Synthesis of an approximate feedback nonlinear control based on optimization methods

WIEM JEBRI JEMAI
Research Unit MACS
High School of Engineering
Gabès, 6029, Tunisia
jeb_wiem@yhaoo.fr

HOUSSEM JERBI
Laboratory LECAP
High School of Engineering
Gabès, 6029, Tunisia
houssem.jerbi@fss.rnu.tn

MOHAMED NACEUR ABDELKRIM
Research Unit MACS
High School of Engineering
Gabès, 6029, Tunisia
naceur.abdelkrim@enig.rnu.tn

Abstract: In the present work, we propose a novel polynomial approach to approximate the Input-State feedback linearization control. The aim of this new method is to simplify the implementation complexity of the exact Input-State feedback linearization.
Indeed, the present approach leads to an analytical control law *via* analytical nonlinear transformations without need to resolve a set of partial differential equations. In fact, the analytical control law, determined *via* the proposed work, is dependent to an arbitrary choice of some parameters. So and in order to ensure a satisfactory evolution of the control input, we resort to optimization methods to have the optimal values of parameters. A study simulation is presented to show the effectiveness of the proposed approach.

Key–Words: Nonlinear system control, Input-State feedback linearization, Analytical representations, probabilistic algorithms, Monte Carlo optimization.

1 Introduction

The Input-State feedback linearization is one of the most popular ways to nonlinear control design [1–5]. The main idea of this technique is to transform nonlinear dynamics into a linear form by using state feedback [6–8]. However, the Input-State feedback linearization approach requires rigorous conditions summarized as involutivity, controllability and smoothness [1, 2, 8, 9]. Thus, finding a suitable diffeomorphism is a very difficult, since it mainly based on the resolution of a set of partial differential equations [1, 2, 8–10]. All these rigorous and difficulties, underlying the Input-State feedback linearization, return a complexity in its practical exploitation and implementation and non sure task [11–13]. To solve this problem, various approaches to approximate feedback linearization are developped [14–19].

The first idea of this work is developed a new approach in order to simplify the practical implementation complexity of the Input-State feedback linearization. This new approach is mainly based on the use of approximate polynomial developments of the functions characterizing the original nonlinear model and the powerful mathematical tools of the Kronecker product. Thus, this approach consists in synthesizing a polynomial approximate state feedback *via* a polynomial approximate nonlinear transformation.
In a very recent work [20], we have shown that some parameters of the determined polynomial control can be chosen arbitrary, so a second problem may arises: which are the best parameters that ensure the effectiveness of nonlinear approximate transformation and guarantee the performances of the polynomial control?
As a result, the second purpose of the present work is to exploit the unconstraint and constraint optimization algorithms in order to solve the stated problem.

This work is structured as follows: In section 2, the problem of synthesis of an approximate state feedback *via* a polynomial nonlinear polynomial transformation is solved. Section 3 studies the optimization problem. A background about optimization methods is presented and the choice of Monte Carlo algorithm to solve considered optimization problem is explained. The results of a numerical simulation study are presented in section 4. In section 5, the study problem is summarized *via* a conceptual algorithm. Section 6 concludes this paper.

2 Synthesis of an approximate feedback linearization

In this paper, we consider the control-affine nonlinear SISO system in the following analytical form:

$$\dot{x} = \sum_{i \geqslant 1} f_i x^{[i]} + g_0 u = \sum_{i \geqslant 1} \tilde{f}_i \tilde{x}^{[i]} + g_0 u \qquad (1)$$

where $x \in \mathbb{R}^n$ is the vector of state variables and $u \in \mathbb{R}$ is the input variable. It is assumed that $f(x) = \sum_{i\geqslant 1} f_i x^{[i]}$ is analytical vector fields on $\mathbb{R}^n$ and can be developed as generalized Taylor series using the Kronecker product and power of vectors. $x^{[i]}$ is the *i-th* redundant Kronecker power of vector x [21], and $\tilde{x}^{[i]}$ is the *i-th* non-redundant Kronecker power of vector x [22]. The relation between the non-redundant Kronecker power $\tilde{x}^{[i]}$ and the redundant Kronecker power $x^{[i]}$ of vector x can be written as [22]: $x^{[i]} = R_n^i \tilde{x}^{[i]}$ or $\tilde{x}^{[i]} = \left(R_n^i\right)^{\dagger} x^{[i]}$ where $R_n^i \in \Re^{n^i \times n_i}$ is called matrix of redundance with $n_i = C_{n+i-1}^i$ and $\left(R_n^i\right)^{\dagger}$ is the Moore-Penrose pseudo-inverse of R_n^i.
The purpose of the polynomial input-state feedback linearization technique, proposed in this work, is the synthesis of a plynomial control described by:

$$u = \alpha(x) + \beta(x) v \tag{2}$$

via a nonlinear analytical transformation:

$$z = \phi(x) \tag{3}$$

Thus, the original system (1) will be transformed into a linear one in the controllable canonical form given by:

$$\dot{z} = Az + bv \tag{4}$$

where $v \in R$ is a new external input, $(A; b)$ is a controllable pair of constant matrices of appropriate dimensions.
Note that, in literature [1, 3, 23], the new linear system have a Brunowsky canonical form. In the present work, the original system (1) is transformed to a linear one provided that it be stable.
Since the transformed system (4) is linear, the next step is to employ linear pole-placement techniques in order to arbitrarily assign the poles of the closed-loop system. In the particular case, one can calculate a constant gain vector , like the static state feedback law:

$$v = -kz \tag{5}$$

induces the closed-loop dynamics :

$$\dot{z} = (A - bk) z \tag{6}$$

when applied to the linear system (4).
Using the analytical nonlinear transformation given by:

$$z = \phi(x) = \sum_{k\geqslant 1} \phi_k x^{[k]} \tag{7}$$

we will determine an analytical static state feedback written as:

$$u = \sum_{i\geqslant 1} \alpha_i x^{[i]} + \sum_{j\geqslant 0} \beta_j x^{[j]} v \tag{8}$$

Consequently, the studied problem consists in determining the unknown coefficients ϕ_k, α_i and β_j which characterize, respectively, the nonlinear transformation $\phi(.)$ and the state feedback u.
Moreover, the seeking of components $\phi(.)$ of the nonlinear transformation can be solved via the determination of the unknown components of the analytical nonlinear reverse transformation defined by:

$$x = \phi^{-1}(z) = S(z) = \sum_{i\geqslant 1} S_i^1 z^{[i]} \tag{9}$$

To solve this problem, The key idea can be stated as follows:
Starting from the analytical expression (1) and using the powerful mathematical tool of the kronecker power of the vector x given by:

$$x^{[k]} = \sum_{i\geqslant k} S_i^k z^{[i]} \tag{10}$$

where:

$$S_p^n = \sum_{j=1}^{p-n+1} \left(S_{p-j}^{n-1} \otimes S_j^1\right) \tag{11}$$

we obtain a new representation of the considered system (1) in terms of the new variable z, written as:

$$\begin{aligned} \dot{x} = \sum_{i\geqslant 1} f_1 S_i^1 z^{[i]} + \sum_{i\geqslant 2} f_2 S_i^2 z^{[i]} + \ \dots \\ + \sum_{i\geqslant p} f_p S_i^p z^{[i]} + g_0 u \end{aligned} \tag{12}$$

Moreover, if we use the following derivative of *i-th* Kronecker's power of vector x [21]:

$$\left(x^{[i]}\right)_x = \frac{dx^{[i]}}{dt} = V^{[i]} \left(I_n \otimes x^{[i-1]}\right) \tag{13}$$

where $V^{[i]} = \sum_{j=0}^{i-1} \left(U_{n^j \times n} \otimes I_{n^{(i-j-1)}}\right)$ and $U_{p\times q}$ designates the Kronecker permutation matrix [21], one obtains another expression of the derivative given by:

$$\dot{x} = \sum_{i\geqslant 1} S_i^1 V^{[i]} \left(I_n \otimes z^{[i-1]}\right) \dot{z} \tag{14}$$

Replacing $\dot{z}$ by its expression (6), one has:

$$\dot{x} = \sum_{i\geqslant 1} S_i^1 v^{[i]} \left((A - bk) \otimes I_{n^{(i-1)}}\right) z^{[i]} \tag{15}$$

When expressing the input control u in terms of z, we obtain the new following expression:

$$u = \sum_{i\geqslant 1} \alpha_1 S_i^1 z^{[i]} + \ \ldots\ldots \ + \sum_{i\geqslant p} \alpha_p S_i^p z^{[i]} - \left(\beta_0 kz + \ \ldots\ldots + \sum_{i\geqslant p} \left(\beta_p S_i^p \otimes k\right) z^{[i+1]} \right) \tag{16}$$

Replacing u by its new expression (16) in (12), yields the expression of the derivative $\dot{x}$ in terms of z. The identification of elements of this equation with those of equation (15) leads to the following recurrent algorithm which exposes the solution of the studied problem:

1. For $p = 1$

$$\begin{cases} S_1^1 = I_n \\ f_1 + g_0\alpha_1 = A - bk \\ A - bk \quad \text{is stable} \\ \beta_0 \text{ arbitrary} \end{cases} \tag{17}$$

