Optimal Switching Function Discrete Sliding-mode Control Based on Unmatched Uncertainties States Observer

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Abstract: - For a class of discrete unmatched uncertain systems with incompletely measured states, a stable closed-loop sliding-mode control method based on the unmatched uncertain states observer is proposed. The output error feedback matrix is obtained by transformation of uncertainties state equation and solving the linear matrix inequalities(LMI). The stability of the states observer is proved. A reaching-law sliding-mode controller is presented for this discrete system by employing the results of the observer. To enhance the stability of the reaching-law sliding-mode controller, the stability margin is defined by using the min-max eigenvalue ration method. By deducing the min-max Lyapunov difference function, solving linear programming and optimizing the switching function, the closed-loop sliding-mode controller can get the maximum stability margin. The simulation results show the effectiveness of the proposed control method.

Key Words: - discrete system; unmatched uncertainties states observer; reaching-law slide mode controller; stability margin

1 Introduction

[1]Variable structure sliding-mode control algorithms is a good solution to the effects of external perturbation on the control system, which have been widely used in various control system. Provided the controller is designed appropriately, the motion of the states is completely insensitive to so-called matched uncertainty when reached to the sliding-mode surface. This algorithm improves the robustness of the controlled system. One kinds of the effective continuous sliding-mode controller need to maintain an ideal sliding motion is known as the equivalent control [1,2]. However, the control signal is held constant during the sample period in digital system, so it is impossible to attain ideal sliding-mode surface unless control signal switches at infinite frequency. As a result, the invariance properties of continuous sliding-mode controller are lost. The ideal of discrete sliding-mode controller has been proposed in Reference [3-5].

Sliding-mode controller for unmatched uncertainties system is more complex. Commonly, linear transformations and assumptions of unmatched uncertainties are used to design the sliding-mode controller [6, 7]. In practice, the unmatched uncertainties system often encountered in the following problems: on the one hand, how to design sliding-mode controller if some states are unmeasured; on the other hand, how to improve the stability of the closed-loop sliding-mode controller.

Traditionally, sliding-mode controller requires all the system states are available to design. But it is not very realistic for practical engineering system. Reference[8,9] only using measurable outputs to design the sliding-mode controller, avoiding dependence on states. However, this method is complicated to derivate when system contains unmatched uncertainties. Reference [10-12]using sliding-mode observer to estimate the unmeasurable states, and design full-states feedback sliding-mode controller. However, this method may lead to states chattering.

It is very important to choose switching function in the discrete sliding-mode controller. To improve the controller performance, designing fuzzy [13,14]or adaptive [15,16]methods to adjust switching function according to the dynamic characteristics of system. These approaches are generally based on the characteristics of the controlled system and debugging experience to determine the fuzzy or

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adaptive rules, so they are not the general approaches.

The theoretical foundation of parameter uncertainty in time domain stability of control system is Lyapunov stability theory. Riccati equation approach to such issues is an important tool, but there are various shortcomings. With the proposal of interior point methods, LMI overcomes the deficiencies of Riccati equation approach. In recently, LMI has been widely used in control design optimization and stability conditions analysis[17,18].

The paper proposes a novel discrete sliding-mode control method for an unmatched uncertainties discrete system with states incomplete measurable. We use parts of measurable states and outputs to estimate the unmeasureable states and get all the system states. Next, we design reaching-law sliding-mode controller, and optimize the switching function. For the unmeasurable states, designing reduced-order states observer by linear transformation, and estimating unmeasured states. To prevent unmatched uncertainties undermine the stability of the observer, proposed a method to select output error feedback matrix L for states observer. Using LMI to solve matrix L, and proved the asymptotic stability of the observer. To improve the stability of the closed-loop sliding-mode controller, defining the closed-loop control stability margin by using of the min-max eigenvalue ration method, and obtaining optimal switching function by means of solving linear programming to improve the stability margin of the control system. Finally, the simulation shows the effectiveness of the observer and control method mentioned above.

The structure of the paper is as follows. Section 2 will describe the system model and give the relevant assumption. Section 3 will introduce the process of system linear transformation. In Section 4, the unmatched uncertainties states observer will be proposed, and the stability existing condition of the observer will be proved. Section 5 will use all the states to design reaching-law sliding-mode controller, and prove the existence and reaching of the sliding surface. Section 6 will analyse the closed-loop control stability margin and optimize the switching function. Section 7 will illustrate two examples, one shows the results of the unmatched uncertainties observer, the other one shows the effectiveness of optimal switching function sliding-mode controller. Conclusions are presented in Section 8.

2 System Model Description

Consider the following discrete unmatched uncertainties system:

$$\begin{cases} x(k+1) = (A+\Delta A)x(k) + B[u(k)+\xi(k)] \\ y(k) = Cx(k) \end{cases}$$

(1)

where $x \in R^n$ is the state vector, $y \in R^l$ is the measurable output, $u \in R^m$ is control input, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{l\times n}$ are constant matrices, $\Delta A \in R^{n\times n}$ is unmatched uncertainties time-varying matrix, $\xi(k) \in R^m$ is bounded external uncertain perturbation, it can be expressed as

$$\xi(k) = [\xi_1(k), \xi_2(k), \cdots \xi_m(k)]^T .$$

The following assumptions of system (1) are given as:

Assumption 1. The states of system (1) can be expressed as $x(k) = \begin{pmatrix} x_1(k) & x_2(k) \end{pmatrix}^T$, $x_1(k) \in R^p$ is unmeasurable state, $x_2 \in R^{n-p}$ is measurable state.

