A Theoretical and Empirical Analysis of Convergence Related Particle Swarm Optimization

MILAN R. RAPAIĆ, ŽELJKO KANOVIĆ, ZORAN D. JELIČIĆ
Automation and Control Systems Department
University of Novi Sad
Trg Dositeja Obradovića 6, 21000 Novi Sad
SERBIA
rapaja@uns.ac.rs

Abstract: - In this paper an extensive theoretical and empirical analysis of recently introduced Particle Swarm Optimization algorithm with Convergence Related parameters (CR-PSO) is presented. The convergence of the classical PSO algorithm is addressed in detail. The conditions that should be imposed on parameters of the algorithm in order for it to converge in mean-square have been derived. The practical implications of these conditions have been discussed. Based on these implications a novel, recently proposed parameterization scheme for the PSO has been introduced. The novel optimizer is tested on an extended set of benchmarks and the results are compared to the PSO with time-varying acceleration coefficients (TVAC-PSO) and the standard genetic algorithm (GA).

Key-Words: - Global Optimization, Swarm Intelligence, Evolutionary Computation, Particle Swarm Optimization (PSO), Algorithm Analysis

1 Introduction

The natural world is extraordinary complex and often provides us with remarkably elegant, robust and beautiful solutions to even the toughest problems. The field of global optimization has prospered much from such nature-inspired techniques. Since the first introduction of the evolutionary computing in the mid twentieth century (see [1] and references therein) a number of highly successful optimizers emerged. Genetic Algorithms (GAs), exploiting the ideas of Darwinian evolution, are arguably the most well known among these [2]. Deep connection between statistical mechanics and optimization was the source of another successful stochastic optimizer, Simulated Annealing (SA) [3]. More recently, the Ant Colony Optimization (ACO) algorithm inspired by shortest path search strategies utilized by ants during foraging was proposed [4,5] and utilized in various application [6]. The field of global optimization is, however, evolving rapidly and new optimizers are constantly emerging. Among the latest developments are Artificial Bees Colony (ABC) optimizer [7] and Bacteria Foraging Optimization (BFO) [8].

Among these nature-inspired strategies, Particle Swarm Optimization (PSO) algorithm is relatively novel, yet well studied and proven optimizer. Since its original introduction by Kennedy and Eberhart in 1995 [9], PSO raised a considerable interest among researchers. Originating in an attempt to mimic simplified social behavior of animals moving in large groups (birds in particular), PSO is grown to be a successful global optimization technique, well fit for solving complex, multimodal problems. Compared to other evolutionary techniques, most notably GA, PSO is simple and elegant in concept, it has but a few adjustable parameters, it is computationally inexpensive, very easy to implement and can easily be parallelized on massive parallel processing machines [10, 11, 12].

PSO operates on a set of particles. Each particle is characterized by its position (x) and velocity (v). Position of each particle is a potential solution, and each particle is capable of memorizing the best position it ever achieved in the course of optimization process (p). This position is referred to as the *personal best* position. The swarm as a whole memorizes the best position ever achieved by any of its particles (g), known as the *global best* position. In the k^{th} iteration, the position and velocity of each particle are updated as

$$v[k] = w[k]\cdot v[k-1] + cp[k]\cdot rp[k]\cdot (p[k]-x[k]) + \\ + cg[k]\cdot rg[k]\cdot (g[k]-x[k]) \quad (1)$$

$$x[k+1] = x[k] + v[k] \quad (2)$$

where w, cp and cg are parameters of the algorithm, *inertia*, *cognitive factor* and *social factor* respectively, while rp and rg are independent, uniformly distributed random numbers in the range [0, 1].

The idea behind PSO is the following. Each particle investigates a portion of a search space. During its investigation, a particle has a tendency to revisit the good areas, that is the neighbourhood of the points with small objective value. However, each particle is not totally independent. There is some communication inside the swarm. In particular, each particle communicates its personal best position. Knowing the best position found by any other particle, each member of the swarm will be attracted to this globally known best point.