2. For $p \geq 2$

$$\begin{cases} vec(S_p^1) = -pinv\left(A_p\right) vec\left(\sum\limits_{i=2}^{p} f_i S_p^i R_n^p\right) \\ \alpha_p = pinv\left(g_0\right) g_0 \left[\left(-\sum\limits_{i=2}^{p-1} \alpha_i S_p^i R_n^p\right) + \left(\sum\limits_{i=1}^{p-1}\left(\beta_i S_{p-1}^i \otimes k\right) R_n^p\right)\right] pinv\left(S_p^p R_n^p\right) \\ Ap = \left(\left(R_n^p\right)^T \otimes \left(f_1 + g_0\alpha_1\right)\right) - \left(\left[V^{(p)}\left[(A - bk) \otimes \left(I_{n^{p-1}}\right)\right] R_n^p\right]^T \otimes I_n\right) \\ \beta_{p-1} \ \text{ arbitrary} \end{cases} \tag{18}$$

where $vec(.)$ designates the vectorization operator and $pinv(.)$ designates the Moore-Penrose inverse.
Using this algorithm, we can deduce the components of the nonlinear transformation defined by (7) as:

$$\begin{cases} \phi_1^1 = \left(S_1^1\right)^{-1} \\ \phi_k^1 = -\left(S_1^1\right)^{-1}\left(\sum\limits_{i=2}^{k} S_i^1 \phi_k^i\right) \\ \phi_k^i = \sum\limits_{j=1}^{k-i+1}\left(\phi_{k-j}^{i-1} \otimes S_j^1\right) \end{cases} \tag{19}$$

According to the solution given by the above recurrent algorithm, we can deduce that the dynamical behavior of the considered system (1) controlled by (8), is strongly related to the choice of parameters α_i and β_j. Indeed, the original system (1) must be equivalent to linear one (4). So and in order to ensure this equivalence, the nonlinear transformation (7) must yield the same dynamical behavior as obtained for the exact vector z of (4).
In the next section, we will resort to optimization methods to ensure the effectiveness of the nonlinear transformation by the determination of the optimal values of considered parameters β_j.

3 Optimization Methods

3.1 Background

We have shown in the previous section that the evolution of the analytical control, and consequently the efficiency of the nonlinear transformation, is controlled by the choice of parameters α_i and β_j. Thus, the problem, considered in this section, becomes an optimization problem.
Solving an optimization problem requires not only the comprehension of the problem but also knowledge about the optimization tools. So, and firstly, we will give, in the sequel, the basic concepts of an optimization problem [24]:

- ***Objective function***:
  It is a mathematical function which present what one aims to optimize (maximize or minimize). The objective function is called variously as: cost function, energy function.

- ***Decision variables***:
  Decision variables are the variables of the cost function. These variables (or parameters) are adjusted by the optimization algorithm to obtain the optimal values

- ***Search space/Choice set***:
  The search space of an optimization problem is a set containing all elements which could be its solution called candidate solutions or feasible solutions.

- ***Constraints***:
  The set of constraints, imposed to the objective function, can be equalities or inequalities. These latter must be satisfied by the elements of the search. So, these constraints limits the choice set.

There is a wide variety of optimization algorithms. Generally, the optimization algorithms can be divided in two basic classes: deterministic and probabilistic algorithms.
Deterministic algorithms are most often used if a clear

relation between the characteristics of the possible solutions and their utility for a given problem exists. Then, the search space can efficiently be explored using for example a divide and conquer scheme. If the relation between a solution candidate and its "fitness" are not so obvious or too complicated, or the dimensionality of the search space is very high, it becomes harder to solve a problem deterministically.
Then, probabilistic algorithms come into play. The initial work in this area which now has become one of most important research fields in optimization [25–28] was started about 59 years ago [29].
An especially relevant family of probabilistic algorithms are the Monte Carlo-based approaches. They trade in guaranteed correctness of the solution for a shorter runtime.

3.2 Monte Carlo's algorithm

The Monte Carlo (MC) optimization algorithm is a class of randomized algorithms (probabilistic algorithms) [30–33]. Such algorithms includes at least one instruction that acts on the basis of random numbers. Moreover, the MC algorithm as defined by Conley for mathematical programming problems consists of generating a random sample of many feasible solutions and selecting the best one.
In the considered problem, we have not any idea about the values of the parameters in question. Thus, we will use the MC optimization method and generate randomly and independently 50 vectors of β_p (search space).
Note That these 50 randomly parameters are different then those taken in [20].
In each simulation, we calculate the Normalized Square Error (NSE) which presents the objective function of considered optimization problem, defined as:

$$NSE_z = \frac{\|z-\hat{z}\|_2{}^2}{\|z\|_2{}^2} \tag{20}$$

where z is the exact solution given by:

$$\begin{cases} \dot{z} = Az + bv \\ z(0) = \phi(x(0)) \\ v = -kz \end{cases} \tag{21}$$

and $\hat{z}$ is the approximate solution given by (7) and (19):

$$\hat{z} = \sum_{k\geqslant 1} \phi_k x^{[k]} \tag{22}$$

$\|.\|_2$ denotes the norm 2.
The optimization principle of the studied problem can be schematized as follows:

