Realizations of 2D Continuous-Discrete Systems with Boundary Conditions over Spaces of Regulated Functions

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Abstract: 2D hybrid continuous-discrete systems with boundary conditions are studied, in the general approach of the coefficient matrices and controls over spaces of functions of bounded variation or of regulated functions. The formulæ of the state and of the general response of these systems are provided, both in the case of causal and acausal cases. It is shown that the behaviour of the systems with boundary conditions is characterized by some generalized 2D semiseparable kernels. The existence of realizations of generalized 2D semiseparable kernels is proved and necessary and sufficient conditions for the minimality of the realizations are obtained.

Key–Words: 2D continuous-discrete systems, input-output map, regulated functions, functions of bounded variation, semiseparable kernels, realizations

1 Introduction

In the last two decades, the study of the 2D continuous-discrete control systems became an important branch of Systems and Control Theory (see [7], [12], [15], [16]), due to their applications in many domains such as linear repetitive processes [1], [17], iterative learning control synthesis [10] or long-wall coal cutting and metal rolling.

In this paper we extend the study of the 2D continuous-discrete systems to the general framework represented by the space of regulated functions. The topic of regulated functions was studied in a series of monographs or papers (e.g. [2], [6], [11], [19], [20]). We use the properties of the Perron-Stieltjes integral with respect to regulated functions and the differential equation in this framework. A class of 2D generalized hybrid linear control systems is introduced, having the controls over the space of regulated functions, the drift matrix with respect to the continuous variable of bounded variation and the other coefficient matrices being regulated matrix functions. This class is the 2D hybrid counterpart of the 1D continuous-time acausal systems introduced by Krener [8], [9] and developed by Gohberg, Kaashoek and Lerer [3], [4], [5]. Some extended models were studied in [13] and [16].

The present paper provides a generalized variaation-of-parameters formula for a 2D generalized differential-difference equation. Using this formula, the expressions of the state and of the general response of the 2D generalized hybrid linear control systems are provided, both in the case of causal and acausal cases. It is shown that the behaviour of the systems with boundary conditions is characterized by some generalized 2D semiseparable kernels. The existence of realizations of generalized 2D semiseparable kernels is proved and necessary and sufficient conditions for the minimality of the realizations are obtained.

We shall use the following definitions and notations. A function $f : [a, b] \to \mathbf{R}$ which posseses finite one sided limits $f(t-)$ and $f(t+)$ for any $t \in [a, b]$ (where by definition $f(a-) = f(a)$ and $f(b+) = f(b)$) is said to be *regulated* on $[a, b]$. The set of all regulated functions denoted by $G(a, b)$, endowed with the supremal norm, is a Banach space; the set $BV(a, b)$ of functions of bounded variation on $[a, b]$ with the norm $||f|| = |f(a)| + \text{var}_a^b f$ is also a Banach space; the Banach space of n-vector valued functions belonging to $G(a, b)$ and $BV(a, b)$ respectively are denoted by $G^n(a, b)$ and $BV^n(a, b)$ (or simply G^n and BV^n); $BV^{n\times m}$ denotes the space of $n \times m$ matrices with entries in $BV(a, b)$. The set of functions $f : [a, b] \times \mathbf{Z} \to \mathbf{R}$ such that $\forall k \in \mathbf{Z}, f(\cdot, k) \in G(a, b)$ $(BV(a, b))$ will be denoted $G_1(a, b)$ $(BV_1(a, b))$ and similar significances will have the above mentioned spaces with subscript 1 (G_1^n, BV_1^n, $BV_1^{n\times m}$).

A pair $D = (d, s)$ where $d = \{t_0, t_1, \ldots, t_m\}$ is a division of $[a, b]$ (i.e. $a = t_0 < t_1 < \ldots < t_m = b$) and $s = \{s_1, \ldots, s_m\}$ verifies $t_{j-1} \le s_j \le t_j$, $j =$

$1, \ldots, m$ is called a *partition* of $[a, b]$.

A function $\delta : [a, b] \to (0, +\infty)$ is called a *gauge* on $[a, b]$.

Given a gauge δ, the partition (d, s) is said to be *δ-fine* if

$$[t_{j-1}, t_j] \subset (s_j - \delta(s_j), s_j + \delta(s_j)), \quad j = 1, \ldots, m.$$

Given the functions $f, g : [a, b] \to \mathbf{R}$ and a partition $D = (d, s)$ of $[a, b]$ let us associate the integral sum

$$S_D(f\Delta g) = \sum_{j=1}^{m} f(s_j)(g(t_j) - g(t_{j-1})).$$

Definition 1 The number $I \in \mathbf{R}$ is said to be the *Perron-Stieltjes* (*Kurzweil*) *integral of f with respect to g from a to b* and it is denoted as $\int_a^b f \mathrm{d}g$ or $\int_a^b f(t)\mathrm{d}g(t)$ if for any $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that $|I - S_D(f\Delta g)| < \varepsilon$ for all δ-fine partitions D of $[a, b]$.

Given $f \in G(a, b)$ and $g \in G([a, b] \times [a, b])$ we define the differences $\Delta^+, \Delta^-, \Delta$ and $\Delta_s^+, \Delta_s^-, \Delta_s$ by $\Delta^+ f(t) = f(t+) - f(t)$, $\Delta^- f(t) = f(t) - f(t-)$, $\Delta f(t) = f(t+) - f(t-)$, $\Delta_s^+ g(t, s) = g(t, s+) - g(t, s)$, $\Delta_s^- g(t, s) = g(t, s) - g(t, s-)$; $\mathbf{D}^-(f)$, $\mathbf{D}^+(f)$ denote respectively the set of the left and right discontinuities of f in $[a, b]$ and similarly for g we can define $\mathbf{D}_t^-(g)$, $\mathbf{D}_t^+(g)$ with respect to the argument t. We denote by $\sum_t$ the sum $\sum_{t \in \mathbf{D}}$ where $\mathbf{D} = \mathbf{D}^-(f) \cup \mathbf{D}^+(f) \cup \mathbf{D}^-(g) \cup \mathbf{D}^+(g)$.