Numerous studies have been published addressing PSO both empirically and theoretically, resulting in many modification of the algorithm. Among the first and most important of them is the study reported by Shi and Eberhart [13]. An account of early PSO development can be found in van der Berg's PhD thesis [14]. Over the years, the effectiveness of the algorithm was proven on various engineering problems [15, 16, 17, 18, 19, 20, 21, 22, 23]. However, the theoretical justification of the PSO procedure long remained open. A sound theoretical analysis of the algorithm was needed in order to address this issue properly. First formal theoretical analyses are due to Ozcan and Mohan [24, 25]. They addressed the dynamics of simplified, one-dimensional, deterministic PSO model. Clerc and Kennedy also analyzed PSO in [26] focusing on swarm stability and explosion. Jiang *et al* [27] were the first to analyze stochastic nature of the algorithm.

Several modifications of the original algorithm were proposed [14], including the ones designed for combinatorial optimization problems [28, 29]. Recently, Rapaić and Kanović explicitly addressed the time-varying nature of most practical PSO implementations in [30]. The result was a new PSO modification with parameters based on convergence analysis of the algorithm. Several variants of this modification, named Convergence Related PSO (CR-PSO), were tested on four standard benchmarks and compared to four other common PSO schemes. It was concluded that CR-PSO outperforms other PSO variants in most of the considered cases.

In this paper a more extensive empirical analysis of newly introduced algorithm is presented. The extended set of standard benchmark problems is used to test the performance of the algorithm. The obtained results are compared to the results of PSO with time-varying acceleration coefficients (TVAC-PSO) proposed by Ratnaweera, *et al* [11] and standard genetic algorithm (GA) [2], both applied on the same set of benchmarks.

The paper is organized as follows. A method to analyze convergence of particle swarms is revisited in Section 2. CR-PSO algorithm is presented in Section 3. Empirical results, including an account of benchmarks used to test the algorithms and the discussion of numerical experiments are given in Section 4. Section 5 contains the concluding remarks.

2 Convergence analysis of Particle Swarm Optimizers

Particle Swarm Optimization is a stochastic optimization procedure. Equations (1) and (2) can be considered as equations describing the motion of a discrete-time, linear stochastic system with two external inputs p and g.

In general there are different ways to define the notion of convergence for stochastic sequences. The notion of "mean-square" convergence is utilized in recent theoretical studies of PSO [27, 30]. In this setting, a stochastic sequence is said to converge to a real number a if and only if its mathematical expectation converges to a and simultaneously its variance convergence to zero. In this way, the investigation of convergence of a stochastic sequence can be made by analysis of two separate deterministic ones.

The main problem when analyzing the PSO algorithm lies in the fact that its parameters vary in time. If one would fix the PSO parameters, assuming that the algorithm does converge, the limit position of each particle would be

$$\mu = \frac{cp \cdot p + cg \cdot g}{cp + cg}, \tag{3}$$

where p and g are the limit values of the personal best and global best positions, respectively. In other words, the algorithm would converge to a point within a line segment connecting the limit values of the two attractor points. The distance from the limit position of a particle and each attractor would be proportional to the corresponding acceleration factor, cp and cg, respectively. Let us denote the mathematical expectation of a stochastic sequence by $\mathbf{E}$. The idea utilized in [30] is to analyze $\mathbf{E}(y[k])^2$, with

$$y[k] = x[k] - \mu[k] \tag{4}$$

and

$$\mu[k] = \frac{cp[k] \cdot p[k] + cg[k] \cdot g[k]}{cp[k] + cg[k]}. \tag{5}$$

If this newly introduced deterministic sequence converges to zero, the original stochastic sequence converges to μ. To see this, note that asymptotically $\mu[k]$ is equal to μ (the mathematical expectation of the limit value of x) and $\mathbf{E}(y[k])^2$ is asymptotically equal to the variance of a particle position.

It is readily obtained that (1) and (2) are equivalent to

$$x[k+1] = \big(1 + w[k] - cp[k]\cdot rp[k] - cg[k]\cdot rg[k]\big)x[k] - w[k]\cdot x[k-1] + cp[k]\cdot rp[k]\cdot p[k] + cg[k]\cdot cp[k]\cdot g[k] \quad (6)$$

and, due to (4)

$$y[k+1] = (\Psi[k] - R[k])y[k] - w[k]y[k-1] + Q[k]. \quad (7)$$

where the following abbreviations were introduced in order to shorten the notation

$$\Psi[k] = 1 + w[k] - 0.5(cp[k] + cg[k]), \quad (8)$$

$$R[k] = cp[k]rp[k] + cg[k]rg[k] - 0.5(cp[k] + cg[k]), \quad (9)$$

$$Q[k] = cp[k]cg[k](g[k] - p[k])\frac{rg[k] - rp[k]}{cp[k] + cg[k]}. \quad (10)$$