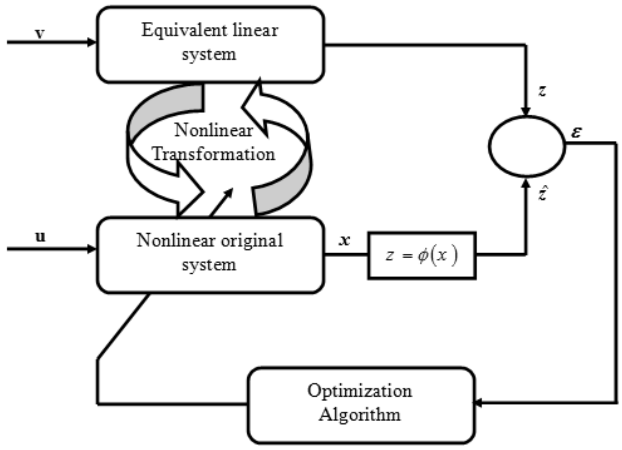

Figure 1: Block diagram of the optimization algorithm

4 Illustrative example and simulation results

Consider the following single-input nonlinear dynamic system [34]:

$$\begin{cases} \dot{x}_1 = 3\sin x_2 \\ \dot{x}_2 = -x_1 x_2 + u \end{cases} \tag{23}$$

The study is considered in the neighborhood of the equilibrium point $x_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.
The system (23) developed as generalized Taylor series, truncated to the third order, yields the following polynomial system:

$$\begin{cases} \dot{x}_1 = 3x_2 - 0.5x_2^3 \\ \dot{x}_2 = -x_1 x_2 + u \end{cases} \tag{24}$$

The analytical form of the system (24) is given by:

$$\dot{x} = f_1 x + f_2 x^{[2]} + f_3 x^{[3]} + g_0 u \tag{25}$$

with
$f_1 = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$;
$f_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$;
$f_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and
$g_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

4.1 First study: Unconstrained optimization

In this first study, we will discuss the effectiveness of the Monte Carlo's algorithm to determine the non-

linear analytical transformation and the static feedback. In this subsection, we will solve the optimization problem without imposed any constraint to the objective function defined by the equation (20).
The nonlinear system (25) will be changed to a linear one given by (4) with:
$A = \begin{bmatrix} 0 & 3 \\ 1 & -0.5 \end{bmatrix}; b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$
and the new control law $v = -k\hat{z}$ is chosen like the linear system is defined by poles -1.5 and -2 which correspond to a gain matrix $k = \begin{bmatrix} 2 & 3 \end{bmatrix}$.
The polynomial feedback control and the state transformation are truncated to the order 3.
Then, according to equation (9), the analytical state reverse transformation is given by:

$$\hat{x} = S(z) = S_1 z + S_2 z^{[2]} + S_3 z^{[3]} \qquad (26)$$

and the polynomial feedback control law is written as:

$$\begin{aligned} u &= \alpha(x) + \beta(x) v \\ &= \alpha_1 x + \alpha_2 x^{[2]} + \alpha_3 x^{[3]} + \left(\beta_0 + \beta_1 x + \beta_2 x^{[2]}\right) v \end{aligned} \qquad (27)$$

The initial conditions have been chosen as:
$x(0) = [0.5 \text{ - } 0.5]^T$.
For arbitrary choices of β_p such as:
$\beta_0 = 0.0986$;
$\beta_1 = \begin{bmatrix} 6.8656 & 0.5063 \end{bmatrix}$;
$\beta_2 = \begin{bmatrix} 9.2868 & 0.8746 & 0.2371 & 1.1250 \end{bmatrix}$.
we have obtain the following parameters α_i:
$\alpha_1 = \begin{bmatrix} -0.8027 & -3.2041 \end{bmatrix}$;
$\alpha_2 = \begin{bmatrix} 13.7312 & 10.8047 & 10.8047 & 1.5188 \end{bmatrix}$;
$\alpha_3 = \begin{bmatrix} \alpha_{31} & \alpha_{32} \end{bmatrix}$ with
$\alpha_{31} = \begin{bmatrix} 28.9287 & 14.4385 & 14.4385 & 4.3193 \end{bmatrix}$,
$\alpha_{32} = \begin{bmatrix} 14.4385 & 4.3193 & 4.3193 & 3.8509 \end{bmatrix}$.
Using (18) and (19), we can deduce and calculate the values of the compoments of nonlinear transformation defined by (22) as:

$$\hat{z} = \phi_1 x + \phi_2 x^{[2]} + \phi_3 x^{[3]} \qquad (28)$$

with
$\phi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$;
$\phi_2 = \begin{bmatrix} -4.3874 & -5.1417 & -5.1417 & -6.3581 \\ 3.4278 & 3.7305 & 3.7305 & 4.5521 \end{bmatrix}$;
$\phi_3 = \begin{bmatrix} \phi_{31} & \phi_{32} \end{bmatrix}$ with
$\phi_{31} = \begin{bmatrix} -5.2593 & -8.5150 & -5.9021 & -8.3910 \\ 5.9021 & 8.9830 & 6.0391 & 8.7160 \end{bmatrix}$,
$\phi_{32} = \begin{bmatrix} -3.2893 & -6.6188 & -4.8466 & -7.3906 \\ 3.0952 & 6.1032 & 3.4903 & 5.8441 \end{bmatrix}$.

We simulate the variations of the exact and approximate states $(z_1; \hat{z}_1)$ and $(z_2; \hat{z}_2)$, represented, respectively, in figure 2 and figure 3.

It is obvious, in these figures, that the exact variables of system (21) and the approached one of system (22) have not the same dynamical behavior. However, one can note that the variables of the two systems are asymptotically stable although arbitrary choices of parameters β_0, β_1 and β_2. The important difference between $(z_1; \hat{z}_1)$ and $(z_2; \hat{z}_2)$ will cause the change of the operating conditions for the state vector and then the whole local performances are not achieved.

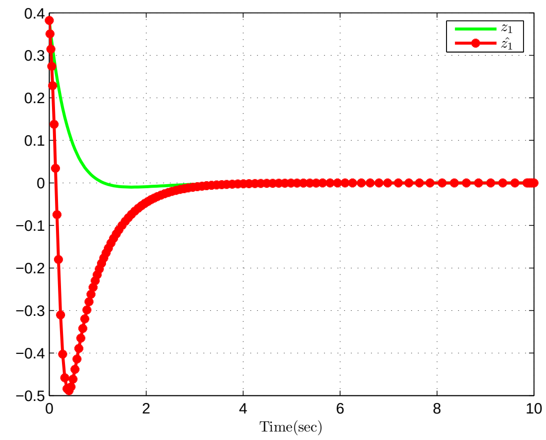

Figure 2: Evolution of the variables z_1 and $\hat{z}_1$.

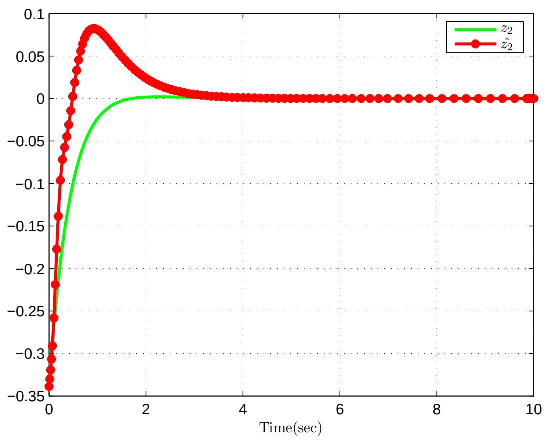

Figure 3: Evolution of the variables z_2 and $\hat{z}_2$.

These results are confirmed in figures 4 and 5 which represent the behaviors of the state variables $x = [x_1 \ x_2]$, solution of equation (25), and $\hat{x} = [\hat{x}_1 \ \hat{x}_2]$, given by the equation (26).
It appears in the below figures the convergence of the state variables, x and $\hat{x}$, to the origin. But, they have not the same dynamical evolution. Then, we can confirm, again, the non validity of the nonlinear transformation. Consequently, we check the bad influence of bad choice of the parameters β_j.

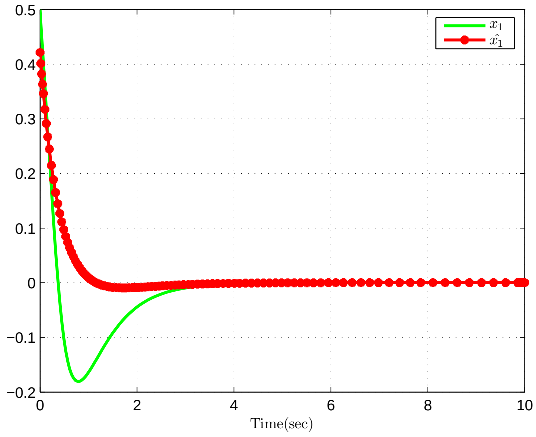

Figure 4: Evolution of the variables x_1 and $\hat{x}_1$.