Let us recall some basic properties of the Perron-Stieltjes integral, by following [18] and [19]. The existence theorem of the Perron-Stieltjes integral $\int_a^b f\mathrm{d}g$ for $f \in BV(a, b)$ and $g \in G(a, b)$, due to Tvrdý [19] is essential for our treatment.

Theorem 2 ([19, Theorems 2.8 and 2.15]) *If $f \in G(a, b)$ and $g \in BV(a, b)$ then the Perron-Stieltjes integrals $\int_a^b f\mathrm{d}g$ and $\int_a^b g\mathrm{d}f$ exist* and

$$\int_a^b f\mathrm{d}g + \int_a^b g\mathrm{d}f = f(b)g(b) - f(a)g(a) + \\ + \sum_t [\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)]. \quad (1)$$

Theorem 3 ([19, Proposition 2.16]) *If $\int_a^b f\mathrm{d}g$ exists, then the function $h(t) = \int_a^t f\mathrm{d}g$ is defined on $[a, b]$ and*

i) if $g \in G(a, b)$ then $h \in G(a, b)$ and, for any $t \in [a, b]$

$$\Delta^+ h(t) = f(t)\Delta^+ g(t), \ \Delta^- h(t) = f(t)\Delta^- g(t) \quad (2)$$

ii) if $g \in BV(a, b)$ and f is bounded on $[a, b]$, then $h \in BV(a, b)$.

Theorem 4 (substitution, [19, Theorem 2.19]) *Let f, g, h be such that h is bounded on $[a, b]$ and the integral $\int_a^b f\mathrm{d}g$ exists. Then the integral $\int_a^b h(t)f(t)\mathrm{d}g(t)$ exists if and only if the integral $\int_a^b h(t)\mathrm{d}\left[\int_a^t f(s)\mathrm{d}g(s)\right]$ exists, and in this case*

$$\int_a^b h(t)f(t)\mathrm{d}g(t) = \int_a^b h(t)\mathrm{d}\left[\int_a^t f(s)\mathrm{d}g(s)\right]. \quad (3)$$

Theorem 5 (Dirichlet formula, [18, Theorem I.4.32]) *If $h : [a, b] \times [a, b] \to \mathbf{R}$ is a bounded function and $var_a^b h(s, \cdot) + var_a^b h(\cdot, t) < \infty$, $\forall t, s \in [a, b]$, then for any $f, g \in BV(a, b)$*

$$\int_a^b \mathrm{d}g(t)\left(\int_a^t h(s, t)\mathrm{d}f(s)\right) = \\ = \int_a^b \left(\int_s^b \mathrm{d}g(t)h(s, t)\right)\mathrm{d}f(s) + \\ + \sum_t [\Delta^- g(t)h(t, t)\Delta^- f(t) - \\ -\Delta^+ g(t)h(t, t)\Delta^+ f(t)]. \quad (4)$$

2 Generalized Differential Equations

The symbol

$$\mathrm{d}x = \mathrm{d}[A]x + \mathrm{d}g \quad (5)$$

where $A \in BV^{n\times n}$ and $g \in G^n(a, b)$ is said to be a *generalized linear differential equation* (GLDE) *in the space of regulated functions*.

Definition 6 A function $x : [a, b] \to \mathbf{R}^n$ is said to be a *solution* of GLDE (5) if for any $t, t_0 \in [a, b]$ it verifies the equality

$$x(t) = x(t_0) + \int_{t_0}^t \mathrm{d}[A(s)]x(s) + g(t) - g(t_0). \quad (6)$$

If x satisfies the initial condition

$$x(t_0) = x_0 \quad (7)$$

for given $t_0 \in [a, b]$ and $x_0 \in \mathbf{R}^n$ then x is called the *solution of the initial value problem* (5), (7).

Theorem 7 ([10, Theorem III.2.10]) *Assume that for any* $t \in [a, b]$ *the matrix* $A \in BV^{n\times n}$ *verifies the condition*

$$\det[I + \Delta^+ A(t)] \det[I - \Delta^- A(t)] \neq 0. \quad (8)$$

Then there exists a unique matrix valued function $U : [a, b] \times [a, b] \to \mathbf{R}^{n\times n}$ *such that, for any* $(t, s) \in [a, b] \times [a, b]$

$$U(t, s) = I + \int_s^t d[A(\tau)]U(\tau, s). \quad (9)$$

$U(t, s)$ *is called the fundamental matrix solution of the homogeneous equation*

$$\mathrm{d}x = \mathrm{d}[A]x \quad (10)$$

(or the fundamental matrix of A) and it has the following properties, for any $\tau, t, s \in [a, b]$*:*

$$U(t, s) = U(t, \tau)U(\tau, s); \quad (11)$$

$$U(t, t) = I; \quad (12)$$

$$\begin{aligned} U(t+, s) &= [I + \Delta^+ A(t)]U(t, s),\\ U(t-, s) &= [I - \Delta^- A(t)]U(t, s);\\ U(t, s+) &= U(t, s)[I + \Delta^+ A(s)]^{-1},\\ U(t, s-) &= U(t, s)[I - \Delta^- A(s)]^{-1}; \end{aligned} \quad (13)$$

$$U(t, s)^{-1} = U(s, t); \quad (14)$$

there exists a constant $M > 0$ *such that*

$$|U(t, s)| + \mathrm{var}_a^b U(t, \cdot) + \mathrm{var}_a^b U(\cdot, s) + \mathrm{v}(U) < M \quad (15)$$

where $\mathrm{v}(U)$ *is the twodimensional Vitali variation of* U *on* $[a, b] \times [a, b]$ ([18, *Definition* I.6.1]).