Assumption 2. ΔA is unmatched uncertainties item, it can be composed as $\Delta A = D\Sigma(k)E$, D and E are known constant matrices with appropriate dimensions, $\Sigma(k)$ is an uncertain time-varying matrix, and such that $\|\Sigma(k)\| < 1$, where norm $\|\cdot\|$ is expressed as the maximum singular value. So $\Sigma(k)$ has the property of $\Sigma^T(k)\Sigma(k) \le I$.

3 System linear transformation

Select n dimensional linear transformation matrix $T = \begin{bmatrix} T_i & T_o \end{bmatrix}$ and it's inverse matrix $T^{-1} = \begin{bmatrix} \overline{T}_i \\ \overline{T}_o \end{bmatrix}$, such that:

Condition 1. $\overline{T}_i B = 0$;

Condition 2. elements of the first p column in matrix $\overline{T}_o$ are zero.

By linear transformation $x = Tz$, system (1) is expressed as

$$\begin{cases} \begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} \overline{A} + \Delta\overline{A} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \overline{B}\begin{bmatrix} u(k)+\xi(k) \end{bmatrix} \\ y(k) = CT_i z_1(k) + CT_o z_2(k) \end{cases}$$

(2)

where $\overline{A}_{11} = \overline{T}_i A T_i$; $\overline{A}_{12} = \overline{T}_i A T_o$;

$\bar{A}_{21} = \bar{T}_o A T_i$; $\bar{A}_{22} = \bar{T}_o A T_o$;

$\bar{B} = \begin{bmatrix} 0 & B' \end{bmatrix}^T$; $B' = \bar{T}_o B$;

$\Delta\bar{A}_{11} = \bar{T}_i D\Sigma(k) E T_i$; $\Delta\bar{A}_{12} = \bar{T}_i D\Sigma(k) E T_o$;

$\Delta\bar{A}_{21} = \bar{T}_o D\Sigma(k) E T_i$; $\Delta\bar{A}_{22} = \bar{T}_o D\Sigma(k) E T_o$.

Linear transformation $x = Tz$ can be write as $\begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} = \begin{bmatrix} \bar{T}_i \\ \bar{T}_o \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$, so $z_2(k)$ only related to $x_2(k)$ when Condition 2 holds. So $z_2(k)$ is measurable state.

4 Design of unmatched uncertainties states observer

To design sliding-mode control law, it is necessary to obtain the estimation of unmeasurable state $z_1(k)$. So we use the measurement of $z_2(k)$ and $y(k)$ to design unmatched uncertainties states observer.

Deduce the following reduced-order state equation from system (2):

$$\begin{cases} z_1(k+1) = (\bar{A}_{11} + \Delta\bar{A}_{11}) z_1(k) + (\bar{A}_{12} + \Delta\bar{A}_{12}) z_2(k) \\ y(k) = CT_i z_1(k) + CT_o z_2(k) \end{cases} \tag{3}$$

The states observer of $z_1(k)$ can be designed by equation (3):

$$\hat{z}_1(k+1) = \bar{A}_{11}\hat{z}_1(k) + LCT_i\tilde{z}_1(k) + \bar{A}_{12} z_2(k) \tag{4}$$

where $CT_i\tilde{z}_1(k) = CT_i z_1(k) - CT_i\hat{z}_1(k)$

$$= y(k) - CT_o z_2(k) - CT_i\hat{z}_1(k)$$

L is the output error feedback matrix. How to determine L such that equation (4) is asymptotic stability under the influence of unmatched uncertainties factors is the key to the design of the observer.

Theorem 1. The designed states observer (4) is asymptotic stability, if there exist matrices L and symmetric positive definite matrix P with appropriate dimensions such that the equation (5) holds.

$$\begin{bmatrix} A'^T PA' - P + C'^T C' & A'^T PB' & C'^T D' \\ B'^T PA' & B'^T PB' - I & 0 \\ D'^T C' & 0 & D'^T D' \end{bmatrix} < 0 \tag{5}$$

where $A' = \begin{bmatrix} \bar{A}_{11} & LCT_i \\ 0 & \bar{A}_{11} - LCT_i \end{bmatrix}, B' = \begin{bmatrix} 0 \\ \bar{T}_i D \end{bmatrix}$,

$C' = \begin{bmatrix} ET_i & ET_i \end{bmatrix}$, $D' = ET_o$.

Proof. For the introduction of state error vector, equation (6) can be obtained by subtracting equation (4) from equation (3) :

$$\begin{aligned} \tilde{z}_1(k+1) = \bar{A}_{11}\tilde{z}_1(k) + \Delta\bar{A}_{11}\hat{z}_1(k) + \Delta\bar{A}_{11}\tilde{z}_1(k) \\ -LCT_i\tilde{z}_1(k) + \Delta\bar{A}_{12} z_2(k) \end{aligned} \tag{6}$$

Extended-dimensional state equation (7) can be derived from equation (4) and equation (6):

$$\begin{bmatrix} \hat{z}_1(k+1) \\ \tilde{z}_1(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & LCT_i \\ \Delta\bar{A}_{11} & \bar{A}_{11} + \Delta\bar{A}_{11} - LCT_i \end{bmatrix} \begin{bmatrix} \hat{z}_1(k) \\ \tilde{z}_1(k) \end{bmatrix} + \begin{bmatrix} \bar{A}_{12} \\ \Delta\bar{A}_{12} \end{bmatrix} z_2(k) \tag{7}$$

State $z_2(k)$ is considered as the control vector. Equation (7) can be written as a linear fractional model (8) when we regardless of the determined item $\bar{A}_{12}$:

$$\begin{cases} \varepsilon(k+1) = A'\varepsilon(k) + B'\omega(k) \\ q(k) = C'\varepsilon(k) + D' z_2(k) \\ \omega(k) = \Sigma(k) q(k) \end{cases} \tag{8}$$

where $\varepsilon(k) = \begin{bmatrix} \hat{z}_1(k) & \tilde{z}_1(k) \end{bmatrix}^T$.