By application of a series of elementary, yet somewhat cumbersome steps, the following expressions can be derived from (7)

$$\mathbf{E}y[k+1] = \Psi[k]\mathbf{E}y[k] - w[k]\mathbf{E}y[k-1], \quad (11)$$

$$\mathbf{E}y^2[k+1] = (\Psi^2[k] + \mathbf{E}R^2[k])\mathbf{E}y^2[k] + w^2[k]\mathbf{E}y^2[k-1] - 2w[k]\Psi[k]\mathbf{E}(y[k]y[k-1]) - 2\mathbf{E}(Q[k]R[k])\mathbf{E}y[k] + \mathbf{E}Q^2[k], \quad (12)$$

$$\mathbf{E}(y[k+1]y[k]) = \Psi[k]\mathbf{E}y^2[k] - w[k]\mathbf{E}(y[k]y[k-1]). \quad (13)$$

The key point is that equations (7), (11), (12) and (13) define the dynamics of a linear time-varying (LTV) system. To see this, let us introduce

$$\alpha_1[k] = \mathbf{E}y[k-1], \quad (14)$$

$$\alpha_2[k] = \mathbf{E}y[k], \quad (15)$$

$$\alpha_3[k] = \mathbf{E}y^2[k-1], \quad (16)$$

$$\alpha_4[k] = \mathbf{E}y^2[k], \quad (17)$$

$$\alpha_5[k] = \mathbf{E}\big(y[k]y[k-1]\big). \quad (18)$$

$$\boldsymbol{\alpha} = \begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5\end{bmatrix}. \quad (19)$$

It now obvious that (7), (11), (12) and (13) can be expressed in a more compact fashion using matrix notation

$$\boldsymbol{\alpha}[k+1] = \mathbf{A}[k]\boldsymbol{\alpha}[k] + \mathbf{b}[k] \quad (20)$$

with

$$\mathbf{A}[k] = \begin{bmatrix}\mathbf{A}_1[k] & \mathbf{0} \\ \mathbf{A}_3[k] & \mathbf{A}_2[k]\end{bmatrix}, \quad (21)$$

$$\mathbf{A}_1[k] = \begin{bmatrix}0 & 1 \\ -w[k] & \Psi[k]\end{bmatrix}, \quad (22)$$

$$\mathbf{A}_2[k] = \begin{bmatrix}0 & 1 & 0 \\ w^2[k] & \Psi^2[k] + \mathbf{E}R^2[k] & -2w[k]\Psi[k] \\ 0 & \Psi[k] & -w[k]\end{bmatrix}, \quad (23)$$

$$\mathbf{A}_3[k] = \begin{bmatrix}0 & 0 & 0 \\ 0 & -2\mathbf{E}\big(Q[k]R[k]\big) & w^2[k] \\ 0 & 0 & 0\end{bmatrix}, \quad (24)$$

$$\mathbf{b}[k] = \begin{bmatrix}0 \\ 0 \\ 0 \\ \mathbf{E}Q^2[k] \\ 0\end{bmatrix}. \quad (25)$$

If the system (20) is *stable* in the sense usually applied in control theory, the PSO algorithm converges. Further discussion regarding this statement can be found in [13], [14] and [15]. Related discussion can also be found in [10], [11] and [12].

The investigation of stability of LTV systems is not trivial, although it was addressed many times in literature. In general, it is common to apply Lyapunov methods [31], but other techniques have been utilized as well [32]. The following statement was formulated and proven in [30].

Lemma 1. *Consider a sequence generated by first-order difference equation*

$$\mathbf{x}[k+1] = \mathbf{A}[k]\mathbf{x}[k] + \mathbf{b}[k], \quad \mathbf{x}[0] = \mathbf{x}_0 \quad (26)$$

Assume $\mathbf{A}[k]$ *converges to and* $\mathbf{b}[k]$ *converges to* $\mathbf{b}$*. Then, if all eigenvalues of* $\mathbf{A}$ *lie within the unit circle, the sequence* $\mathbf{x}[k]$ *converges to* $\big(\mathbf{I} - \mathbf{A}\big)^{-1}\mathbf{b}$.