Some methods for the calculus of the fundamental matrix $U(t, s)$ were provided in [11].

From [18, Theorem III.3.1] and [20, Proposition 2.5], one obtains

Theorem 8 (Variation-of-parameters formula) *If* $A \in BV^{n\times n}$ *satisfies the condition* (8), *then the initial value problem* (5), (7) *has a unique solution given by*

$$\begin{aligned} x(t) &= U(t, t_0)x_0 + g(t) - g(t_0) -\\ &- \int_{t_0}^t \mathrm{d}_s[U(t, s)](g(s) - g(t_0)). \end{aligned} \quad (16)$$

If $g \in G^n$ $(g \in BV^n)$ *then* $x \in G^n$ $(x \in BV^n)$.

3 Input-output maps of 2D generalized systems

The linear spaces $X = G_1^n$, $U = G_1^m$ and $Y = G_1^p$ are called respectively the *state*, *input* and *output spaces*. The *time set* is $T = [a_1, b_1] \times \{a_2, a_2 + 1, \ldots, b_2\}$, where $[a_1, b_1] \subset \mathbf{R}$ and $a_2, b_2 \in \mathbf{Z}$.

Definition 9 A 2D *generalized continuous-discrete system* (2Dgcd) is an ensemble

$$\begin{gathered} \Sigma = (A_1(t, k), A_2(t, k), B(t, k), C(t, k),\\ D(t, k), N_1, N_2, M_1, M_2) \in\\ \in BV_1^{n\times n} \times G_1^{n\times n} \times G_1^{n\times m} \times G_1^{p\times n} \times\\ \times G_1^{p\times m} \times \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n} \end{gathered}$$

where $A_1(t, k)A_2(t, k) = A_2(t, k)A_1(t, k)$, $\forall(t, k) \in T$, with the following state equation, output equation, boundary condition and output vector equation:

$$\begin{aligned} &\mathrm{d}x(t, k + 1) = \mathrm{d}[A_1(t, k + 1)]x(t, k + 1)+\\ &+A_2(t, k)\mathrm{d}x(t, k) - \mathrm{d}[A_1(t, k)]A_2(t, k)x(t, k)+\\ &+B(t, k)\mathrm{d}u(t, k), \end{aligned} \quad (17)$$

$$y(t, k) = C(t, k)x(t, k) + D(t, k)u(t, k), \quad (18)$$

$$N_1x(a_1, a_2) + N_2x(b_1, b_2) = v, \quad (19)$$

$$z = M_1x(a_1, a_2) + M_2x(b_1, b_2). \quad (20)$$

n is called the *dimension* of the system Σ and it is denoted dimΣ.

Let $U(t, t_0; k)$ be the fundamental matrix of $A_1(t, k)$, $k \in \{a_2, a_2 + 1, \ldots, b_2\}$ and $F(t; k, k_0)$ the discrete fundamental matrix of $A_2(t, k)$, $t \in [a, b]$, i.e.

$$F(t; k, k_0) =$$

$$= \begin{cases} A_2(t, k-1)A_2(t, k-2)\cdots A_2(t, k_0) & \text{for } k > k_0\\ I_n & \text{for } k = k_0. \end{cases}$$

Since $A_1(t, k)$ and $A_2(t, k)$ are commutative matrices for any $(t, k) \in T$, by the Peano-Baker type formula for U [11] and by the definition of F it results that $U(t, t_0; k)$ and $F(t; k, k_0)$ are commutative matrices too. We shall use the following notations: $\Delta^+ f(s, l) = f(s+, l) - f(s, l)$, $\Delta_s^+ U(t, s; k) = U(t, s+; k) - U(t, s; k)$ and similarly we define $\Delta^- f(s, l)$ and $\Delta_s^- U(t, s; k)$.

Definition 10 A vector $x_0 \in X$ is called the *initial state* of Σ at the moment $(t_0, k_0) \in T$ if $\forall(t, k) \in T$

with $(t,k) \geq (t_0,k_0)$

$$\begin{aligned} x(t,k_0) &= U(t,t_0;k_0)x_0, \\ x(t_0,k) &= F(t_0;k,k_0)x_0. \end{aligned} \tag{21}$$

Proposition 11 (2D *generalized variation of parameters formula). If*

$$\det[(I-\Delta^- A_i(t,k))(I+\Delta^+ A_i(t,k))] \neq 0,\ i=1,2, \tag{22}$$

$\forall t \in [a,b],\ k \in \mathbf{Z}$, then the solution of the generalized differential-difference equation

$$\begin{aligned} \mathrm{d}x(t,k+1) &= \mathrm{d}[A_1(t,k+1)]x(t,k+1)+ \\ &+ A_2(t,k)\mathrm{d}x(t,k)- \\ &- \mathrm{d}[A_1(t,k)]A_2(t,k)x(t,k)+ \\ &+ \mathrm{d}f(t,k) \end{aligned} \tag{23}$$

with the initial conditions (19) *is*

$$\begin{aligned} x(t,k) &= U(t,t_0;k)F(t_0;k,k_0)x_0+ \\ &+ \int_{t_0}^{t}\sum_{l=k_0}^{k-1} U(t,s;k)F(s;k,l+1)\mathrm{d}f(s,l)+ \\ &+ \sum_{a\leq s<t}\Delta_s^+ U(t,s;k)\sum_{l=k_0}^{k-1}F(s;k,l+1)\cdot \\ &\cdot\ \Delta^+ f(s,l)- \\ &- \sum_{a<s\leq t}\Delta_s^- U(t,s;k)\sum_{l=k_0}^{k-1}F(s;k,l+1)\cdot \\ &\cdot\ \Delta^- f(s,l). \end{aligned} \tag{24}$$