Select Lyapunov function as $V(k) = \varepsilon^T(k) P \varepsilon(k)$, where P is the symmetric positive definite matrix. So equation (9) is the sufficient condition for asymptotic stability of equation (8):

$$\begin{aligned} \Delta V(k) = \varepsilon^T(k+1) P\varepsilon(k+1) - \varepsilon^T(k) P\varepsilon(k) \\ -q^T(k) q(k) + q^T(k) q(k) \le \varepsilon^T(k+1) P\varepsilon(k+1) \\ -\varepsilon^T(k) P\varepsilon(k) - \omega^T(k)\omega(k) + q^T(k) q(k) < 0 \end{aligned} \tag{9}$$

Equation (9) can be expressed as LMI:

$$\Delta V(k) \le \begin{bmatrix} \varepsilon(k) & \omega(k) & z_2(k) \end{bmatrix} Q \begin{bmatrix} \varepsilon(k) & \omega(k) & z_2(k) \end{bmatrix}^T < 0 \tag{10}$$

where

$$Q = \begin{bmatrix} A'^T PA' - P + C'^T C' & A'^T PB' & C'^T D' \\ B'^T PA' & B'^T PB' - I & 0 \\ D'^T C' & 0 & D'^T D' \end{bmatrix}$$

.

If there exist P and L such that Q become a negative definite matrix, the asymptotic stability of equation (8) is guaranteed. So the asymptotic stability of states observer (4) is guaranteed also. This completes the proof.

In accordance with the above analysis, the block diagram of the unmatched uncertainties states observer is shown in figure 1.

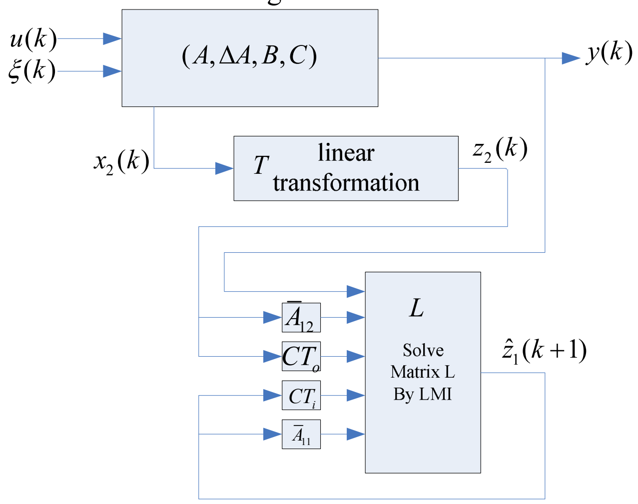

Figure 1 Block diagram of the states observer

5 Design of the variable structure controller

States can be observed stably from equation (4), they can be considered that : $z_1(k) = \hat{z}_1(k)$, so we can design reaching-law sliding-mode controller by means of all the states. We choose switching function as $\sigma(k) = \bar{B}^T \tilde{P} z(k)$, where $\tilde{P}$ is a symmetric positive definite matrix. According discrete system reaching-law:

$\sigma(k+1) - \sigma(k) = -\alpha \Delta T \operatorname{sgn}(\sigma(k)) - \beta \Delta T \sigma(k)$,

where ΔT is sampling period, α and β are constant, which determine the reaching rate and chattering amplitude of sliding-mode controller[3,19].

By system (1) and Assumption 2 we can see the uncertainties and perturbation are bounded functions[20], so they are satisfy

$$d_l(z(k)) \le (\bar{B}^T \tilde{P} \bar{B})^{-1} \bar{B}^T \tilde{P} \Delta A z(k) + \xi(k) \le d_u(z(k)) \tag{11}$$

Define mean of perturbation as

$$d_o = (d_u(z(k)) + d_l(z(k)))/2$$

and deviation of perturbation as

$$\delta_d = (d_u(z(k)) - d_l(z(k)))/2$$

By substituting switching function and discrete reaching-law into system (2), the discrete sliding-mode control law can be expressed as

$$u(k) = u_{eq}(k) + u_w(k) \tag{12}$$

where
$$u_{eq}(k) = \left(\bar{B}^T \tilde{P} \bar{B}\right)^{-1} [\bar{B}^T \tilde{P} z(k) - \bar{B}^T \tilde{P} \bar{A} z(k) - (1-\beta\Delta T)\sigma(k)]$$

$$u_w(k) = -\left(\bar{B}^T \tilde{P} \bar{B}\right)^{-1} [\alpha \Delta T \operatorname{sgn}(\sigma(k))] - (d_0 + \delta_d \operatorname{sgn} \sigma(k))$$

Theorem 2. Under the drive of control law (12), the sliding surface $\sigma(k) = \bar{B}^T \tilde{P} z(k)$ of controlled system (2) will be existent and reachable.