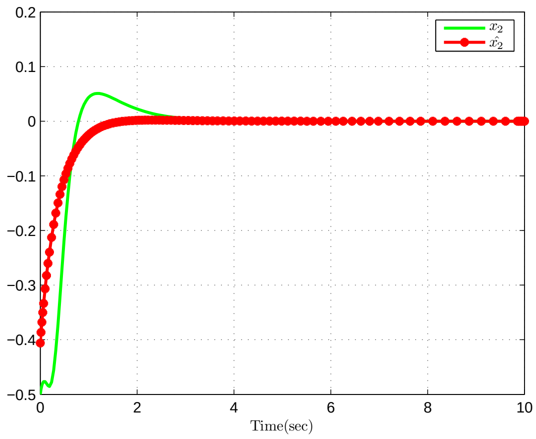

Figure 5: Evolution of the variables x_2 and $\hat{x}_2$.

When implementing the MC optimization method, we have obtained these following parameters among 50 candidates, according to objective function (20) :

$\beta_0 = 0.0858$;
$\beta_1 = \begin{bmatrix} 1.8341 & 3.8404 \end{bmatrix}$;
$\beta_2 = \begin{bmatrix} 9.7181 & 7.0742 & 6.9618 & 9.0388 \end{bmatrix}$.
which correspond to the following α_i:
$\alpha_1 = \begin{bmatrix} -0.8285 & -3.2427 \end{bmatrix}$;
$\alpha_2 = \begin{bmatrix} 3.6681 & 6.5915 & 6.5915 & 11.5212 \end{bmatrix}$;
$\alpha_3 = \begin{bmatrix} \alpha_{31} & \alpha_{32} \end{bmatrix}$ with
$\alpha_{31} = \begin{bmatrix} 22.3325 & 22.2969 & 22.2969 & 23.1994 \end{bmatrix}$,
$\alpha_{32} = \begin{bmatrix} 22.2969 & 23.1994 & 23.1994 & 31.0437 \end{bmatrix}$.
Thus, the components of the nonlinear transformation $\phi(x)$ are:

$$\phi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$\phi_2 = \begin{bmatrix} -4.5168 & -5.3063 & -5.3063 & -6.5720 \\ 3.5375 & 3.8645 & 3.8645 & 4.7221 \end{bmatrix};$$

$\phi_3 = \begin{bmatrix} \phi_{31} & \phi_{32} \end{bmatrix}$ with

$$\phi_{31} = \begin{bmatrix} -5.4867 & -8.9122 & -6.1698 & -8.8008 \\ 6.1698 & 9.4187 & 6.3329 & 9.1619 \end{bmatrix},$$

$$\phi_{32} = \begin{bmatrix} -3.4274 & -6.9321 & -5.0635 & -7.7524 \\ 3.2470 & 6.4195 & 3.6771 & 6.1705 \end{bmatrix}.$$

In figure 6, we represent the variation of the variables (z_1;$\hat{z}_1$) , and in figure 7, we represent the variables (z_2;$\hat{z}_2$). These figures show the good concordance between the state variables given by (21) and those defined by (22). This confirms the validity of the approximate transformation. Consequently, we deduce the importance of the implementation of MC algorithm.

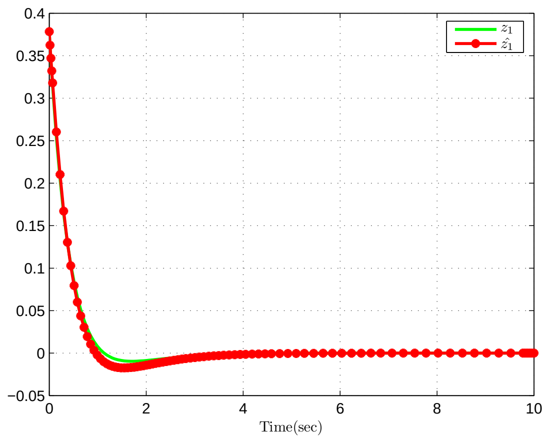

Figure 6: Evolution of the variables z_1 and $\hat{z}_1$.

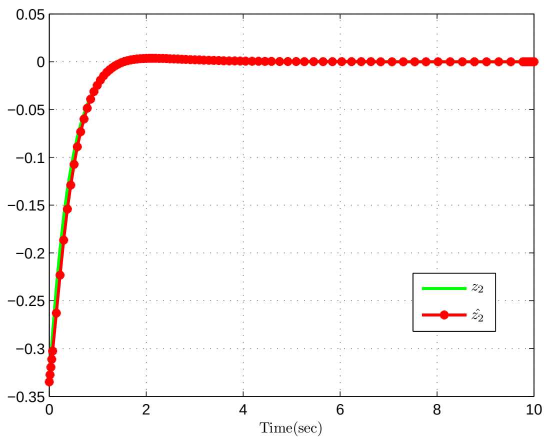

Figure 7: Evolution of the variables z_2 and $\hat{z}_2$.