Proof. We shall use the notation

$$\mathrm{d}g(t,k) = \mathrm{d}x(t,k) - \mathrm{d}[A_1(t,k)]x(t,k). \tag{25}$$

The equation (24) becomes

$$\mathrm{d}g(t,k+1) = A_2(t,k)\mathrm{d}g(t,k) + \mathrm{d}f(t,k). \tag{26}$$

Then

$$\begin{aligned} \mathrm{d}g(t,k_0+1) &= A_2(t,k_0)\mathrm{d}g(t,k_0) + \mathrm{d}f(t,k_0) = \\ &= F(t;k_0+1,k_0)\mathrm{d}g(t,k_0)+ \\ &+ F(t;k_0+1,k_0+1)\mathrm{d}f(t,k_0). \end{aligned}$$

Let us assume that

$$\begin{aligned} &\mathrm{d}g(t,k) = F(t;k,k_0)\mathrm{d}g(t,k_0)+ \\ &+\sum_{l=k_0}^{k-1} F(t;k,l+1)\mathrm{d}f(t,l). \end{aligned} \tag{27}$$

Then, by (26), (27) and by the definition of $F(t;k,k_0)$, we get

$$\begin{aligned} \mathrm{d}g(t,k+1) &= A_2(t,k)F(t;k,k_0)\mathrm{d}g(t,k_0)+ \\ &+ \sum_{l=k_0}^{k-1} A_2(t,k)F(t;k,l+1)\mathrm{d}f(t,l)+ \\ &+ \mathrm{d}f(t,k) = \\ &= F(t;k+1,k_0)\mathrm{d}g(t,k_0)+ \\ &+ \sum_{l=k_0}^{k} F(t;k+1,l+1)\mathrm{d}f(t,l) \end{aligned}$$

hence (27) is true $\forall k > k_0$. Moreover, from (19), (25) and (10) one obtains

$$\begin{aligned} \mathrm{d}g(t,k_0) &= \mathrm{d}x(t,k_0)- \\ &- \mathrm{d}[A_1(t,k_0)]x(t,k_0) = \\ &= \mathrm{d}[U(t,t_0;k_0)]x_0 - \mathrm{d}[A_1(t,k_0)]x(t,k_0) = \\ &= \mathrm{d}[A_1(t,k_0)]U(t,t_0;k_0)x_0- \\ &- \mathrm{d}[A_1(t,k_0)]U(t,t_0;k_0)x_0 = 0 \end{aligned}$$

hence (27) becomes

$$\mathrm{d}g(t,k) = \sum_{l=k_0}^{k-1} F(t;k,l+1)\mathrm{d}f(t,l). \tag{28}$$

Equation (25) is equivalent to the generalized differential equation

$$\mathrm{d}x(t,k) = \mathrm{d}[A_1(t,k)]x(t,k) + \mathrm{d}g(t,k)$$

with the solution given by Theorem 8

$$\begin{aligned} x(t,k) &= U(t,t_0;k)x(t_0,k)- \\ &- \int_{t_0}^{t}\mathrm{d}_s[U(t,s;k)]\int_{t_0}^{s}\mathrm{d}g(\tau,k)+ \\ &+ \int_{t_0}^{t}\mathrm{d}g(s,k). \end{aligned} \tag{29}$$

By Theorem 3, (29) becomes

$$\begin{aligned} x(t,k) &= U(t,t_0;k)x(t_0,k)+ \\ &+ \int_{t_0}^{t} U(t,s;k)\mathrm{d}\int_{t_0}^{s}\mathrm{d}g(\tau,k)+ \\ &+ \sum_{a\leq s<t}\Delta_s^+ U(t,s;k)\Delta^+\int_{t_0}^{s}\mathrm{d}g(\tau,k)- \\ &- \sum_{a<s\leq t}\Delta_s^- U(t,s;k)\Delta^-\int_{t_0}^{s}\mathrm{d}g(\tau,k). \end{aligned} \tag{30}$$

We replace (28) in (30). One obtains the formula of the state of the system Σ (24) from (30) taking into account the following equality

$$\int_{t_0}^{t} \mathrm{d}g(s,k) = \sum_{l=k_0}^{k-1} \int_{t_0}^{t} F(s;k,l+1)\mathrm{d}f(s,l)$$

and also (19) and Theorem 3, Theorem 4 and Theorem 5.

□

Proposition 12 *If* (22) *holds, then the state of the system at the moment* $(t,k) \in T$ *determined by the initial state* x_0 *at the moment* $(t_0,k_0) \in T$ *and the control* $u : [t_0,t] \times \{k_0, k_0+1, \ldots, k-1\} \to \mathbf{R}^m$ *is given by the following formula:*

$$\begin{aligned} x(t,k) &= U(t,t_0;k)F(t_0;k,k_0)x_0 + \\ &+ \int_{t_0}^{t} \sum_{l=k_0}^{k-1} U(t,s;k)\cdot \\ &\cdot F(s;k,l+1)B(s,l)\mathrm{d}u(s,l) + \\ &+ \sum_{a\le s<t} \Delta_s^+ U(t,s;k) \sum_{l=k_0}^{k-1} F(s;k,l+1)\cdot \\ &\cdot B(s,l)\Delta^+ u(s,l) - \\ &- \sum_{a<s\le t} \Delta_s^- U(t,s;k) \sum_{l=k_0}^{k-1} F(s;k,l+1)\cdot \\ &\cdot B(s,l)\Delta^- u(s,l). \end{aligned} \tag{31}$$

Proof. The state equation (17) can be obtained from (19) by replacing $f(t,k)$ by

$$f(t,k) = \int_{t_0}^{t} B(s,k)\mathrm{d}u(s,k)$$

. Then (31) results from (24) and (2).