Proof. Equation(13) is the necessary condition for the existence and reaching of sliding surface in discrete control systems.

$$[\sigma(k+1) - \sigma(k)]\sigma(k) < 0 \tag{13}$$

Substitute system (2) into $\sigma(k+1)$, we get:

$$\begin{aligned} \sigma(k+1) &= \bar{B}^T \tilde{P} z(k+1) \\ &= \bar{B}^T \tilde{P}(A + \Delta A) z(k) + \bar{B}^T \tilde{P} \bar{B}[u(k) + \xi(k)] \end{aligned} \tag{14}$$

By substituting the control law (12) into equation (14), $\sigma(k+1)$ can be expressed as:

$$\begin{aligned} \sigma(k+1) &= \bar{B}^T \tilde{P}(A + \Delta A) z(k) + \bar{B}^T \tilde{P} z(k) \\ &- \bar{B}^T \tilde{P} A z(k) - (1-\beta\Delta T)\sigma(k) - \alpha \Delta T \operatorname{sgn} \sigma(k) \\ &+ \bar{B}^T \tilde{P} \bar{B}[-d_o - \delta_d \operatorname{sgn} \sigma(k) + \xi(k)] \end{aligned} \tag{15}$$

Then we get:

$$\begin{aligned} \sigma(k+1) - \sigma(k) &= \underbrace{-(1-\beta\Delta T)\sigma(k) - \alpha \Delta T \operatorname{sgn} \sigma(k)}_{part\ 1} \\ &\underbrace{+ \bar{B}^T \tilde{P} \Delta A z(k) + \bar{B}^T \tilde{P} \bar{B}[-d_o - \delta_d \operatorname{sgn} \sigma(k) + \xi(k)]}_{part\ 2} \end{aligned} \tag{16}$$

Firstly, we consider the part 1 of equation (16). Because of too short sample time ΔT, it can be deduced that $\Delta T \ll \dfrac{1}{\beta}$, hence

$$\left[-(1-\beta\Delta T)\sigma(k) - \alpha \Delta T \operatorname{sgn} \sigma(k)\right]\sigma(k) < 0$$

Next, we consider the part 2 of equation (16). According to Assumption 2 and equation (11), the part 2 of equation (16) can be expressed as

$$\begin{aligned} &\bar{B}^T \tilde{P} \Delta A z(k) + \bar{B}^T \tilde{P} \bar{B} \\ &[-d_o - \delta_d \operatorname{sgn} \sigma(k) + \xi(k)] \end{aligned} \begin{cases} < 0 & \sigma(k) > 0 \\ > 0 & \sigma(k) < 0 \end{cases}$$

By calculating of the above two-step, we get $[\sigma(k+1) - \sigma(k)]\sigma(k) < 0$, so Theorem 2 is proven.

6 Optimize the switching function parameters

Although the sliding surface is existent and reachable, there is a certain extent blindness to select matrix $\tilde{P}$. If choose $\tilde{P}$ improperly, the control system may become unstable, especially on the influence of uncertain items and external perturbation. This section studies how to optimize matrix $\tilde{P}$ to improve stability margin of closed-loop control system. To avoid the complex derivation, only consider the influence of $u_w(k)$ and $\xi(k)$ on the system, the unmatched uncertainties are considered as constant $\Delta A \equiv 0$.

Substitute control law (12) into system (2), we get

$$z(k+1) = \tilde{A}z(k) - \bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1}\tilde{\xi}_1(k) + \bar{B}\tilde{\xi}_2(k) \tag{17}$$

where

$$\tilde{A} = \bar{A} + \bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1}\bar{B}^T\tilde{P}[I - \bar{A} - (1-\beta\Delta T)],$$

$$\tilde{\xi}_1(k) = \alpha\Delta T\,\mathrm{sgn}(\sigma(k)),$$

$$\tilde{\xi}_2(k) = (d_0 + \delta_d\,\mathrm{sgn}\,\sigma(k)) + \xi(k).$$

Assumption 3. $\tilde{\xi}(k)$ is defined as

$$\tilde{\xi}(k) = \left[\tilde{\xi}_1(k) \quad \tilde{\xi}_2(k)\right]$$

there exist constants ρ_0 and ρ_1, such that $\tilde{\xi}(k) < \rho_0 + \rho_1\|z(k)\|$ [9], where ρ_0, ρ_1 are determined by the characteristic of $\tilde{\xi}(k)$.

Theorem 3. Consider Lyapunov function as $V(k) = z^T(k)\tilde{P}z(k)$, and the states trajectory remain on sliding surface under the control law (12), the min-max Lyapunov difference function can be deduced as

$$\min_{u_{eq}}\left[\max_{\tilde{\xi}_1,\tilde{\xi}_2}\Delta V(k+1)\right] = -z^T(k)(\tilde{P} - \tilde{A}^T\tilde{P}\tilde{A})z(k) + \begin{bmatrix}\tilde{\xi}_1(k)\\ \tilde{\xi}_2(k)\end{bmatrix}^T Y \begin{bmatrix}\tilde{\xi}_1(k)\\ \tilde{\xi}_2(k)\end{bmatrix}$$

where

$$Y = \begin{bmatrix} -\left(\bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1}\right)^T \\ \bar{B}^T\end{bmatrix}\tilde{P}\left[-\bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1} \quad \bar{B}\right]$$