Moreover, the effectiveness of the use of the MC optimization method to obtain an available nonlinear approximate transformation, is checked by the same dynamical behaviors of the state vectors x and $\hat{x}$ presented in figures 8 and 9.

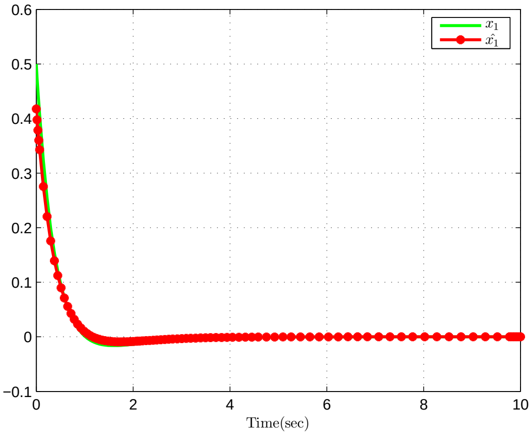

Figure 8: Evolution of the variables x_1 and $\hat{x}_1$.

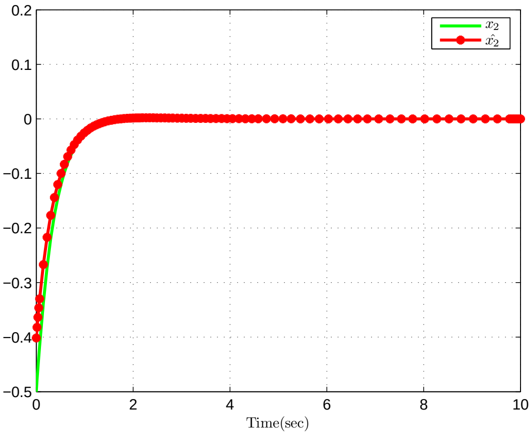

Figure 9: Evolution of the variables x_2 and $\hat{x}_2$.

In table 1, we compare the NSE, defined by (20) for the parameters β_p ($p = 0, 1, 2$) determined arbitrarily in a first study, and by the use of MC in a second one. It appears that the NSE is very reduced in the case of MC optimization which explain the accurate concordance of variables of systems (21) and (22). This second comparison study will conclude the big advantage of implementing MC optimization routine when determining the β_p parameters.

	Arbitrary choices	MC choices
NSE_z	2.4772	0.0029

Table 1: Comparison of NSE

In Figure 10, we represent the dynamical evolution of the control inputs respectively obtained with variable of transformation (21) and (22). An important over shoot is observed in the case of arbitrary choice of β_p parameters. This will induce saturation behavior of control variable in the case of physical processes. However, a satisfactory behavior is obtained with optimized β_p parameters.

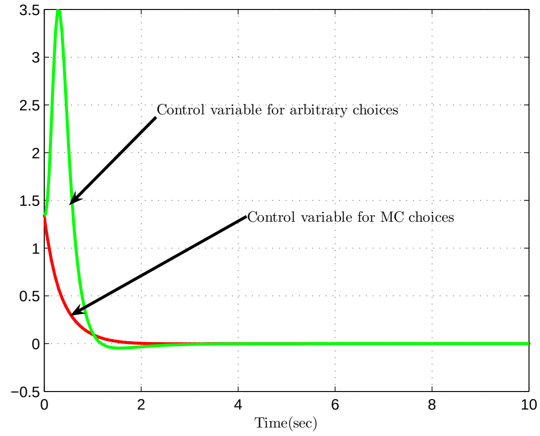

Figure 10: Evolution of Control inputs.

From this simulation, a new problem may arises. Thus, the above solution can lead to a valid nonlinear approximate transformation. But, one can not ensure that the obtained control input is saturated or not. To solve this problem, we resort, in the next subsection, to an optimization algorithm where some constraints, concerning the non-saturation of the control input, are imposed to the objective function.

4.2 Second study: Constrained Optimization

In this study, we consider the same objective function where some constraints are imposed. These latter will be on the control variable.
Then, the considered problem becomes:
Minimize the objective function given by:

$$NSE_z = \frac{\|z - \hat{z}\|_2{}^2}{\|z\|_2{}^2}$$

under constraints:

$$u_{min} \leq u \leq u_{max} \tag{29}$$

In the present simulation:
$u_{min} = 0$ and $u_{max} = 1$,
The following parameters are the best parameters which yield the minimum value of the objective function and satisfy the imposed constraints.