□

Now we replace the state $x(t,k)$ given by (31) into the output equation of Σ (18). One obtains the formula of the general response of the system Σ

Theorem 13 *Under the hypothesis* (22) *the input-output map of the* 2Dgh *system* Σ (17), (18) *is*

$$\begin{aligned} & y(t,k) = C(t,k)U(t,t_0;k)F(t_0;k,k_0)x_0 + \\ & + \int_{t_0}^{t} \sum_{l=k_0}^{k-1} C(t,k)U(t,s;k)F(s;k,l+1)\cdot \\ & \cdot B(s,l)\mathrm{d}u(s,l) + D(t,k)u(t,k) + \\ & + \sum_{a\le s<t} C(t,k)\Delta_s^+ U(t,s;k)\cdot \\ & \cdot \sum_{l=k_0}^{k-1} F(s;k,l+1)B(s,l)\Delta^+ u(s,l) - \\ & - \sum_{a<s\le t} C(t,k)\Delta_s^- U(t,s;k)\cdot \\ & \cdot \sum_{l=k_0}^{k-1} F(s;k,l+1)B(s,l)\Delta^- u(s,l). \end{aligned} \tag{32}$$

Corollary 14 *If* $u \in G_1^m$ $(u \in BV_1^m)$ *then* $x \in G_1^n$ *and* $y \in G_1^p$ $(x \in BV_1^n$ *and* $y \in BV_1^p)$.

Proof. We apply Theorems 8 and 13 and Proposition 12.

□

Definition 15 The boundary condition (7) is said to be *well-posed* if the homogeneous problem corresponding to (17) and (19) (i.e. with $u \equiv 0$ and $v = 0$) has the unique solution $x = 0$.

Proposition 16 *The boundary condition* (19) *is well-posed if and only if the matrix* $R = N_1 + N_2U(b_1,a_1;b_2)F(a_1;b_2,a_2)$ *is nonsingular*.

Proof: By (31) with $u \equiv 0$ we get

$$x(b_1,b_2) = U(b_1,a_1;b_2)F(a_1;b_2,a_2)x(a_1,a_2);$$

we replace $x(b_1,b_2)$ and $v = 0$ in (19). It results that (19) is well-posed if and only if the equation $[N_1 + N_2U(b_1,a_1;b_2)F(a_1;b_2,a_2)]x(a_1,a_2) = 0$ has the unique solution $x(a_1,a_2) = 0$, condition which is equivalent to R nonsingular.

□

In the sequel we shall consider systems with well-posed boundary condition (19) and which verify (22). Moreover, the discrete-time character of Σ with respect to the variable k imposes the following assumption: the matrices A_2 depend only on k and $A_2(k)$ are nonsingular for any $k \in \{a_2, a_2+1, \ldots, b_2\}$.

Then the discrete fundamental matrix of A_2 becomes $F(k,l)$ and we can define, for this fundamental matrix even for the case $k < l$, by the following formula:

$$F(k,l) = [A_2(l-1)A_2(l-2)\cdots A_2(k+1)A_2(k)]^{-1}.$$

In this case the semigroup property $F(k,l)F(l,i) = F(k,i)$ is true for any $k,l,i \in \{a_2, a_2+1, \ldots, b_2\}$.

□

Definition 17 The matrix $P = P_\Sigma = R^{-1}N_2U(b_1,a_1;b_2)F(b_2,a_2)$ is called the *canonical boundary value operator* of the system Σ with well-posed boundary condition .

Theorem 18 *If the system is with well-posed boundary condition then the state of the system Σ determined by the control $u : T \to \mathbf{R}^m$ and by the input vector $v \in \mathbf{R^n}$ is*

$$\begin{aligned}
&x(t,k) = U(t,a_1;k)F(k,a_2)R^{-1}v-\\
&-\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1} U(t,a_1;k)F(k,a_2)\cdot\\
&\cdot PU(a_1,s;b_2)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l)+\\
&+\int_{a_1}^{t}\sum_{l=a_2}^{k-1} U(t,s;k)F(k,l+1)B(s,l)\mathrm{d}u(s,l)+\\
&-U(t,a_1;k)F(k,a_2)P\cdot\\
&\cdot\Bigg(\sum_{a_1\le s<b_1}\Delta_s^+U(a_1,s;b_2)\sum_{l=a_2}^{b_2-1}F(a_2,l+1)\cdot\\
&\cdot B(s,l)\Delta^+u(s,l) - \sum_{a_1<s\le t}\Delta_s^-U(a_1,s;b_2)\cdot\\
&\cdot\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^-u(s,l)\Bigg)+\\
&+\sum_{a_1\le s<t}\Delta_s^+U(t,s;k)\sum_{l=a_2}^{k-1}F(k,l+1)\cdot\\
&\cdot B(s,l)\Delta^+u(s,l) - \sum_{a_1<s\le t}\Delta_s^-U(t,s;k)\cdot\\
&\cdot\sum_{l=a_2}^{k-1}F(k,l+1)B(s,l)\Delta^-u(s,l).
\end{aligned}\tag{33}$$

Proof: We replace $x(b_1,b_2)$ given by (31) in the boundary condition (19). We get

$$\begin{aligned}
&[N_1 + N_2U(b_1,a_1;b_2)F(b_2,a_2)]x_0+\\
&+N_2\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}U(b_1,s;b_2)F(b_2,l+1)B(s,l)\mathrm{d}u(s,l)+\\
&+\sum_{a_1\le s<b_1}\Delta_s^+U(b_1,s;b_2)\sum_{l=a_2}^{b_2-1}F(b_2,l+1)\cdot\\
&\cdot B(s,l)\Delta^+u(s,l) - \sum_{a_1<s\le b_1}\Delta_s^-U(b_1,s;b_2)\cdot\\
&\cdot\sum_{l=a_2}^{b_2-1}F(b_2,l+1)B(s,l)\Delta^-u(s,l) = v
\end{aligned}$$

hence, by the semigroup properties of the fundamental matrices $U(t,s;k)$ and $F(k,l)$, we obtain the initial state of the system Σ

$$\begin{aligned}
&x_0 = R^{-1}v - P\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}U(a_1,s;b_2)F(a_2,l+1)\cdot\\
&\cdot B(s,l)\mathrm{d}u(s,l) - P\sum_{a_1\le s<b_1}\Delta_s^+U(a_1,s;b_2)\cdot\\
&\cdot\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^+u(s,l)+\\
&+P\sum_{a_1<s\le b_1}\Delta_s^-U(a_1,s;b_2)\cdot\\
&\cdot\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^-u(s,l).
\end{aligned}\tag{34}$$

