of a signal on $\pi/2$ and at $-\infty < t < \infty$ is assign by convolution $s(t)$ with function $HT(t) = 1/(\pi t)$:

$$\tilde{s}(t) = HT[s(t)] = s(t) * \frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{t - \tau} d\tau \quad (3)$$

Function $\frac{1}{t-\tau}$ called as a core of a Hilbert transform, and the integral in the equation (3) is defined by a Cauchy principal value. The instant phases difference [6] for two signals $s_1(t)$ and $s_2(t)$ can be define with use of a Hilbert transform as:

$$\Delta\phi_{12} = \phi_1(t) - \phi_2(t) = t g^{-1} \frac{\tilde{s}_1(t) \cdot s_2(t) - s_1(t) \cdot \tilde{s}_2(t)}{s_1(t) \cdot s_2(t) + \tilde{s}_1(t) \cdot \tilde{s}_2(t)} \quad (4)$$

For *SISO* control system:

$$\Delta\phi_{yr} = \phi_y(t) - \phi_r(t) = t g^{-1} \frac{\tilde{y}(t) \cdot r(t) - y(t) \cdot \tilde{r}(t)}{y(t) \cdot r(t) + \tilde{y}(t) \cdot \tilde{r}(t)} \quad (5)$$

For *MIMO* control system:

$$\begin{aligned} \Delta\phi_{mn} &= \phi_{y_n(t)}(t) - \phi_{r_m(t)}(t) = \\ &= t g^{-1} \frac{\tilde{y}_n(t) \cdot r_m(t) - y_n(t) \cdot \tilde{r}_m(t)}{y_n(t) \cdot r_m(t) + \tilde{y}_n(t) \cdot \tilde{r}_m(t)} \end{aligned} \quad (6)$$

where m – inputs of control system, n - outputs of control system.

Definition 2. State surface of a complex error S_e of SISO control system is set origins of vectors a complex error e , which coordinates are defined concerning an input (reference) vector r in the Cartesian coordinates system, introduced in a definition 1.

Let us define as \mathbf{e}_θ a vector of the complex error of control system that to be formed owing to influence of disturbances in an initial moment of time t_0 , and through $\boldsymbol{\varepsilon}(\mathbf{0})$ – an area of reachability for all vectors of a complex error \mathbf{e}_i with a centre $\mathbf{0}$ in the end of transient by duration t_l . Viewing a change of origin coordinates a vector of a complex error \mathbf{e}_θ in time $t_0 \leq t \leq t_l$, we shall get a curve on a state surface of the complex error \mathcal{S}_e insert in the three-dimensional space \mathbf{R}^3 in which the reference point $\mathbf{0}$ with an area of reachability $\boldsymbol{\varepsilon}(\mathbf{0})$ is introduced, and a basis $(\mathbf{e}_R, \mathbf{e}_I, t)$, which in aggregate to define the Cartesian coordinates system. Any point of this state surface defines a state of *SISO* control system. The curve on a state surface of a complex error is defined by vector function $\mathbf{e}(t) = \mathbf{e}(e_R(t), e_I(t))$ or equations system: $e_R = e_R(t)$, $e_I = e_I(t)$. In view of the introduced extra coordinate - time t , arrivals of control system to area of reachability $\boldsymbol{\varepsilon}(\mathbf{0})$, parameters of a complex error will be defined through the measured values input and output of control system and an angle $\Delta\varphi$ between them as:

$$e_0^2 = r_0^2 + y_0^2 - 2 \cdot r_0 \cdot y_0 \cdot \cos(\Delta\phi) = e_{R0}^2 + e_{I0}^2 + \Delta t^2;$$

$$\delta_0 = \sin^{-1} \left(\frac{\sqrt{y_0^2 \cdot \sin^2(\Delta\varphi) - \Delta t^2}}{e_0} \right); \quad (7)$$

$$e_{R0}^2 = e_0^2 - y_0^2 \cdot \sin^2(\Delta\varphi) = e_0^2 \cdot \cos^2 \delta_0 - \Delta t^2;$$

$$e_{I0}^2 = y_0^2 \cdot \sin^2(\Delta\varphi) - \Delta t^2 = e_0^2 \cdot \sin^2 \delta_0;$$

where $\Delta t = (t_1 - t_0)$ - achievement time of control system the area of reachability $\varepsilon(\theta)$, which from system (7) should not exceed an estimation $\Delta t \leq y_0 \sin(\Delta \varphi)$. The state surface of a complex error is shown on fig. 3,

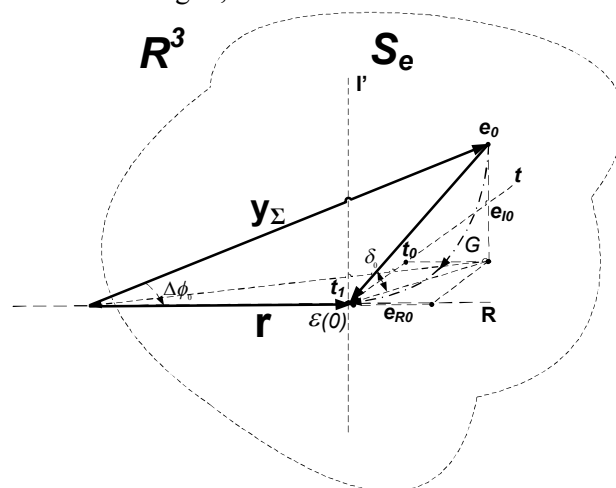


Fig. 3. State surface of a complex error \mathbf{S}_e , described a complex error \mathbf{e}_0 ((e_{R0}, e_{I0}) or (e_0, δ_0)) at time t_0 to a point, defining necessary state $\mathbf{e}(0)$ (area of reachability) of a control system on state surface at time t_1 .

Definition 3. State space of a complex error M_e of MIMO control system is formed by set of surfaces (leaves) according to definition 2. For everyone m - n surfaces of complex error S_{emn} for MIMO plant are defined the Cartesian coordinates systems of concerning the general for all surfaces area of reachability $\varepsilon(0)$. The complex error $e_{mn}(t)$ of MIMO control system is defined as a difference vector between an input $r_m(t)$ and an output $y_n(t)$ according to (6) and (7). Thus, using definitions 1-3 for MIMO plant, probably to display it $R^{(m \cdot n)}$ -dimensional state space in set $(m \cdot n)$ surfaces (leaves), considered in

coordinates (e_R, e_I, t) , that is $R^{(m \cdot n)} \rightarrow \sum_1^{m \cdot n} R^3$.

Further statement and possible decision of problems of the analysis, identification and optimal control of SISO plant in parameters of internal geometry of a state surface of a complex error will be considered.