In order to apply Lemma 1 to the systems described by (20), it is necessary to investigate locations of the eigenvalues of matrix $\mathbf{A}[k]$. As it is well known, eigenvalues of a matrix are zeros of its characteristic polynomial.

$$f(z,k) = \det(z\mathbf{I} - \mathbf{A}[k]) \quad (27)$$

Note that the $\mathbf{A}[k]$ matrix is lower block-triangular for any k. Its characteristic polynomial can therefore be found as

$$f(z,k) = f_1(z,k) f_2(z,k) \quad (28)$$

where $f_1(z,k)$ and $f_2(z,k)$ are the characteristic polynomials of matrices $\mathbf{A}_1[k]$ and $\mathbf{A}_2[k]$. These polynomials are

$$f_1(z,k) = z^2 - \Psi[k]z + w[k], \quad (29)$$

$$\begin{aligned} f_2(z,k) = z^3 &- \left(\Psi^2[k] + \mathbf{E}R^2[k] - w[k]\right)z^2 \\ &+ w[k]\left(\Psi^2[k] - \mathbf{E}R^2[k] - w[k]\right)z \;. \\ &- w^3[k] \end{aligned} \quad (30)$$

The location of roots of these polynomials can be investigated by various methods, including for example Jury's criterium widely utilized in control theory [33]. However, the conditions for the roots of (29) and (30) to lie within the unit circle have already been investigated in [27]. These conditions are

$$0 < w[k] < 1, \quad (31)$$

$$cp[k] + cg[k] < 4(1 + w[k]), \quad (32)$$

$$f_2(1,k) > 0. \quad (33)$$

In fact, it was proven in [27, 30] and also demonstrated in the sequel of this paper that (33) implies (32).
Combining Lemma 1 with the above conditions regarding the location of eigenvalues, the following statement can be derived

Theorem 1. *Assume that both $p[k]$ and $g[k]$ converge to g. Assume also that the PSO parameters $w[k]$, $cp[k]$ and $cg[k]$ converge to their respective limit values w, cp and cg. If the conditions(31)and (33) hold for all k larger than some k_0 the PSO algorithm defined by (1) and (2) converges to g.*

Theorem 1 is a direct consequence of Lemma 1. To see this, note that if global best and personal best particle position both converge to the same limit point g, it follows from (10) that $\mathbf{E}Q^2[k]$ is zero in limit. Therefore, by Lemma 1 α zero in limit also. By definition of α is clear now that both mean value and variance of y are asymptotically zero. As a direct consequence, the mean value of $x[k]$ tends to μ and its variance is zero in limit. However, from the equality of personal and global best position limits it follows that asymptotically $\mu = g$.

The assumption that the personal best and local best particle position converge is reasonable. It is, in fact, related to the shape of the optimality criteria. If an optimum does exists it is usually the case that these attractor points will converge. In practice, at the end of the search the entire swarm usually clusters in a small area of the search space. There is however, no guarantee that the swarm will converge to either global or local optimum. Van den Berg suggested a modification of PSO that guarantees that the limit point will be at least locally optimal [2].

2 PSO with convergence related parameters

Various authors report numerous different recommendations for the selection of the PSO parameters. These recommendations can be roughly classified as either empirical or theoretical. Empirical analyses are typically conducted using a set of benchmark problems. Theoretical analyses usually address the convergence of the optimizers, as well as the communication topology of the swarm.

It is generally true for all population-based optimizers that high population diversity is desirable at initial phases of the optimization process. The particles should initially be allowed to roam freely through the search-space. In the later stages, as the optimization process reaches its end, the particles should become more constrained and the swarm needs to be more concentrated. The swarm should not attempt to reach uncovered areas of the search space, but should rather try to fine-tune the good solutions found so far. Consequently, the dynamics of each individual particle should be less stable at the beginning of the search and become more stable at the end. According to condition (31) this means that the inertia factor (w) should decrease during the search. It is common practice to use inertia factor decreasing from 0.9 to 0.4. It is also plausible to use a bit higher initial inertia value, 1.2 for example. In this case, the swarm will be unstable initially, and consequently would diversify considerably in the first few iterations of the optimization procedure.