We replace the initial state $x_0 = x(a_1,a_2)$ in (31); then (33) results by using the semigroup property of the fundamental matrices, i.e.

$$U(b_1,s;b_2) = U(b_1,a_1;b_2)U(a_1,s;b_2)$$

and

$$F(b_2,l+1) = F(b_2,a_2)F(a_2,l+1)$$

.

□

Theorem 19 *The input-output map of the* 2Dgcd *system Σ is*

$$H : G_1^m \times \mathbf{R}^n \to G_1^p \times \mathbf{R}^n,$$

$H(u,v)=(y,z)$ where

$$\begin{aligned}
&y(t,k)=C(t,k)U(t,a_1;k)F(k,a_2)R^{-1}v-\\
&-\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}C(t,k)U(t,a_1;k)F(k,a_2)\cdot\\
&\cdot PU(a_1,s;b_2)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l)+\\
&+\int_{a_1}^{t}\sum_{l=a_2}^{k-1}C(t,k)U(t,s;k)F(k,l+1)B(s,l)\mathrm{d}u(s,l)+\\
&+D(t,k)u(t,k)-C(t,k)U(t,a_1;k)F(k,a_2)P\cdot\\
&\cdot\left(\sum_{a_1\le s<b_1}\Delta_s^+U(a_1,s;b_2)\sum_{l=a_2}^{b_2-1}F(a_2,l+1)\cdot\right.\\
&\cdot B(s,l)\Delta^+u(s,l)-\sum_{a_1<s\le t}\Delta_s^-U(a_1,s;b_2)\cdot\\
&\left.\cdot\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^-u(s,l)\right)+\\
&+C(t,k)\sum_{a_1\le s<t}\Delta_s^+U(t,s;k)\sum_{l=a_2}^{k-1}F(k,l+1)\cdot\\
&\cdot B(s,l)\Delta^+u(s,l)-C(t,k)\sum_{a_1<s\le t}\Delta_s^-U(t,s;k)\cdot\\
&\cdot\sum_{l=a_2}^{k-1}F(k,l+1)B(s,l)\Delta^-u(s,l).
\end{aligned}\tag{35}$$

and, by denoting $Q = M_1 + M_2U(b_1,a_1;b_2)\cdot$ $\cdot F(b_2,a_2), S=Q(I-P)-M_1$,

$$\begin{aligned}
&z=QR^{-1}v+S\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}U(a_1,s;b_2)F(a_2,l+1)\cdot\\
&\cdot B(s,l)\mathrm{d}u(s,l)+S\left(\sum_{a_1\le s<b_1}\Delta_s^+U(a_1,s;b_2)\cdot\right.\\
&\cdot\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^+u(s,l)-\\
&-\sum_{a_1<s\le b_1}\Delta_s^-U(a_1,s;b_2)\cdot\\
&\left.\cdot\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^-u(s,l)\right).
\end{aligned}\tag{36}$$

Proof: We obtain (35) by replacing the state $x(t,k)$ given by (33) in the output equation (18). Then, by replacing $x(a_1,a_2)=x_0$ (34) and $x(b_1,b_2)$ given by (33) in (20) and by a long calculus which uses the semigroup property and which is omitted, we get (36). □

Corollary 20 *If $u\in G_1^m$ then $x\in G_1^n$ and $y\in G_1^p$. If $A_2\in BV_1^{n\times n}, B\in BV_1^{n\times m}, C\in BV_1^{p\times n}$, $D\in BV_1^{n\times m}$ and $u\in BV_1^m$ then $x\in BV_1^n$ and $y\in BV_1^p$.*

Proof: We apply Theorems 8, 18 and 19. □

Definition 21 The *space of admissible controls* is the set

$$\mathcal{U}=\{u\in G_1^m(a,b)|\mathbf{D}_t^+(A_i(\cdot,k)\cap\mathbf{D}_t^+(u(\cdot,k))=\emptyset,$$
$$\mathbf{D}_t^-(A_i(\cdot,k))\cap\mathbf{D}_t^-(u(\cdot,k))=\emptyset,\ i=1,2,\ \forall k\in\mathbf{Z}\}.$$

Corollary 22 *If $u\in\mathcal{U}$, then the state and the output of the system Σ are given by the following formulæ:*

$$\begin{aligned}
&x(t,k)=U(t,a_1;k)F(k,a_2)R^{-1}v-\\
&-\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}U(t,a_1;k)F(k,a_2)\cdot\\
&\cdot PU(a_1,s;b_2)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l)+\\
&+\int_{a_1}^{t}\sum_{l=a_2}^{k-1}U(t,s;k)F(k,l+1)B(s,l)\mathrm{d}u(s,l),
\end{aligned}\tag{37}$$

$$\begin{aligned}
&y(t,k)=C(t,k)U(t,a_1;k)F(k,a_2)R^{-1}v-\\
&-\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}C(t,k)U(t,a_1;k)F(k,a_2)\cdot\\
&\cdot PU(a_1,s;b_2)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l)+\\
&+\int_{a_1}^{t}\sum_{l=a_2}^{k-1}C(t,k)U(t,s;k)F(k,l+1)\cdot\\
&\cdot B(s,l)\mathrm{d}u(s,l)+D(t,k)u(t,k).
\end{aligned}\tag{38}$$