For control system, which consider on a state surface of a complex error can be define the next properties:

1. Reachability - an opportunity of movement the control system at control $u^*(t)$ from a point of a current state of the plant, described a complex error e_0 to the area (point) defining the demanded state of the plant on a state surface of a complex error $\varepsilon(0)$, where ε -vicinity defines an admissible control error;
2. Observability - for each value of a complex error e_i exists, at least, one control input u_i which allows to distinguish given state of a control system between all nearest states e_j ;

The given properties can be defining, if the state surface and state space of a complex error are normed metric [7], and for them following axioms are carried out:

$$\begin{aligned}
 1. & \rho(e_1, e_2) = \|e_1 - e_2\|, \forall e_1, e_2 \in S_e(M_e) \\
 2. & \|e\| \geq 0, \rho(e_1, e_2) \geq 0, \forall e_1, e_2 \in S_e(M_e), \\
 & \|e\| = 0 \Leftrightarrow e = 0, \rho(e_1, e_2) = 0 \Leftrightarrow e_1 = e_2; \\
 3. & \rho(e_1, e_2) = \rho(e_2, e_1), \forall e_1, e_2 \in S_e(M_e); \\
 4. & \|\alpha \cdot e_1\| = |\alpha| \cdot \|e_1\|, \forall e_1 \in S_e(M_e), \forall \alpha \in R; \\
 5. & \|e_1 + e_2\| \leq \|e_1\| + \|e_2\|, \\
 & \rho(e_0, e_1) \leq \rho(e_0, e_2) + \rho(e_2, e_1), \\
 & \forall e_0, e_1, e_2 \in S_e(M_e);
 \end{aligned} \tag{8}$$

where: $\|\cdot\|$ - complex error norm, ρ - distance between points on a state surface S_e or in a state spaces M_e of a complex error.

Properties and parameters of a state surface of a complex error, which do not change at its bending (deformations without compression and stretching), name internal geometry of a surface and define with use of the first I and second II fundamental forms [8]. To such parameters of a surface to carry: lengths of curves, angles between curves, square of areas on a surface, Gaussian curvature of a surface and geodesic curvature of a curve on a surface. For plant control let us used the curve G of minimal length - **geodesic** [8] between points e_0 and $\varepsilon(0)$, laying on a state surface S_e . Then **the analysis** of control system based on a complex error to be concluded in definition the internal geometry and, accordingly, parameters of I and II fundamental forms of a state surface. The problem of **synthesis** of a control system based on a complex error consists in definition the law of optimum control as the equation of the geodesic curve connecting a signal of control with the equation geodesic, provided that given geodesic connected the boundary points e_0 and $\varepsilon(0)$. Thus, the problem of control to be reduced to definition of movement the radius-vector of a complex error e on a state surface from a boundary point of the initial, not certain value e_0 , depending from the disturbances influencing on plant, in the beginning of the entered coordinates system (the area of reachability $\varepsilon(0)$) at minimal time and minimal oscillations along a trajectory of movement. We assume, that the state surfaces of a complex error are one-coherent, suppose continuous bending and have constant Gaussian curvature $K = const$. Curves on them are smooth and suppose differentiation necessary number of times, except for the beginning of coordinates with a vicinity $\varepsilon(0)$. Identification of plant based on characteristics of its state surface of a complex error, using an isometry [8], i.e. transformation of one surface to another with preservation of length, angles and square of areas. For this purpose, the existing state surface of a complex error will be transform to other surface having more simple metrics, but equal I fundamental form of the connected surfaces, that can be considered as:

$$I = E \cdot de_R^2 + 2 \cdot F \cdot de_R \cdot de_I + G \cdot de_I^2 \tag{9}$$

where its gains are defined as partial derivatives of a complex error for the real and imaginary components:

$$\begin{aligned}
 E &= \frac{\partial \bar{e}}{\partial e_R} \cdot \frac{\partial \bar{e}}{\partial e_R} = e_{e_R e_R}^2, F = \frac{\partial \bar{e}}{\partial e_R} \cdot \frac{\partial \bar{e}}{\partial e_I} = e_{e_R} \cdot e_{e_I}, \\
 G &= \frac{\partial \bar{e}}{\partial e_I} \cdot \frac{\partial \bar{e}}{\partial e_I} = e_{e_I e_I}^2
 \end{aligned} \tag{10}$$

The regularity condition of a state surface of a complex error is defined by an inequality [8]

$$E \cdot G - F^2 > 0 \quad (11)$$

For definition 2 components of a complex error e_R and e_I is orthogonal that $F = 0$. Therefore a condition of regularity $E \cdot G > 0 \Rightarrow \frac{\partial^2 \bar{e}}{\partial e_R^2} \cdot \frac{\partial^2 \bar{e}}{\partial e_I^2} > 0$ it

is carried out for all state surface of a complex error, except for the beginning $\varepsilon(0)$. The I fundamental form describes a state surface of a complex error as a first approximation when the small area of a surface is replaced on an area of a tangent plane. The II fundamental form describes a surface in the second approximation, showing as the surface deviates at a tangent plane, i. e. defines curvature of a state surface of a complex error:

$$II = 2 \cdot h = L \cdot de_R^2 + 2 \cdot M \cdot de_R \cdot de_I + N \cdot de_I^2 \quad (12)$$

where h - distance between a point e_0 surfaces S_e with coordinates $(e_{R0} + de_R, e_{I0} + de_I)$ and a point e_T of a tangent plane T with coordinates (e_{R0}, e_{I0}) in view of a sign, depending on an arrangement of a point e_T . Gains of a II fundamental form are defined as a scalar product:

$$L = (\bar{e}_{e_R e_R}, \bar{n}), M = (\bar{e}_{e_R e_I}, \bar{n}), N = (\bar{e}_{e_I e_I}, \bar{n}) \quad (13)$$

where \bar{n} - the ort of a normal to a state surface of a complex error in the given point. Classification of points (structure of point vicinity) [8] of a state surface of a complex error taking into account gains of II fundamental form can be used for identification of plant in small («differential identification»):

1) point e_0 of a regular state surface of a complex error S_e be called *elliptic* (accordingly plant of elliptic type), if discriminant of II fundamental form in this point $LN - M^2 > 0$;

2) point e_0 of a regular state surface of a complex error S_e be called *hyperbolic* (accordingly plant of hyperbolic type), if discriminant of II fundamental form in this point $LN - M^2 < 0$;

3) point e_0 of a regular state surface of a complex error S_e be called *parabolic* (accordingly plant of parabolic type), if discriminant of II fundamental form in this point $LN - M^2 = 0$ at $L^2 + N^2 \neq 0$;

4) point e_0 of a regular state surface of a complex error S_e be called *planar*, if in this point $L = N = M = 0$. For a planar point of a state surface of a complex error the plant cannot be identified, as in its vicinity not probably to define change of a complex error on a state surface concerning a tangent plane.