Proof. Lyapunov difference function as follows:

$$\Delta V(k+1) = z^T(k+1)\tilde{P}z(k+1) - z^T(k)\tilde{P}z(k) \tag{18}$$

Substitute system (17) into equation (18), we get:

$$\begin{aligned}\Delta V(k+1) &= z^T(k)(\tilde{A}^T\tilde{P}\tilde{A} - \tilde{P})z(k)\\ &+ z^T(k)\tilde{A}^T\tilde{P}[-\bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1}\tilde{\xi}_1(k) + \bar{B}\tilde{\xi}_2(k)]\\ &+ [-\tilde{\xi}_1^T(k)\left(\bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1}\right)^T + \tilde{\xi}_2^T(k)\bar{B}^T]\tilde{P}\tilde{A}z(k)\\ &+ [-\tilde{\xi}_1^T(k)\left(\bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1}\right)^T + \tilde{\xi}_2^T(k)\bar{B}^T]\\ &\cdot[-\bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1}\tilde{\xi}_1(k) + \bar{B}\tilde{\xi}_2(k)]\end{aligned} \tag{19}$$

For the nominal linear system when $u_w(k) \equiv 0$, $\Delta A \equiv 0$, $\xi(k) \equiv 0$, an ideal sliding motion is obtained. Hence, while states trajectory of system (17) reaches to sliding surface, the switching function is considered as zero:

$$\begin{aligned}\sigma(k+1) &= \bar{B}^T\tilde{P}z(k+1)\\ &= \bar{B}^T\tilde{P}\left[\bar{A}z(k) + \bar{B}u_{eq}\right]\\ &= \bar{B}^T\tilde{P}\bar{A}z(k) + \bar{B}^T\tilde{P}\bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1}[\bar{B}^T\tilde{P}z(k) -\\ &\quad \bar{B}^T\tilde{P}\bar{A}z(k) - (1-\beta\Delta T)\sigma(k)]\\ &= \bar{B}^T\tilde{P}\tilde{A}z(k) = 0\end{aligned} \tag{20}$$

Substituting equation (20) into (19), the following items can be deduced to:

$$z^T(k)\tilde{A}^T\tilde{P}[-\bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1}\tilde{\xi}_1(k) + \bar{B}\tilde{\xi}_2(k)] = 0$$

$$[-\tilde{\xi}_1^T(k)\left(\bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1}\right)^T + \tilde{\xi}_2^T(k)\bar{B}^T]\tilde{P}\tilde{A}z(k) = 0$$

So equation (19) can be wrote as:

$$\begin{aligned}\Delta V(k+1) &= z^T(k)(\tilde{A}^T\tilde{P}\tilde{A} - \tilde{P})z(k)\\ &+ [-\tilde{\xi}_1^T(k)\left(\bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1}\right)^T + \tilde{\xi}_2^T(k)\bar{B}^T]\\ &\cdot[-\bar{B}\left(\bar{B}^T\tilde{P}\bar{B}\right)^{-1}\tilde{\xi}_1(k) + \bar{B}\tilde{\xi}_2(k)]\end{aligned}$$

Making some simple matrix operations, we can proof theorem 3.

Theorem 4.[21,22] If $\tilde{P} - \tilde{A}^T\tilde{P}\tilde{A}$, Y are positive definite matrices, variable μ can be defined as $\frac{1}{\mu} = \sqrt{\frac{\lambda_{\min}(\tilde{P} - \tilde{A}^T\tilde{P}\tilde{A})}{\lambda_{\max}(Y)}}$. System(17) can be guaranteed closed-loop control asymptotically stable with perturbation $\tilde{\xi}(k)$ when inequality $\rho_1 < \sqrt{1/\mu}$ holds.

$\sqrt{1/\mu}$ can be considered as the closed-loop stability margin of system(17) which is driven by control law(12). Where $\lambda_{\min}(\bullet)$ and $\lambda_{\max}(\bullet)$ indicate the minimum or maximum eigenvalues of corresponding matrices.

In section 4, $\tilde{P}$ is defined as a symmetric positive definite matrix; to ensure the stability of the closed-loop system, $\tilde{P}-\tilde{A}^T\tilde{P}\tilde{A}$ is requested a positive definite matrix, so they are satisfy the prerequisite of Theorem 4. With Theorem 3 and Theorem 4, optimize matrix $\tilde{P}$ by solving the following linear programming:

$$\begin{aligned} & \textit{find} \quad \tilde{P} \quad \min \mu \\ s.t. \quad & (i)\ \tilde{P}-\tilde{A}^T\tilde{P}\tilde{A} > I \\ & (ii)\ Y < \mu I \qquad (21) \\ & (iii)\ \tilde{P} > 0 \end{aligned}$$

By means of Theorem 4, we can determine whether the closed-loop control system is stability. System (2) can obtain the maximum stability margin by substituting optimization matrix $\tilde{P}$ into sliding-mode control law (12).

Based on the above analysis, this paper discusses the sliding mode controller design approach involves the following steps: linear transformation, LMI solution, optimize matrix $\tilde{P}$, calculation of sliding mode switching function and control law. The relationship between the various steps is shown in Figure 2.