It is also known that particles should be more-or-less independent in the initial phases of the search. This enables the swarm to spread widely across the search space. However, as the optimization process finishes, the swarm should act more as a community, *i.e.* the particles should move in accordance to the global knowledge of the swarm. This means that cognitive factor (*cp*) should dominate social factor (*cg*) in the beginning, yet social factor should prevail near the end of the search. It has been proven in [30] that *cp*=*cg* is the best choice when only the speed of convergence of the algorithm is regarded. However, this is not the only issue one should think of when adjusting parameters of a PSO scheme. The TVAC-PSO scheme, for example, uses cognitive factor decreasing from 2.5 to 0.5 and social factor simultaneously increasing from 0.5 to 2.5.

Based on these recommendations and on detailed convergence analysis presented earlier, a new set of PSO parameters is proposed in [30]. These parameters are the **acceleration constriction ratio** (ξ) defined by

$$cp[k]+cg[k]=4\xi[k]\cdot(1+w[k]) \tag{34}$$

and the **acceleration ratio** (η) such that

$$cg[k]=\eta[k]\cdot cp[k]\ . \tag{35}$$

Note that (34) formally resembles (32). The original PSO parameters (w, cp, cg) can be replaced by a set of new ones (w, ξ, η). The proposed parameters simplify convergence conditions (31), (32), (33), and enable direct control over the diversity of the swarm in the course of the optimization.

According to [30], the CR-PSO is convergent provided that (31),

$$0\le\xi[k]\le 1, \tag{36}$$

$$\frac{1-w^2[k]}{(1+w[k])^2}\frac{1-\xi[k]}{\xi[k]}>g(\eta[k])=\frac{1}{3}\frac{1+\eta^2[k]}{(1+\eta[k])^2}\ . \tag{37}$$

The condition (36) is a direct consequence of the definition of acceleration constriction ratio (34) and previously introduced condition (32). The condition (37) is derived by simple, yet tidious calculations from (33). Interesting conclusions can be obtained by ploting the right hand side of (37) as a function of η. This plot is presented in Fig 1.

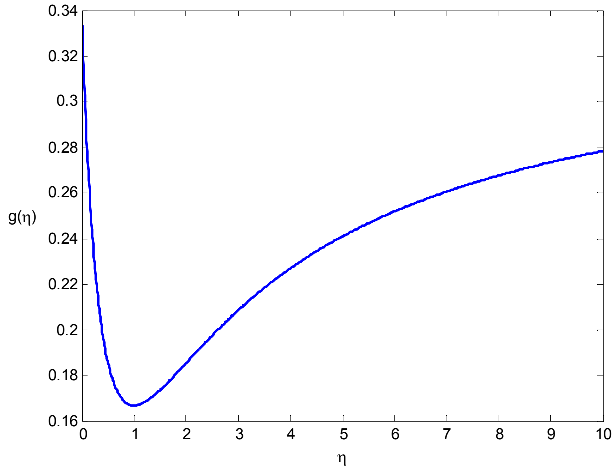

Fig 1. *The right-hand side of condition (37) as a function of acceleration ratio η.*

Notice that *g(η)* is allways positive. Therefore, in order for the condition (37) to hold the left hand side must also be positive. This implies (36). Therefore (37) implies (36), and consequently (33) implies (32). It is also interesting to note that when (32) is satisfied the condition (33) is not too restrictive. The following figure depicts the area of convergence of CR-PSO in w-ξ plane.

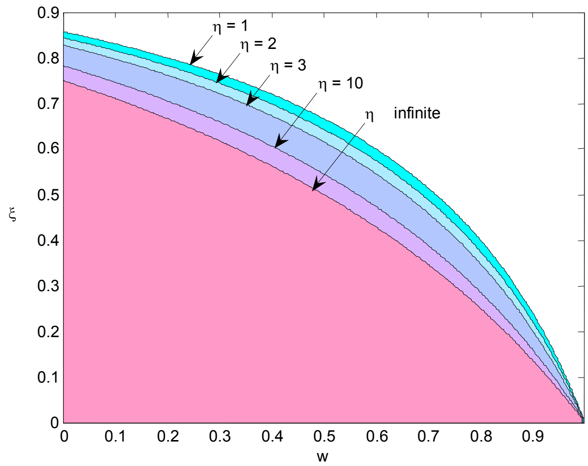

Fig 2. *The area of convergence of CR-PSO in w-ξ plane*

Having in mind these conditions, several parameter adjustment schemes can be adopted. For instance, by keeping ξ and η fixed while decreasing w, the algorithm is guaranteed to gradually move deeper in the convergence region. However, other parameter adaptation schemes are also plausible, since all three parameters can be varied simultaneously. A particularly promising idea is to keep ξ fixed, while decreasing w and increasing η. The later is not beneficial regarding the convergence, yet, it is beneficial regarding the explorative and exploitative abilities of the algorithm.