Curvature of a state surface of a complex error is defined by structure and parameters of plant and it is proportional to the disturbances influencing on control system. The curve γ on a state surface of a complex error S_e is geodesic only if the main normal in its each point coincides with a normal to surface S_e in the given point. This corresponds to equality to zero of its geodesic curvature in each point [8]. Thus through any point of a regular state surface of a complex error in any direction can pass exactly one geodesic [8], that can be used for unequivocal definition of control. For this purpose on regular surface S_e in a small vicinity of a point e_0 it is possible to define a semigeodesic coordinates system [8] in which coordinate lines of various families in pairs is orthogonal and one of families consists the geodesic. Let us construct a semigeodesic coordinates system (SCS) in a point e_0 , for what:

- let us define a point e_0 as beginning of SCS on a state surface of a complex error and set an any direction d ;
- lead through a point e_0 in a direction d geodesic γ_1 and to define on it a direction of detour;
- each point e_i on a geodesic γ_1 is uniquely defined by length of an arch $\xi = \tilde{e}_0 \tilde{e}_i$, taken with a sign “+”, if the direction of an arch coincides with a direction of detour and with a sign “-” in a return case;
- through each point e_i on a geodesic γ_1 and to orthogonal it, let us lead to direct geodetic γ_2 . Orientation of a geodetic γ_2 continuously depends on a point e_i and family of geodetic γ_2 for various points e_i geodetic γ_1 do not intercross among themselves;
- each point e_j of a state surfaces of a complex error, through which there transits one of geodetic γ_2 , is uniquely defined the length of an arch $\eta = \tilde{e}_i \tilde{e}_j$ with the certain sign +/-, and coordinates (ξ, η) uniquely define position of a point e_j on a state surface of a complex error in entered SCS concerning a point e_0 .

Thus, the length ξ is a natural parameter [8] on γ_1 , and length η on γ_2 and both are counted from e_0 .

Then I fundamental form of a state surface of a complex error in entered SCS is defined as:

$$ds^2 = d\xi^2 + G(\xi, \eta)d\eta^2 \quad (14)$$

where $G > 0, G(0, \eta) = 1, G_\eta(0, \eta) = 0$. If to accept $\xi \equiv e_R$ and $\eta \equiv e_I$, we shall receive:

$$de^2 = de_R^2 + G(e_R, e_I)de_I^2 \quad (15)$$

Expression for Gaussian curvature in given SCS

$$K = \frac{(\sqrt{G})_{\eta\eta}}{\sqrt{G}} \equiv \frac{(\sqrt{G})_{e_I e_I}}{\sqrt{G}} [8], \text{ that leads to the}$$

differential equation of a kind $\frac{\partial^2 \sqrt{G}}{\partial e_I \partial e_I} + K\sqrt{G} = 0$

with a constant gain $K = \text{const}$. The decision of the given differential equation and, accordingly, the metric form of a state surface of a complex error, depend on a sign of a Gaussian curvature:

- 1) $K > 0 \Rightarrow dl = de_R^2 + \cos^2(\sqrt{K}) \cdot e_R \cdot de_I^2$;
- 2) $K < 0 \Rightarrow dl = de_R^2 + \text{ch}^2(\sqrt{-K}) \cdot e_R \cdot de_I^2$;
- 3) $K = 0 \Rightarrow dl = dl = de_R^2 + de_I^2$;

Classification of the metric form of a state surface of a complex error, depending on a sign of a Gaussian curvature [8], can be used for identification of plant in big («integrated identification »):

1. $K > 0$. The state surface of a complex error is locally isometric to sphere of radius $1/K$ or areas on sphere. Points e_{κ} the given type of a regular state surfaces of a complex error S_e refer to *elliptic* and correspond to discriminant of II fundamental form in this point $L_0 N_0 - M_0^2 > 0$;
2. $K < 0$. The state surface of a complex error is locally isometric to a pseudosphere of radius $1/K$ or areas on a pseudosphere. Points e_{κ} the given type of a regular state surfaces of a complex error S_e refer to *hyperbolic* and correspond to discriminant of II fundamental form in this point $L_0 N_0 - M_0^2 < 0$;
3. $K = 0$. The state surface of a complex error is locally isometric to a plane or areas on a plane. Points $e_{\kappa 0}$ the given type of a regular state surfaces of a complex error S_e refer to *parabolic* and correspond to discriminant of II fundamental form in this point $L_0 N_0 - M_0^2 = 0$.

Thus, considering a state surface of a complex error for *SISO* plant or state space of a complex error for *MIMO* plant, they can be identified as:

1. Plant of *elliptic* type;
2. Plant of *hyperbolic* type;
3. Plant of *parabolic* type.

Let's consider further a **synthesis** of the optimal control law for plant (elliptic, hyperbolic or parabolic type) as the equation of the geodesic passing on a state surface of a complex error between boundary points e_0 and $\varepsilon(0)$. From the point of view of the theory of optimal control [9], the decision of a problem in offered statement is carried out for conditions with a free beginning point e_0 and the fixed final point $\varepsilon(0)$ of optimized trajectories at restrictions:

- time T of achievement a condition $e < \varepsilon(0)$, where $\varepsilon(0)$ - the small area of reachability that defining an precision of control, $T = t_1 - t_0$;
- character of movement along a geodesic, that it is limited a condition $0 \leq \Delta e_i \leq \Delta e_{\max}$ where the value Δe_{\max} is a possible admissible oscillation along geodesic;
- power of control $u \leq u_{\max}$.

The length of curve L defined in the Cartesian coordinates system (e_R, e_I, t) with use I fundamental form:

$$\begin{aligned} e_R &= e_R(t), e_I = e_I(t), t \in [t_0, t_1], \\ L &= \int_{t_0}^{t_1} \sqrt{E \cdot \dot{e}_{Rt}^2 + 2 \cdot F \cdot \dot{e}_{Rt} \cdot \dot{e}_{It} + G \cdot \dot{e}_{It}^2} dt = \\ &= \int_{t_0}^{t_1} \sqrt{g_{ij}(e(t)) \frac{de_R}{dt} \frac{de_I}{dt}} dt \end{aligned} \quad (17)$$

where gains E, F, G are defined according to (10) and $g_{ij}(e(t))$ is a metric tensor [8], representing a scalar product of a standard tangent vectors to e_R and e_I curvilinear coordinate lines on a state surface of a complex error in a point of their crossing. Coordinates of a vector are differentials of increments de_R and de_I . SCS (14) can be presented in a polar coordinates [8], considered family of the geodesic, which are starting with a point e_0 in all directions and having fixed one of these directions (a polar angle $\psi = 0$). Then I fundamental form:

$$I = d\rho^2 + G(\rho, \psi) d\psi^2 \quad (18)$$

where ρ - radius of a geodesic circle (distance from a point e_0), and ψ - a polar angle, fig. 4.:

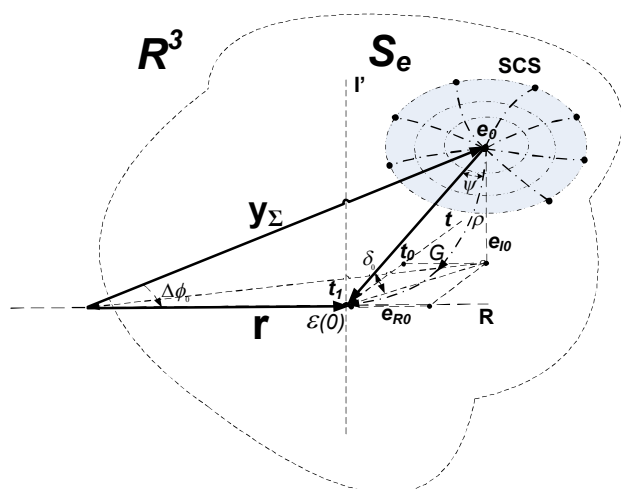


Fig. 4. Semigeodesic coordinates system (SCS) in a polar form, certain in a point e_0 a state surfaces of a complex error S_e .

For SCS in the polar form (18) length of a curve:

$$\begin{aligned} \rho &= \rho(t), \psi = \psi(t), t \in [t_0, t_1], \\ L &= \int_{t_0}^{t_1} \sqrt{\rho^2(t) + G(\rho(t), \psi(t)) \dot{\psi}^2(t)} dt \end{aligned} \quad (19)$$

The necessary condition of an optimal control on a state surface of a complex error is defined by Euler-Lagrange equations [9]:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{e}_R} \right) - \frac{\partial L}{\partial e_R} &= 0, \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{e}_I} \right) - \frac{\partial L}{\partial e_I} = 0, (CCS) \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\rho}} \right) - \frac{\partial L}{\partial \rho} &= 0, \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\omega}} \right) - \frac{\partial L}{\partial \omega} = 0, (SCC) \end{aligned} \quad (20)$$

For the smooth curve $\mathbf{e} = (e_R(t), e_I(t))$ in the Cartesian coordinates system or $\mathbf{e} = (\rho(t), \psi(t))$ in the polar coordinates system it is possible to define tangent vector fields [7] in each point of the given curve:

$$\begin{aligned}\bar{v}_{eD} &= \frac{d\bar{e}}{dt} = \frac{\partial \bar{e}}{\partial e_R} \cdot \frac{\partial e_R}{\partial t} + \frac{\partial \bar{e}}{\partial e_I} \cdot \frac{\partial e_I}{\partial t}, \\ \bar{v}_{eP} &= \frac{d\bar{e}}{dt} = \frac{\partial \bar{e}}{\partial \rho} \cdot \frac{\partial \rho}{\partial t} + \frac{\partial \bar{e}}{\partial \psi} \cdot \frac{\partial \psi}{\partial t}\end{aligned}\tag{21}$$

Movement along geodesic G occurs without acceleration (to constant speed). That vector field (21) has the constant module and a vector of its derivative (i. e. acceleration) $\dot{\vec{v}}_e(t)$ is an orthogonal vector functions to a tangent vector field. The necessary condition of an optimal control providing movement along geodesic on a state surface of a complex error is an equality 0 the scalar products

$\langle \bar{\mathbf{v}}_e(t), \dot{\bar{\mathbf{v}}}_e(t) \rangle = 0$. That the given expression for vector fields (21) can be presented as:

$$\left\langle \left(\frac{\partial \bar{e}}{\partial e_R} \cdot \frac{\partial e_R}{\partial t} + \frac{\partial \bar{e}}{\partial e_I} \cdot \frac{\partial e_I}{\partial t} \right), \right. \\ \left. \left(\frac{\partial \bar{e}}{\partial e_R} \cdot \frac{\partial^2 e_R}{\partial t^2} + \frac{\partial \bar{e}}{\partial e_I} \cdot \frac{\partial^2 e_I}{\partial t^2} \right) \right\rangle = 0, \quad (22)$$

Control of plant is function of the coordinates entered on a state surface of a complex error $u = \xi(e_R, e_I)$ or $u = \dot{\eta}(\rho, \psi)$. It is possible to define inverse functions $e_R = \xi_R(u)$, $e_I = \xi_I(u)$ or $\rho = \dot{\eta}_\rho(u)$, $\psi = \dot{\eta}_\psi(u)$, allowing to formulate a condition (22) as:

$$\left\langle \left(\frac{\partial \bar{e}}{\partial \xi_R(u)} \cdot \frac{\partial \xi_R(u)}{\partial t} + \frac{\partial \bar{e}}{\partial \xi_I(u)} \cdot \frac{\partial \xi_I(u)}{\partial t} \right), \right. \\ \left. \left(\frac{\partial \bar{e}}{\partial \xi_R(u)} \cdot \frac{\partial^2 \xi_R(u)}{\partial t^2} + \frac{\partial \bar{e}}{\partial \xi_I(u)} \cdot \frac{\partial^2 \xi_I(u)}{\partial t^2} \right) \right\rangle = 0, \quad (23)$$

Based on (23), the control of plant is defined. Then for synthesis the optimal control law of plant on a state surface of a complex error it is necessary:

1. To define the internal geometry of a state surface of a complex error via I and II fundamental forms according to (9), (14) and (12) and to identify the plant according to (13) and (16);
2. Using the I fundamental form and a necessary condition (20) to define the equation of a geodesic on a state surface of a complex error between a point of a current condition e_θ and a point of demanded state $\varepsilon(\theta)$;
3. Using a condition (23) for the equation of a geodesic on a state surface of a complex error to define a control signal $u = \zeta(e_\theta, e_i)$ or $u = \dot{\eta}(\rho, \psi)$.