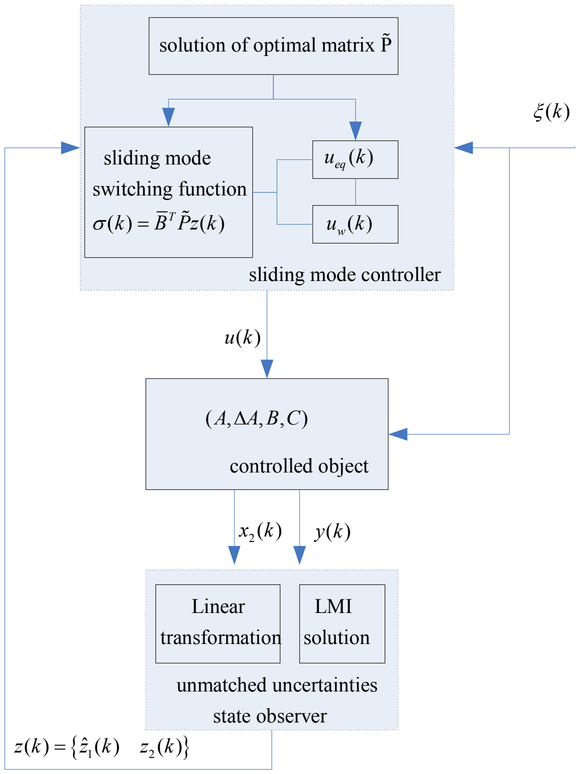

Figure 2 Block diagram
of the sliding-mode controller

7 Examples

In this section, two numerical examples will be considered. The unmatched uncertainties third-order system as follows:

$$\begin{cases} x(k+1)=(A+\Delta A)x(k)+B[u(k)+\xi(k)] \\ y(k)=Cx(k) \end{cases}$$

where

$$A=\begin{bmatrix} -5 & -1 & -1.35 \\ 1.1 & -0.5 & 0.45 \\ -2.4 & 1.5 & -1.65 \end{bmatrix},$$

$$B=\begin{bmatrix} 0 \\ 1 \\ 1.2 \end{bmatrix},$$

$$C=\begin{bmatrix} 1 & 0.6 & -0.7 \\ 0.2 & -0.5 & 0.4 \end{bmatrix},$$

$$\Delta A=\begin{bmatrix} 0 & 0.03\sin(100k) \\ 0.015\cos(50k) & 0 \\ 0.01\cos(50k) & 0 \end{bmatrix}$$

$$\begin{matrix} 0 \\ 0.05 \sum_{n=0}^{\infty} \delta(k-500n) \\ 0 \end{matrix} \Bigg]$$

The system states $x=[x_1,x_2,x_3]^T$, only x_3 is measurable.

The linear transformation matrix is chosen as

$$T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1.2 \end{bmatrix}$$

The unmatched uncertainties can be composed as $\Delta A = D\Sigma(k)E$, where

$$D=\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0.15 \\ 0 & 0 & 0.1 \end{bmatrix},$$

$$E=\begin{bmatrix} 0 & 0.3 & 0 \\ 0 & 0 & 0.5 \\ 0.1 & 0 & 0 \end{bmatrix},$$

$$\Sigma(k)=\begin{bmatrix} \sin(100k) & 0 & 0 \\ 0 & \sum_{n=0}^{\infty} \delta(k-500N) & 0 \\ 0 & 0 & \cos(50k) \end{bmatrix}$$

7.1 States estimation

To verify the effectiveness of the unmatched uncertainties observer design method, compared to this method with the traditional states observer design method of zero-pole placement. Given two output error feedback matrices:

$$L_1=\begin{bmatrix} 0.2674 & 0 \\ 0 & 0.2674 \end{bmatrix},$$

$$L_2=\begin{bmatrix} -0.5 & -2 \\ 0.2 & -0.8 \end{bmatrix}.$$

L_1 is calculated using Matlab LMI toolbox and satisfied inequality (10); L_2 is designed by traditional states observer design method, and the matrix $A_{11}-L_2CT_i$ pole is assigned in $[-4.3560 \quad -1.7640]$, but traditional states observer design method not consider $\Delta\bar{A}_{11}$ and $\Delta\bar{A}_{12}$, and not satisfied inequality (10).

In the numerical simulation, sampling period $\Delta T = 10ms$, simulation time is $40s$. Using L_1 and L_2 to observe states $[z_1 \quad z_2]^T$ respectively, and get the state estimation $[\hat{z}_{1\_L1} \quad \hat{z}_{2\_L1}]^T$ and $[\hat{z}_{1\_L2} \quad \hat{z}_{2\_L2}]^T$. Two numerical simulations are shown in this section.

In simulation one, the input control signal u is set as a ramp with slope of 0.2. Fig.3 shows the u and the theoretical response curve of z_1; Fig.4 shows the actual response curve of $\hat{z}_{1\_L1}$ and $\hat{z}_{1\_L2}$; Fig.5 shows the estimation error of $\tilde{z}_{1\_L1}$ and $\tilde{z}_{1\_L2}$.