Remark 23 The 2D ”classical” continuous-discrete systems [14] with the state equation

$$\begin{aligned}
&\frac{\partial x}{\partial t}(t,k+1)=\tilde{A}_1(t,k+1)x(t,k+1)+\\
&+\tilde{A}_2(t,k)\frac{\partial x}{\partial t}(t,k)-\tilde{A}_1(t,k)\tilde{A}_2(t,k)x(t,k)+\\
&+\tilde{B}(t,k)\tilde{u}(t,k)
\end{aligned}$$

represent particular cases of 2Dgcd (17) with absolutely continuous matrices $A_i(t,k)=\int_a^t\tilde{A}_i(s,k)\mathrm{d}s$, $i=1,2$ and controls $u(t,k)=\int_a^t\tilde{u}(s,k)\mathrm{d}s$.

4 Realizations of 2D semiseparable kernels

Let us consider the case $A_1 : [a_1, b_1] \to \mathbf{R}^{n\times n}$ and $A_2 : \{a_2, a_2+1, \ldots, b_2\} \to \mathbf{R}^{n\times n}$. Therefore the fundamental matrices of A_1 and A_2 are respectively $U(t,s)$ and $F(k,l)$. We denote by $T(t,k)$ and $T^*(t,k)$ the sets $T(t,k) = [a_1,t) \times \{a_2, a_2+1, \ldots, k-1\}$ and $T^*(t,k) = T \setminus T(t,k)$ respectively. We shall consider the notation

$$\diamond_T \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} \overset{def}{=} \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} - \int_{a_1}^{t} \sum_{l=a_2}^{k-1}.$$

Assume $v = 0$. Then, by the semigroup properties $U(t,s) = U(t,a_1)U(a_1,s)$, $F(k,l+1) = F(k,a_2)F(a_2,l+1)$, (38) can be written as

$$\begin{aligned} y(t,k) = & \int_{a_1}^{t} \sum_{l=a_2}^{k-1} C(t,k)U(t,a_1)\cdot \\ & \cdot F(k,a_2)(I-P)\cdot \\ & \cdot U(a_1,s)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l)- \\ & - \diamond_T \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} C(t,k)U(t,a_1)F(k,a_2)\cdot \\ & \cdot PU(a_1,s)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l)+ \\ & + D(t,k)u(t,k), \end{aligned} \quad (30)$$

hence (39) can be written in the form

$$\begin{aligned} y(t,k) &= \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} K(t,s;k,l)\mathrm{d}u(s,l)+ \\ &+ D(t,k)u(t,k). \end{aligned} \quad (40)$$

Definition 24 The function $K(t,s;k,l)$ is called the *kernel* of the equation (40). A kernel is said to be 2D *semiseparable* if it has the form

$$K(t,s;k,l) = \\ = \begin{cases} E_1(t,k)G_1(s,l) & \text{if} \quad (s,l) \in T(t,k) \\ -E_2(t,k)G_2(s,l) & \text{if} \quad (s,l) \in T^*(t,k). \end{cases} \quad (41)$$

where $E_1, E_2 \in BV_1^{p\times n}$, $G_1, G_2 \in BV_1^{n\times m}$. The kernel K obtained from the input-output equation (39) of a system Σ is denoted by K_Σ and it is called the *kernel of the system* Σ.

Proposition 25 *The kernel* K_Σ *of any* 2Dgcd *system with well-posed boundary condition is* 2D *semiseparable*.

Proof: From (39) and (40) we get (41) with $E_1(t,k) = C(t,k)U(t,a_1)F(k,a_2)(I-P)$, $G_1(s,l) = U(a_1,s)F(a_2,l+1)B(s,l)$, $E_2(t,k) = C(t,k)U(t,a_1)F(k,a_2)P$, $G_2(s,l) = U(a_1,s)F(a_2,l+1)B(s,l)$, hence K_Σ is 2D semiseparable.

□

Definition 26 Given a 2D semiseparable kernel K, a 2Dgcd system Σ is said to be a *realization* of K if $K = K_\Sigma$.

Proposition 27 *For any* 2D *semiseparable kernel* K *there exists a realization of* K.

Proof: If K has the form (41), a realization of K is the system Σ given by $A_1 = O_n$ a.e. on $[a_1, b_1]$, $A_2 = I_n$, $B(t,k) = \begin{bmatrix} G_1(t,k) \\ G_2(t,k) \end{bmatrix}$, $C(t,k) = [E_1(t,k) \quad E_2(t,k)]$, $D = O_p^m \in \mathbf{R}^{p\times m}$, $N_1 = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}$, $N_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix}$ with $n_1, n_2 > 0$, $n_1 + n_2 = n$; M_1 and M_2 are arbitrary. Obviously $U(t,s) = I_n$ a.e. on $[a_1, b_1]$, $F(k,l) = I_n$, hence $R = N_1 + N_2 = I_n$; it results that this system has well-posed boundary condition and its canonical operator is $P = R^{-1}N_2 = N_2$. □

Example 28 Let us consider the 2D continuous-discrete Wiener-Hopf equation

$$\begin{aligned} y(t,k) - \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} K(t-s, k-l)\mathrm{d}u(s,l) = \\ = Du(t,s), \quad (t,s) \in T \end{aligned} \quad (42)$$

where $K(t,k) \in BV_1^{p\times m}$ for any $(t,k) \in T$.