3 Model of Plant and it is Stability on a State Surface of a Complex Error.

Control of plant is considered as a movement of beginning the radius-vector of a complex error e on a state surface from a boundary initial point e_0 , depending from the disturbances, influencing on a plant, in the origin the entered coordinates system (area of reachability $\varepsilon(0)$) for minimal time and at minimally oscillations along a trajectory of movement. Then the model of a plant and a problem of it is optimal control can be considered by it as Bolza problem [9]:

Definition 4. Model and a problem of optimal control of SISO plant on a state surface of a complex error:

$$\begin{aligned} \dot{e} &= f(t, e, u), \psi(t_0, e_0, t_1, \varepsilon(0)) = 0, \\ J(e(\cdot), u(\cdot), t_0, t_1) &= \\ &= \int_{t_0}^{t_1} L(t, e(t), \dot{e}(t), u(t)) dt + g(t_0, e_0, t_1, \varepsilon(0)) \rightarrow \min, \\ t &\in +R, e \in S_e \subset R^3, u \in U \subset R, \\ e(t_0) &= e_0, e(t_1) = \varepsilon(0), \\ f: R \times R^3 \times R &\rightarrow R^3; \\ L: R \times R^3 \times R &\rightarrow R; \\ \psi: R \times R^3 \times R \times R^3 &\rightarrow R; \\ g: R \times R^3 \times R \times R^3 &\rightarrow R; \end{aligned} \quad (24)$$

where S_e – a state surface of a complex error, defined a structure and parameters of the plant, e_0 – the point of beginning the radius-vector of a complex error in the initial time moment t_0 ; $\varepsilon(0)$ – area of reachability in the end of a transient t_1 ; f – a vector function, modelling a plant on a state surface of a complex error; L – integrand of a functional J of a problem of the optimal control, defining a trajectory of movement the radius-vector of a complex error on a state surface of its possible conditions, depending on control u and boundary points e_0 and $\varepsilon(0)$; ψ , g – the vector functions, defining a boundary conditions of a problem of optimal control. We assume, that all functions are continuous and are twice differentiated on all sets of variables t , e , u . The vector function f , modelling a plant on a state surface of a complex error, can be presented also by the system of the scalar functions, defined in Cartesian or semigeodesic polar coordinates systems:

$$\begin{aligned} \dot{e}_R(t) &= f_R[t, e_R(t), u(t)], \dot{e}_I(t) = f_I[t, e_I(t), u(t)], \\ \dot{e}_\rho(t) &= f_\rho[t, \rho(t), u(t)], \dot{e}_\psi(t) = f_\psi[t, \psi(t), u(t)], \\ \forall t \geq 0, t \in +R, e_R(t) &\in R, e_I(t) \in R, u(t) \in R, \\ \rho(t) &\in R, \psi(t) \in (0, \pi), \\ f_R: R \times R \times R &\rightarrow R, f_I: R \times R \times R \rightarrow R, \\ f_\rho: R \times R \times (0, \pi) &\rightarrow R, f_\psi: R \times (0, \pi) \times R \rightarrow R \end{aligned} \quad (25)$$

The stability of plant, presented on a state surface or in a state space of a complex error on an interval $T = t_1 - t_0$ is defined by the integrated equation:

$$\int_{t_0}^{t_1} f[t, \bar{e}(t), u(t)] = 0 \quad (26)$$

According to (26) and unlike a classical definition, stability of a plant is considered as dynamic process, with an opportunity to control on an interval $[t_0, t_1]$. At the same time, by analogy to classical definition of Lyapunov stability [10], following definition of the steady state (an equilibrium point) of plant on a state surface or in a state space of a complex error can be formulated:

Definition 5. The steady state of a control system is stable (an equilibrium point is stable) on a state surface or in a state space of a complex error, if for any given moment of time t_0 and any positive α exists positive $\beta = \beta(t_0, \alpha)$ such, that if

$$\|e(t_0) - \varepsilon(0)\| = \|e(t_0)\| < \beta \quad (\text{in a limit } \varepsilon(0) \rightarrow 0), \text{ then}$$

$$\|e(t, t_0) - \varepsilon(0)\| = \|e(t, t_0)\| < \alpha \quad (27)$$

for all moments of time $t \geq t_0$, that is hodograph of the radius-vector of a complex error $e(t)$ at the set initial deviation β in the subsequent, eventually does not leave for the certain border α .

4. Example: Control of 1st Order Plant with a Transport Delay Based on Complex Error

Let us consider the 1st order plant with a transport delay:

$$W(s) = \frac{K_p}{1 + sT} e^{-sL} \quad (28)$$

where s – an independent complex variable, $K_p = 1$ – the gain of plant, $T = 1$ s – a time constant of plant,

$L = 0.3 \text{ s}$ - a transport delay of plant. Used it transitive characteristic at submission a step signal on an input and observe the response the method *CHR* (Chien, Hrones, Reswick) [11] optimal gains of PID controller have been certain: $K = 3$, $T_i = 1 \text{ s}$, $T_d = 0.0452 \text{ s}$. Differential equations of control system with plant (28) were integrated by a method *ode45* operational environment *SIMULINK* of system *MATLAB R2010a*. The control system it is shown on fig. 5:

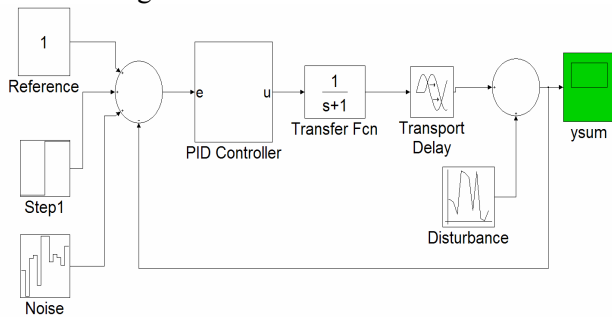


Fig. 5 Simulated control system.

For simulating noise of measurement, according to fig. 1, the block “Band-Limited White Noise” were used with parameters: *Noise power* = 0.001, *Sample time* = 0.01, *Seed* = 23341. Disturbances on an plant output were simulated by the block “Random Number” with parameters: *Mean* = 0, *Variance* = 0.001, *Initial seed* = 0. The used PID controller is shown on fig. 6:

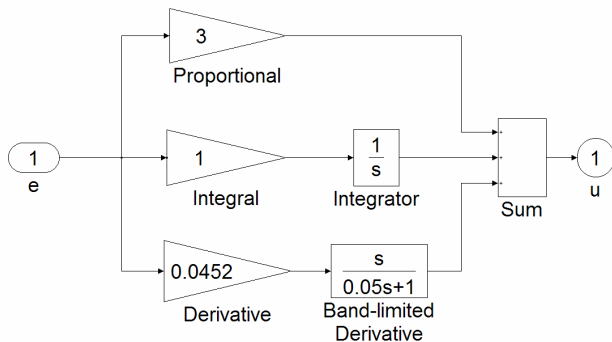


Fig. 6 PID controller for 1st order plant with a transport delay

Control system with adjusted PID controller were investigated at set reference $r = 1$ and the subsequent submission at the $t = 15 \text{ s}$ a step signal. Received transient for plant with nominal parameters is shown in the appendix 1 on fig. 1a. Further, properties of control system were investigated at change of plant parameters $K_p = 0.5 - 2$, $T = (0.5 - 2)\text{s}$, $L = (0.3 - 1)\text{s}$. The received transients resulted in the appendix 1 on fig. 2a-6a and in table 1. It has shown low adaptive and robust

properties of control system with a PID controller which has been adjusted for plant with nominal parameters:

Table 1

Transients of control system with PID controller at change nominal parameters of plant (28)

Changeable parameter	The characteristic of a transient
Increase of a transport delay $L = 0.7 \text{ s}$	Loss of stability
Decrease of a gain $K_p = 0.5$	Delay at change of an output of plant $t_d \approx 10 \text{ s}$
Increase of a gain $K_p = 2$	Loss of stability
Decrease of a time constant $T = 0.5 \text{ s}$	Significant not fading oscillations with frequency $\approx 1 \text{ Hz}$.
Increase of a time constant $T = 2 \text{ s}$	Delay at change of an output of plant $t_d \approx 2 \text{ s}$

Further, for plant (28) it is considered the control system with a controller, that using as an input only the real component e_R of a complex error (1), (7). For measurement a phase difference between reference of a control system and an output of plant (28) according to (1), (7) were used the scheme, using blocks library of *Signal Processing Blockset* [12], shown on fig. 7:

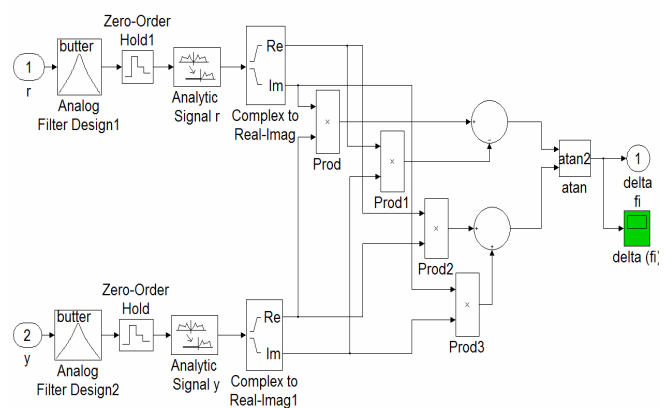
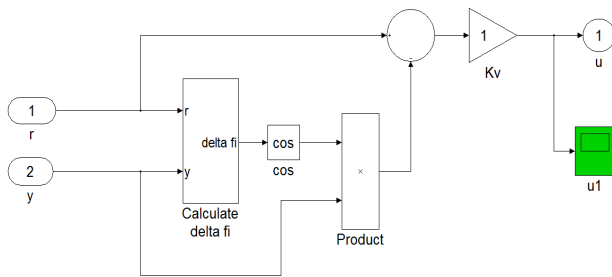


Fig. 7 Scheme for measuring the phase difference, based on a Hilbert transform.

The scheme for measuring a phase difference between reference and output of a plant (28), masked as block “Calculate delta fi”, used in a e_R -controller, fig. 8:

Fig. 8 e_R -controller

Control system with a e_R -controller were investigated under the same conditions, as for the adjusted PID controller. The received transients resulted in the appendix 1 on fig. 1b – 6b and in table 2 and show good adaptive properties:

Table 2

Transients of control system with e_R -controller at change nominal parameters of plant (28)

Changeable parameter	The characteristic of a transient
Increase of a transport delay $L = 0.7s$	Stability transient with an overshoot 0.5 p.u. in the initial moment of time
Decrease of a gain $K_p = 0.5$	The output of plant changes without delay, however it is value decreases proportionally to gain in the steady state ≈ 0.7
Increase of a gain $K_p = 2$	Stability transient, however the output increases proportionally to value of gain ≈ 1.2 times
Decrease of a time constant $T = 0.5 s$	Stability transient with an overshoot 0.2 p.u. in the initial moment of time
Increase of a time constant $T = 2 s$	Delay at change of an output of plant $t_d \approx 2 s$

5 Conclusion

In work presented the formalized mathematical model of a plant on a state surface of a complex error and definitions of it is stability. The identification method of a plant, depending from the internal geometry of a state surface of a complex error is offered. Considering, that the given work is introductory, many questions of control system development with use of a complex error demand the study and decisions. To first such questions are:

1. Detailed mathematical description of a control system and a plant on a state surface and in a

state space of a complex error. Definition of analogies between the entered classes of a plant, based on an internal geometry of a state surface of a complex error and the accepted fundamental types of plant (proportional, integral, differential);

2. Analysis of adaptive properties of a control systems using a state surface or state space of a complex error for synthesis of the optimal control;
3. Development detailed synthesis algorithm for a control system on a state surface or state space of a complex error;
4. Development industrial controllers, using as an input signal a complex error and it is components.

References:

- [1] G. C. Goodwin, S. F. Grebe, M. E. Salgado, "Control system design", Prentice-Hall, Inc, New Jersey, 2001.
- [2] O. N. Agamalov, "Design of Control Systems based on Vector Error", WSEAS Transactions on Systems and Control, issue 9, volume 4, pp. 476-485, September 2009, <http://www.wseas.us/e-library/transactions/control/2009/32-370.pdf>
- [3] R. C. Dorf, R. H. Bishop, "Modern Control Systems: International Edition", 11 Ed., Pearson Higher Education, 2008.
- [4] Hahn Stefan L., "Hilbert transforms in signal processing", Artech House, Inc., Boston, 1996.
- [5] Rabiner R, Gold B. "Theory and Application of Digital Signal Processing", Prentice-Hall, Englewood Cliffs, NJ, 1975.
- [6] Michael Rosenblum, Jurgen Kurths. "Analyzing Synchronization Phenomena from Bivariate Data by Means of the Hilbert Transform", in: Nonlinear Analysis of Physiological Data, Edited by H. Kantz, J. Kurths, and G. Mayer-Kress (Springer, Berlin, 1998), pp. 91-99.
- [7] S. Lang, "Fundamentals of Differential Geometry", Springer, 1998.
- [8] R. S. Millman, G. D. Parker, "Elements of Differential Geometry", Prentice-Hall, 1977.
- [9] D. S. Naidu, "Optimal Control Systems", CRC Press, 2003.
- [10] W. M. Haddad, V. S. Chellaboina, "Nonlinear Dynamical Systems and Control: a Lyapunov-Based Approach", Princeton University Press, 2008.
- [11] Chien K. L., Hrones J. A., Reswick J. B. "On automatic control of generalized passive systems" // Trans. ASME. 1952. Vol. 74. pp. 175-185.
- [12] Signal Processing Toolbox. User's Guide, Version 6.12. The MathWorks, Inc., 2009.