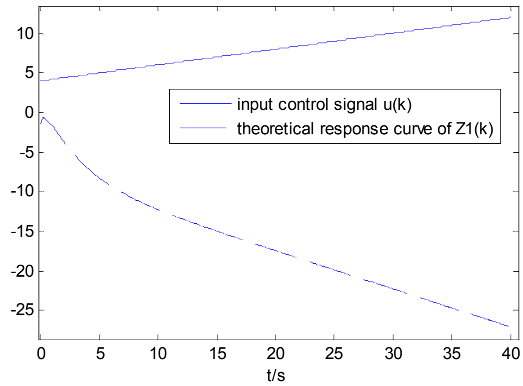

Fig.3 The input control signal and the system's theoretical response of z_1

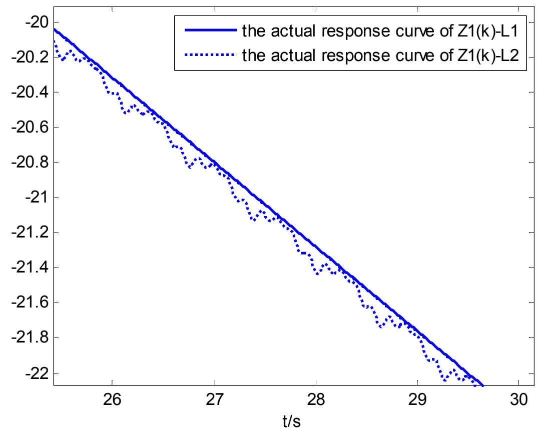

Fig.4 The estimation of observer L_1 and L_2

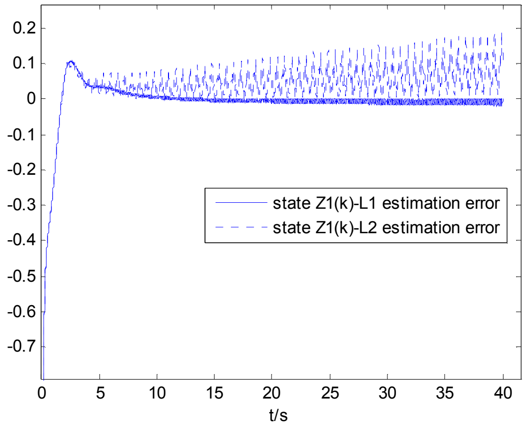

Fig.5 The estimation error of observer L_1 and L_2

In simulation two, the input control signal u is set as a sinusoid. Fig.6 shows the u and the theoretical response curve of z_1; Fig.7 shows the actual response curve of $\hat{z}_{1\_L1}$ and $\hat{z}_{1\_L2}$; Fig.8 shows the estimation error of $\tilde{z}_{1\_L1}$ and $\tilde{z}_{1\_L2}$.

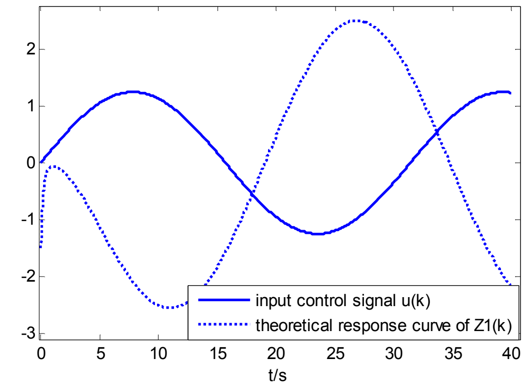

Fig.6 The input control signal and the system's theoretical response of z_1

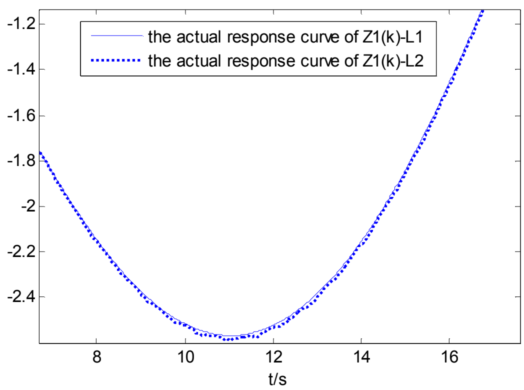

Fig.7 The estimation of observer L_1 and L_2

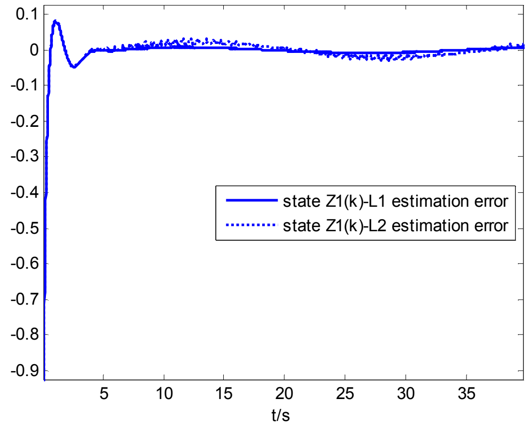

Fig.8 The estimation error of observer L_1 and L_2

It can be analysed from Fig.3-8 that $\hat{z}_{1\_L1}$ approaches to z_1 quickly and $\tilde{z}_{1\_L1}$ fluctuate around zero values less than $\tilde{z}_{1\_L2}$; $\tilde{z}_{1\_L2}$ shows the tendency to deviate from z_1 and gradual diverges. It is indicate that the traditional states observer design method is vulnerable to the influence of unmatched uncertainties, the states observer becomes instability caused by the pole location changing; unmatched uncertainties observer proposed in this paper considers the unmatched uncertainties factors, so it performances better.

7.2 Optimize sliding-mode controller

To demonstrate the effectiveness of the sliding-mode closed-loop controller optimization method, analysis two kinds of switching functions. Given the following numerical simulation dates: external perturbation $\xi(k) < 0.17 + 0.25|z(k)|^2$, sampling period $\Delta T = 10ms$, simulation time is $5s$, $\alpha = 0.5$, $\beta = 10$. Design different switching parameters σ_1 and σ_2 , where

$\sigma_1 = \bar{B}^T \tilde{P}_1 = [-0.5086 \quad 2.2114 \quad 4.5919]$,

$\sigma_2 = \bar{B}^T \tilde{P}_2 = [-0.3698 \quad 1.5897 \quad 1.5896]$.