Assume that $K(t,k)$ can be extended to a function $\bar{K}(t,k)$ defined on $\mathbf{R} \times \mathbf{Z}$ which admits a proper rational 2D continuous-discrete Laplace transform (see [19]) of the form $T(s,z) = \dfrac{\theta(s,z)}{\pi_1(s)\pi_2(z)}$ with $\theta(s,z) \in \mathbf{R}^{p\times n}[s,z]$, $\pi_1(s) \in \mathbf{R}[s]$, $\pi_2(z) \in \mathbf{R}[z]$. Then, using the algorithm of minimal realization described in [14], we can determine the constant matrices A_1, A_2, B, C, D with $A_1A_2 = A_2A_1$ such that $T_\Sigma(s,z) = C(s-A_1)^{-1}(z-A_2)^{-1}B + D$. By considering the matrices $N_1 = I - P$ and $N_2 = P$ where P is a suitable spectral projection and M_1, M_2 arbitrary matrices we obtain a system Σ whose input-output map (38) coincides with the 2Dgcd Wiener-Hopf equation (42).

Definition 29 A realization Σ of a 2D semiseparable kernel K is said to be *minimal* if $\dim\Sigma \leq \dim\hat{\Sigma}$ for any realization $\hat{\Sigma}$ of K.

We introduce the *controllability* and the *observability Gramians* of the system Σ:

$$\mathcal{C}(\Sigma) = \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} U(a_1,s)F(a_2,l)\cdot$$

$$\cdot B(s,l)B(s,l)^T F(a_2,l)^T U(a_1,s)^T ds,$$

$$\mathcal{O}(\Sigma) = \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} U(s,a_1)^T F(l,a_2)^T\cdot$$

$$\cdot C(s,l)^T C(s,l)F(l,a_2)U(s,a_1)ds.$$

The canonical boundary value operator of Σ is

$$P = [N_1 + N_2 U(b_1,a_1)F(b_2,a_2)]^{-1}\cdot$$

$$\cdot N_2 U(b_1,a_1)F(b_2,a_2).$$

Now we shall extend [4, Theorem 3.1] to the case of 2Dgcd acausal systems.

Theorem 30 *A realization Σ of the 2D semiseparable kernel K is minimal if and only if the following conditions hold*:

$$\mathrm{Im}\,[\mathcal{C}(\Sigma)\;\; P\mathcal{C}(\Sigma)] = \mathbf{R}^n, \tag{43}$$

$$\mathrm{Ker}\begin{bmatrix}\mathcal{O}(\Sigma)\\ \mathcal{O}(\Sigma)P\end{bmatrix} = \{0\}, \tag{44}$$

$$\mathrm{Ker}\,\mathcal{O}(\Sigma) \subset \mathrm{Im}\,\mathcal{C}(\Sigma). \tag{45}$$

Proof: *Necessity*. Let us consider an arbitrary direct sum decomposition $\mathbf{R}^n = X_1 \oplus X_2$ with $n_1 = \dim X_1$, $0 < n_1 < n$ and the corresponding partitions of the following operators: $U(a_1,t)F(a_2,k)B(t,k) = \begin{bmatrix} B_1(t,k)\\ B_2(t,k)\end{bmatrix}$, $C(t,k)U(t,a_1)F(k,a_2) = [C_1(t,k)\;\; C_2(t,k)]$, $D(t,k) = \begin{bmatrix} D_1(t,k)\\ D_2(t,k)\end{bmatrix}$, $P = \begin{bmatrix} P_{11} & P_{12}\\ P_{21} & P_{22}\end{bmatrix}$. Let us denote by Σ^1 the 2Dgcd system with well-posed boundary conditions determined by the matrices $A_1^1 = O_{n_1}$, $A_2^1 = I_{n_1}$, $B_1(t,k)$, $C_1(t,k)$, $D_1(t,k)$, $N_1^1 = I - P_{11}$, $N_2^1 = P_{11}$.

If the condition (43) is not fulfilled we take $X_1 = \mathrm{Im}[\mathcal{C}(\Sigma)\;\; P\mathcal{C}(\Sigma)]$ and $X_2 = X_1^{\perp}$; if (44) is not fulfilled we take $X_2 = \mathrm{Ker}\begin{bmatrix}\mathcal{O}(\Sigma)\\ \mathcal{O}(\Sigma)P\end{bmatrix}$ and $X_1 = X_2^{\perp}$; if (45) is not true we consider X_2 the subspace of $\mathrm{Ker}\,\mathcal{C}(\Sigma)$ such that $\mathrm{Im}\,\mathcal{C}(\Sigma) + \mathrm{Ker}\,\mathcal{O}(\Sigma) = \mathrm{Im}\,\mathcal{C}(\Sigma) \oplus X_2$ and X_1 is the complement of X_2 in $\mathbf{R}^n$ which includes $\mathrm{Im}\mathcal{C}(\Sigma)$. As in [9, lemmas 3.2-3.4] we can prove that in all these cases $K_{\Sigma_1} = K$, hence Σ_1 is a realization of K and dim $\Sigma_1 = n_1 < n =$ dim Σ, i.e. Σ is not minimal.

Sufficiency. If the conditions (43)-(45) hold for some realization Σ of K, we consider the direct sum decomposition of the state space $\mathbf{R}^n$ given by $X_2 = \mathrm{Ker}\,\mathcal{O}(\Sigma)$, $X_1 \oplus X_2 = \mathrm{Im}\,\mathcal{C}(\Sigma)$ and $X_1 \oplus X_2 \oplus X_3 = \mathbf{R}^n$. Following the lines of [9, Theorem 3.1] we can prove that $\dim\Sigma \leq \dim\hat{\Sigma}$ for any realization $\hat{\Sigma}$ of K. □

5 Conclusion

The state space representation was studied for a class of time-varying 2D continuous-discrete systems with boundary conditions in the general framework of the state, input and output spaces over the set of regulated functions. The behaviour of these systems was emphasized, and their representation by 2D generalized semiseparable kernels was emphasized. This study can be continued by analysing for this class other important concepts as stability, controllability, observability, the realization problem, the adjoint systems etc.

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