Appendix 1

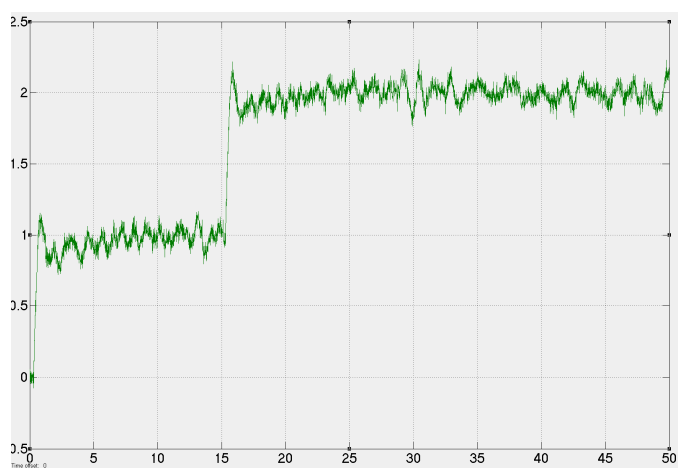


Fig. 1a. Transient of control system with PID controller for plant (28) with nominal parameters.

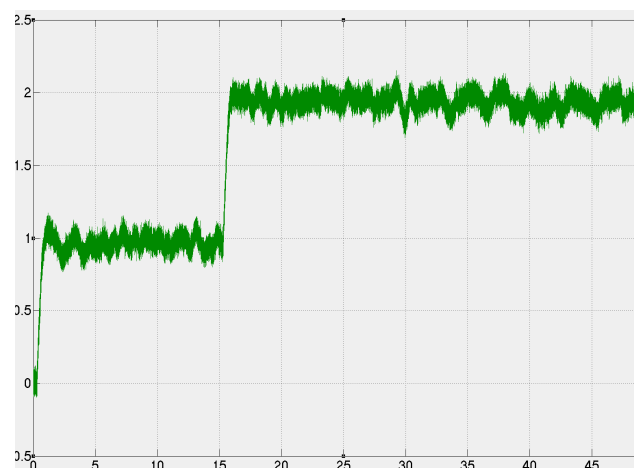


Fig. 1b. Transient of control system with e_R - controller for plant (28) with nominal parameters.

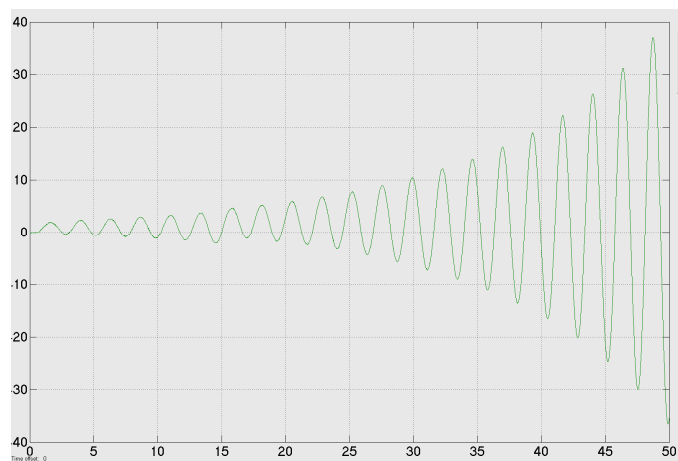


Fig. 2a. Transient of control system with PID controller for plant (28) with $L = 0.7$ s

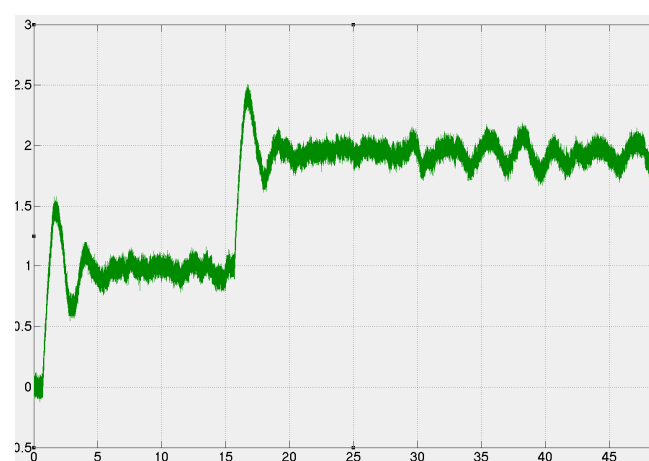


Fig. 2b. Transient of control system with e_R -controller for plant (28) with $L = 0.7$ s.

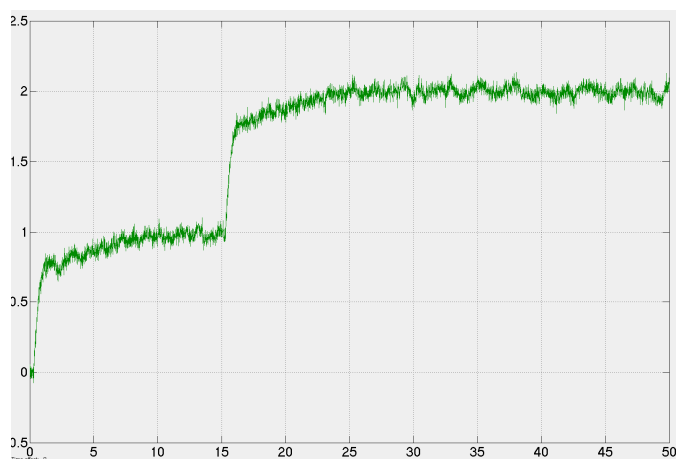


Fig. 3a. Transient of control system with PID controller for plant (28) with $K_P = 0.5$.

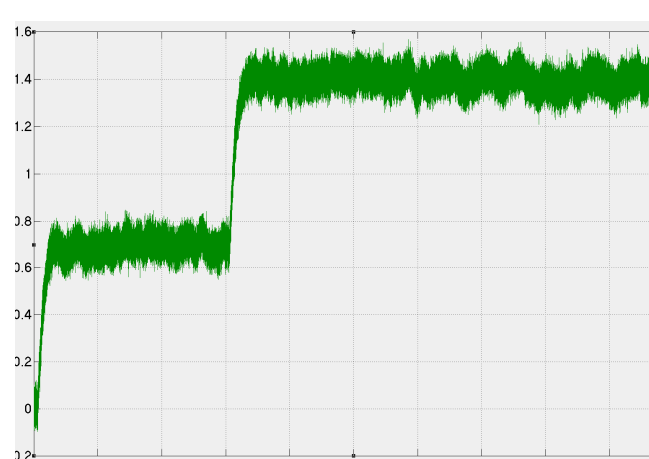


Fig. 3b. Transient of control system with e_R -controller for plant (28) with $K_P = 0.5$.