$\beta_0 = 0.0829;$
$\beta_1 = \begin{bmatrix} 2.2084 & 8.3464 \end{bmatrix};$
$\beta_2 = \begin{bmatrix} 9.4788 & 2.9673 & 5.2031 & 8.5939 \end{bmatrix}.$

these latter lead to the best α_i parameters:

$\alpha_1 = \begin{bmatrix} -0.8342 & -3.2513 \end{bmatrix};$
$\alpha_2 = \begin{bmatrix} 4.4168 & 11.6590 & 11.6590 & 25.0392 \end{bmatrix};$
$\alpha_3 = \begin{bmatrix} \alpha_{31} & \alpha_{32} \end{bmatrix}$ with
$\alpha_{31} = \begin{bmatrix} 22.4813 & 20.8346 & 20.8346 & 20.2213 \end{bmatrix},$
$\alpha_{32} = \begin{bmatrix} 20.8346 & 20.2213 & 20.2213 & 34.4779 \end{bmatrix}.$
and also, we obtain the following components of the approximate nonlinear transformation:

$\phi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$
$\phi_2 = \begin{bmatrix} -4.5471 & -5.3449 & -5.3449 & -6.6223 \\ 3.5633 & 3.8960 & 3.8960 & 4.7621 \end{bmatrix};$
$\phi_3 = \begin{bmatrix} \phi_{31} & \phi_{32} \end{bmatrix}$ with
$\phi_{31} = \begin{bmatrix} -5.5402 & -9.0059 & -6.2329 & -8.8976 \\ 6.2329 & 9.5216 & 6.4022 & 9.2673 \end{bmatrix},$
$\phi_{32} = \begin{bmatrix} -3.4598 & -7.0060 & -5.1145 & -7.8379 \\ 3.2828 & 6.4943 & 3.7212 & 6.2477 \end{bmatrix}.$

So that, we solve the considered problem by the determination of:

- an effectiveness nonlinear approximate transformation. This effectiveness is checked by the small error: $NSE_z = 0.0033$ which confirms a perfect concordance between the exact and approximate nonlinear transformations.

- a satisfactory control without worries about saturation of this latter. Thus, the figure 11 shows that the control input does not exceed the imposed limits.

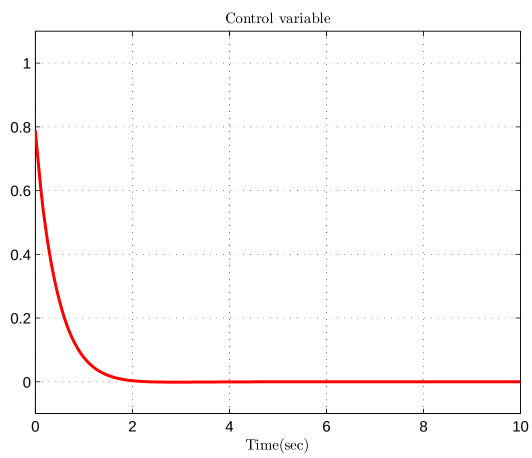

Figure 11: Evolution of Control input.

5 Conceptual Algorithm

This paragraph is reserved to present the conceptual algorithm of the developed method:
Given the control-affine nonlinear system of the following form:

$$\dot{x} = f(x) + g(x)u \tag{30}$$

Step 1: Transform the system (30)to the analytical form described by equation (1).

Step 2: Determine a stable linear equivalent system defined by (4).

Step 3: Initialize randomly and independently the parameters: $\beta_0, \beta_1, \ldots, \beta_q$, where q is a truncation order.

Step 4: Determine the parameters of the nonlinear reverse transformation: $S_1, S_2, \ldots, S_q$, and the parameters of the static feedback: $\alpha_1, \alpha_2, \ldots, \alpha_q$, using (18).

Step 5: Determine the parameters of the nonlinear transformation $\phi_1, \phi_2, \ldots, \phi_q$, using (19).

Step 6: Calculate the value of the objective function (20) for these parameters.

Step 7: Repeat step 3 to step 6 until the constraints $u_{min} \leq u \leq u_{max}$ is not satisfied.

6 Conclusion

The first idea developed, in this paper, is the synthesis of a new polynomial approach to solve practical implementation difficulties of Input-State feedback linearization method. To achieve this goal we have used analytical representations and the Kronecker product and power tools. The new approximate approach consists in: firstly, determining a nonlinear change of coordinates to a linear stable system, secondly, deriving a polynomial approximate control law via this transformation.
The purpose of the second part, is ensuring the effectiveness of such transformation and obtaining a satisfactory dynamical behavior of the control signal. So that, we have exploit the Monte Carlo optimization method without and under contraints to determine the optimal parameters which achieve the second goal. Our future work will focus on the design of a stability analysis by the search of a Lyapunov function. This will ensure the maximization of asymptotical stability regions around the considered operating points.

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