In the following analysis two parameter adjustment schemes are applied, based on experimental results shown in [30]. In both of them, inertia factor w is chosen to be linearly decreased from 0.9 to 0.4, which is common choice in the literature, and factor ξ is kept fixed at value of 0.5. Factor η is increased from 1/5 to 5 in the first scheme (CR-PSO v1), and from 1/2.5 to 2.5 in the second scheme (CR-PSO v2).

3 Experimental results

Performance of the algorithms is compared on a set of benchmarks listed in Table 1. All benchmarks attain a global minimal value of zero. Two experiments were performed, both including 100 trials with 30 particles moving for 100 iterations in a 5-dimensional search space. In the first experiment, the entire initial population was shifted away, so that an extensive exploration was needed in order for a search procedure to localize the global optimum. In the second experiment, particles were initialized within a hypercube centred on the global optimal solution. In order to find the global optimum, algorithms needed to overcome only a relatively small number of local optimal solutions. Edges of the initial hypercube in both experiments are chosen to be of length 20. In the second experiment, the shift applied to each direction is chosen to be 100.

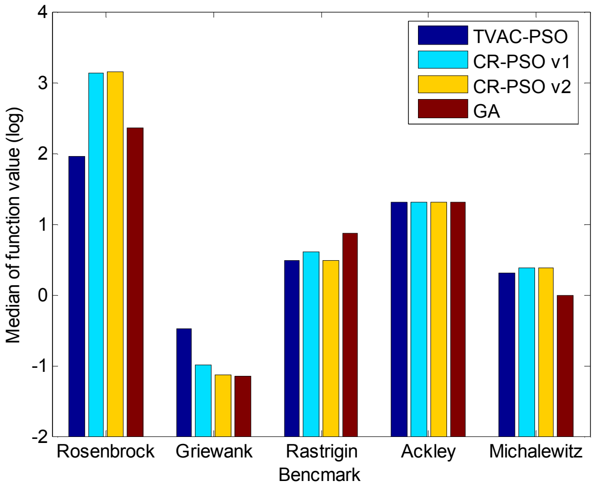

Fig3. *The results of the first experiment, with initial population shifted (benchmarks 1-5)*

For the PSO with time-varying acceleration coefficients (TVAC-PSO) inertia factor w was also linearly decreased from 0.9 to 0.4, cognitive factor cp was decreased from 2.5 to 0.5 and social factor cg was increased from 0.5 to 2.5. For the genetic algorithm, the following options were applied: rank scaling function, stochastic uniform selection, scattered crossover function, elitism with two elite individuals. Standard MATLAB implementation of GA was used. In order for the comparison to be fair, the same number of individuals (30) and generations (100) was used.

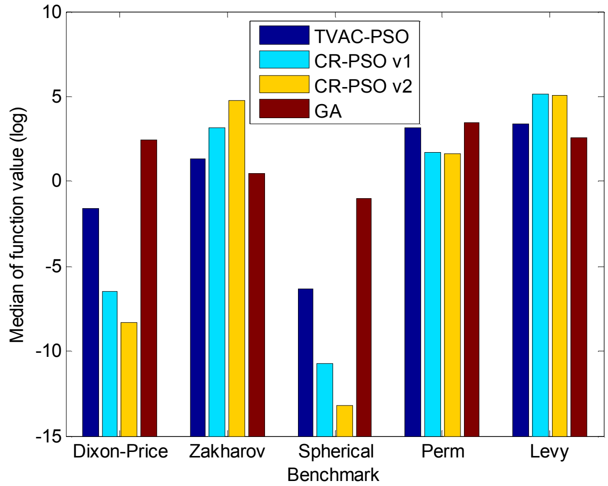

Fig4. *The results of the first experiment, with initial population shifted (benchmarks 6-10)*