σ_1 is obtained from the result of linear programming (21), and its stability margin $\sqrt{1/\mu} = 0.9333$; σ_2 is designed using traditional reaching-law sliding-mode control method, it satisfies $\sigma_2 B \neq 0$ but not satisfies equation(21), and its stability margin $\sqrt{1/\mu} = 0.3273$. Control law

$u_1(k)$ and $u_2(k)$ are deduced from σ_1 and σ_2 respectively. Under the control of $u_1(k)$ and $u_2(k)$, the closed-loop switching functions and state responses of system (2) are showed in Fig.9-12. From Fig.9-10 we can see that the chattering amplitude of switching function $\sigma_1 z(k)$ significantly smaller than $\sigma_2 z(k)$ under the same condition of $\xi(k)$, ΔT, α and β. Fig.11-12 show the fluctuation of the states responses $z(k)$ driven by $u_1(k)$ is smaller than it driven by $u_2(k)$. Although σ_1 and σ_2 can guarantee the stability of sliding mode controller, system (2) can better suppress the impact of external perturbation $\tilde{\xi}(k)$ when the switching parameter is chosen as σ_1, because of higher stability margin.

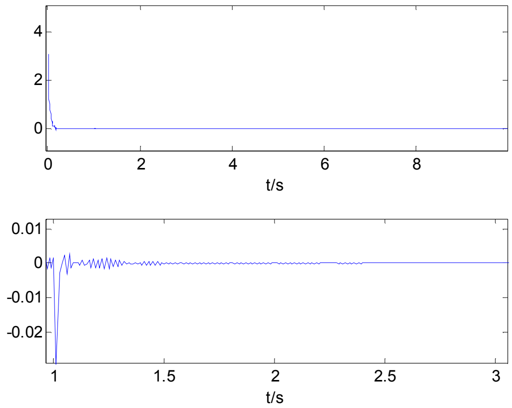

Fig.9 The trajectory of switching function $\sigma_1 z(k)$

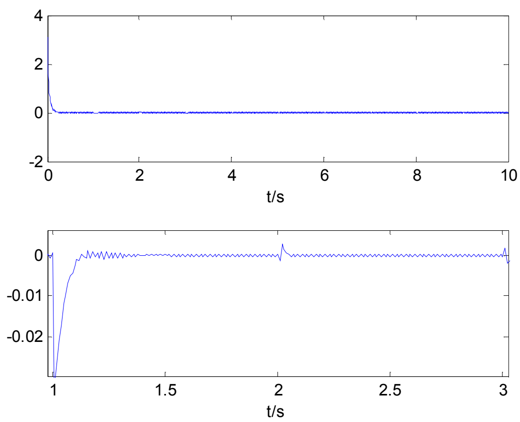

Fig.10 The trajectory of switching function $\sigma_2 z(k)$

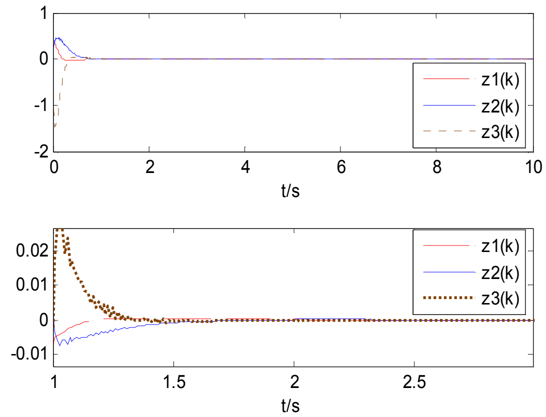

Fig.11 The state responses under the control of $u_1(k)$

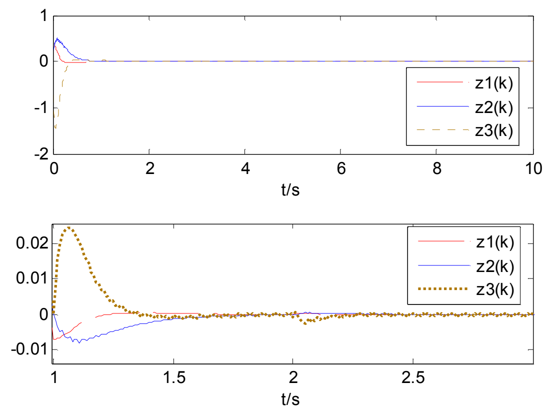

Fig.12 The state responses under the control of $u_2(k)$

8 Conclusions

For an unmatched uncertainties discrete system with states incomplete measurable, proposed a stable closed-loop sliding-mode control method based on an unmatched uncertainties states observer. This control method through the following three steps to achieve: firstly, the states are augmented with the error vector states, and the output error feedback matrix is obtained by solving the LMI; secondly, design reaching-law sliding-mode controller employing the results of the observer; thirdly, by deducing the min-max Lyapunov difference function and optimization sliding-mode switching function $\bar{B}^T \tilde{P} x(k)$, the closed-loop sliding-mode controller can get the maximum stability margin.. In the simulation two groups of parameter are compared, and the results show the effectiveness of the proposed control method.

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