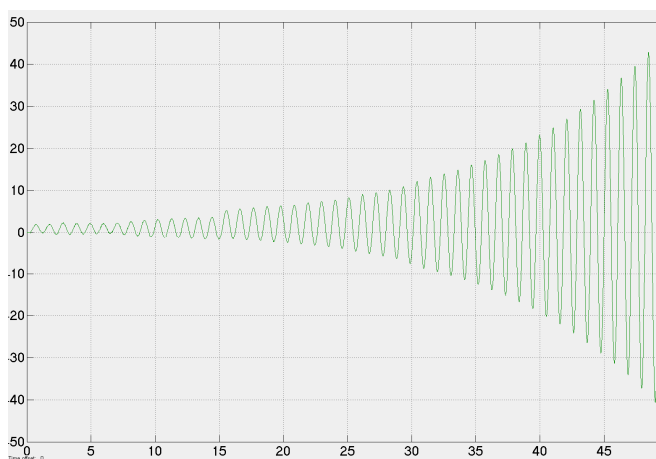


Fig. 4a. Transient of control system with PID controller for plant (28) with $K_P = 2$.

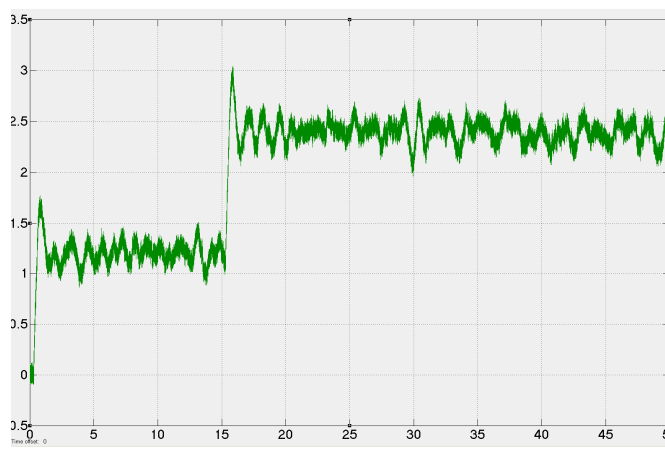


Fig. 4b. Transient of control system with e_R controller for plant (28) with $K_P = 2$.

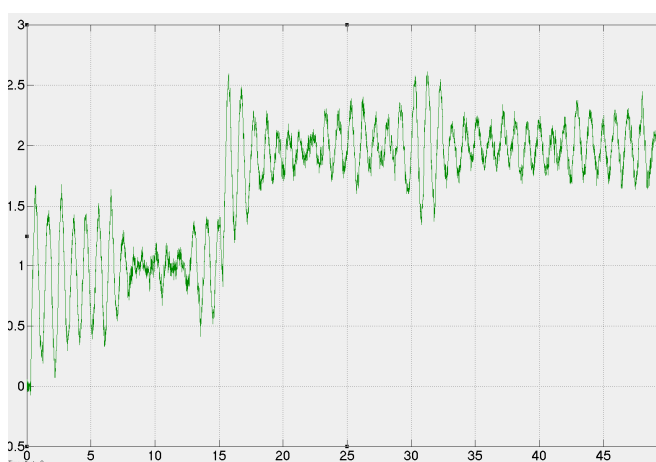


Fig. 5a. Transient of control system with PID controller for plant (28) with $T = 0.5$ s.

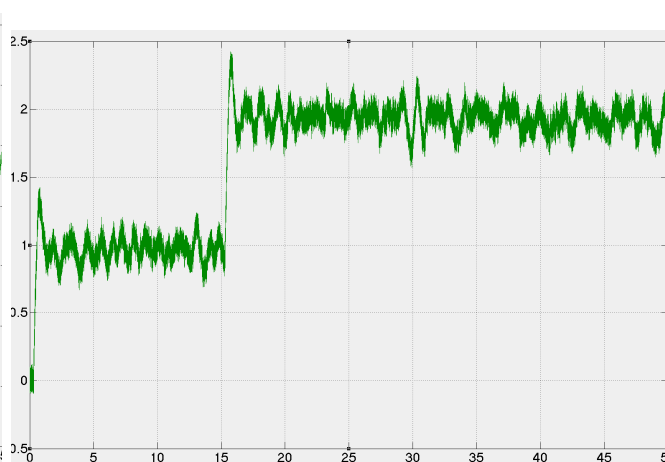


Fig. 5b. Transient of control system with e_R controller for plant (28) with $T = 0.5$ s.

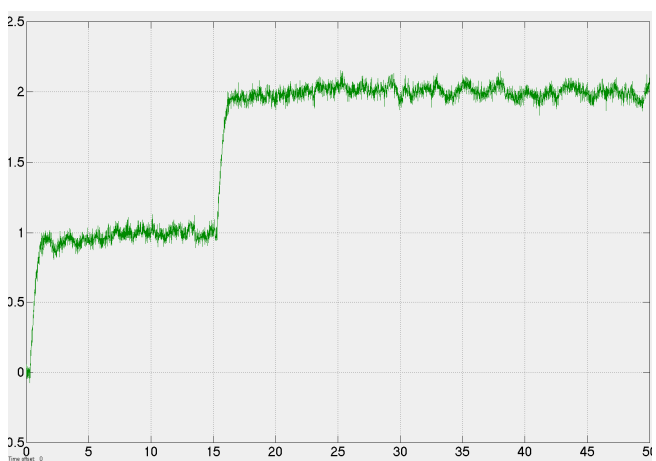


Fig. 6a. Transient of control system with PID controller for plant (28) with $T = 2$ s.

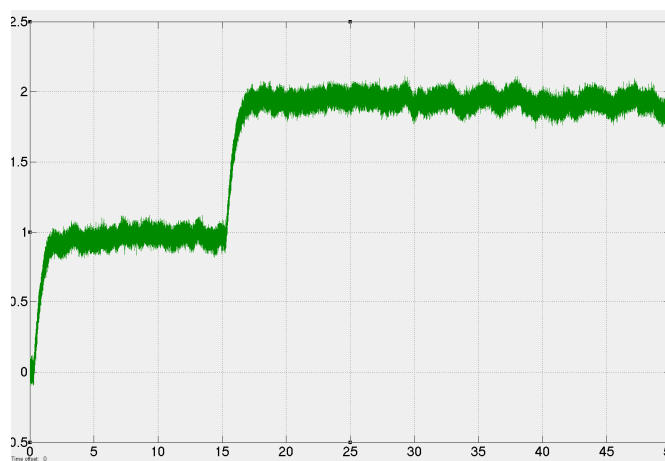


Fig. 6b. Transient of control system with e_R controller for plant (28) with $T = 2$ s.