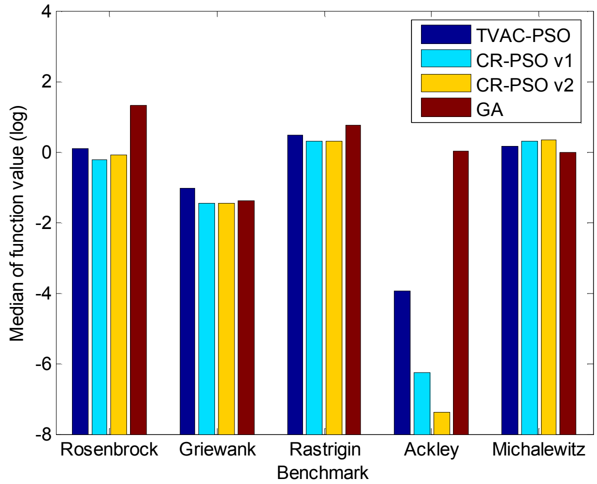

Fig5. *The results of the second experiment, with initial population centred (benchmarks 1-5)*

Fig.3 and Fig.4 show results of the first experiment. Graphs represent the median of the function value in logarithmic scale for TVAC-PSO, both variants of CR-PSO (in both variants) and GA, respectively.

Clearly, both CR-PSO schemes outperform TVAC-PSO on Griewank, Dixon-Price, Spherical and Perm function. However, in the case of Griewank function, all three PSO variants perform worse than GA. All optimizers fail in an attempt to optimize the Ackley's function. CR-PSO seems considerably worse in the case of Rosenbrock, Zakharov and Levy function. It also seems that variant 2 of CR-PSO outperforms variant 1 in most of the considered cases.

Fig.5 and Fig.6 show the results of the second experiment. In this case, both CR-PSO variants show superior performance on all of the considered benchmark, with the exception of Michalewitz function. It can also be concluded that CR-PSO v2 outperforms CR-PSO v1 in most cases.

The results of both experiments are presented in detail in Tables 5 and 6, respectively.

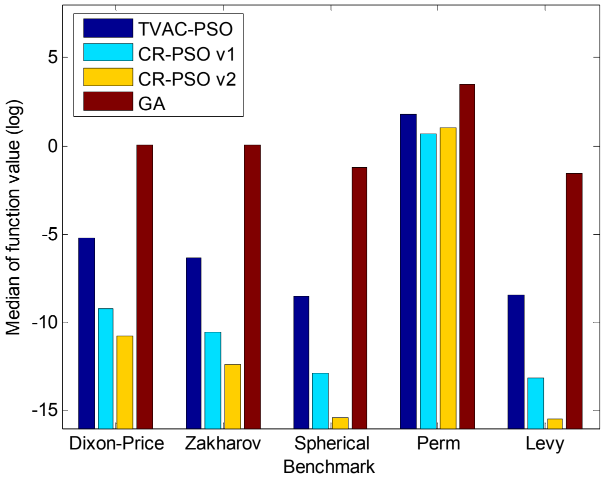

Fig6. *The results of the second experiment, with initial population centred (benchmarks 6-10)*

4 Conclusion

Recently, a novel, convergence-related parameterization of the well-known PSO meta-heuristic was proposed [30]. The presented optimizers were theoretically well based, and initial empirical results seemed promising. A detailed theoretical analysis has been presented in this paper. This theoretical analysis, identical to the one presented in [30], yet a bit more elaborate, provides a significant insight to the stochastic dynamic of the PSO swarm. A detailed empirical analysis has also been presented in this paper. Two of the most successful CR-PSO schemes were benchmarked against TVAC-PSO and GA on ten well-known unconstrained global optimization benchmark problems.

The results presented in this paper confirm those presented previously in [30]. The tighter control over the convergence of the swarm does, in most cases, ensure better performance of the optimizer. However, there is still much space for improvements. In particular, the effect of the acceleration constriction ratio factor is still not well understood. Also, it seems that even more efficient optimizer could be constructed by hybridization of CR-PSO with GA, *i.e.* by introduction of evolutionary operators, such are crossover and mutation, in the original CR-PSO. Further research should address this and other similar issues in more depth.

Table 1. *An account of benchmark functions used for comparison.*

Rosenbrock	$f(\mathbf{x}) = \sum_{i=1}^{n-1}\left[100(x_i^2 - x_{i+1})^2 + (x_i - 1)^2\right]$
Griewank	$f(\mathbf{x}) = \sum_{i=1}^{n} \frac{x_i^2}{4000} - \prod_{i=1}^{n} \cos\left(x_i / \sqrt{i}\right) + 1$
Rastrigin	$f(\mathbf{x}) = 10n + \sum_{i=1}^{n}(x_i^2 - 10\cos(2\pi x_i))$
Ackley	$f(\mathbf{x}) = 20 + e - 20\exp\left(-\frac{1}{5}\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}\right) - \exp\left(-\frac{1}{n}\sum_{i=1}^{n}\cos(2\pi x_i)\right)$
Michalewitz	$f(\mathbf{x}) = 5.2778 - \sum_{i=2}^{n}\sin(x_i)\sin^{2m}\left(\frac{1}{\pi} i x_i^2\right),\ m = 10$
Dixon-Price	$f(\mathbf{x}) = (x_1 - 1)^2 + \sum_{i=2}^{n} i(2x_i^2 - x_{i-1})^2$
Zakharov	$f(\mathbf{x}) = \sum_{i=1}^{n} x_i^2 + \left(\sum_{i=1}^{n} 0.5 i x_i\right)^2 + \left(\sum_{i=1}^{n} 0.5 i x_i\right)^4$
Spherical	$f(\mathbf{x}) = \sum_{i=1}^{n} x_i^2$
Perm	$f(\mathbf{x}) = \sum_{k=1}^{n}\left[\sum_{i=1}^{n}\left(i^k + \beta\right)\left(x_i^k i^{-k} - 1\right)\right]^2$
Levy	$f(\mathbf{x}) = \sin^2(\pi y_1) + \sum_{i=1}^{n-1}\left[(y_i - 1)^2\left(1 + 10\sin^2(\pi y_i + 1)\right)\right] + (y_n - 1)^2\left(1 + 10\sin^2(2\pi y_n)\right),\ y_i = 1 + \frac{x_i - 1}{4}$

Table 2. *The results of the first experiment (initial population shifted from global optimum).*

Benchmark	TVAC - PSO			CR-PSO v1		
	mean	median	std. dev.	mean	median	std. dev.
Dixon-Price	$9.29\cdot10^{-1}$	$2.50\cdot10^{-2}$	$1.82\cdot10^{1}$	$1.86\cdot10^{-1}$	$3.23\cdot10^{-7}$	$3.00\cdot10^{-1}$
Rosenbrock	$3.167\cdot10^{2}$	$9.17\cdot10^{1}$	$6.127\cdot10^{2}$	$1.47\cdot10^{4}$	$1.34\cdot10^{3}$	$2.77\cdot10^{4}$
Zakharov	$7.60\cdot10^{2}$	$2.19\cdot10^{1}$	$2.11\cdot10^{3}$	$3.49\cdot10^{4}$	$1.57\cdot10^{3}$	$8.79\cdot10^{4}$
Griewank	$8.06\cdot10^{-1}$	$3.32\cdot10^{-1}$	$9.25\cdot10^{-1}$	$1.17\cdot10^{-1}$	$1.03\cdot10^{-1}$	$7.14\cdot10^{-2}$
Rastrigin	4.12	3.08	3.25	4.18	3.97	2.70
Ackley	$2.00\cdot10^{1}$	$2.00\cdot10^{1}$	$2.00\cdot10^{-2}$	$2.00\cdot10^{1}$	$2.00\cdot10^{1}$	$5.48\cdot10^{-2}$
Michalewitz	2.05	2.06	$4.49\cdot10^{-1}$	2.26	2.34	$3.85\cdot10^{-1}$
Spherical	$2.45\cdot10^{-6}$	$4.43\cdot10^{-7}$	$1.50\cdot10^{-5}$	$1.37\cdot10^{-10}$	$1.86\cdot10^{-11}$	$4.50\cdot10^{-10}$
Perm	$5.50\cdot10^{14}$	$1.43\cdot10^{3}$	$2.85\cdot10^{15}$	$1.17\cdot10^{21}$	$4.94\cdot10^{1}$	$9.46\cdot10^{21}$
Levy	$2.38\cdot10^{3}$	$2.40\cdot10^{3}$	$5.13\cdot10^{2}$	$1.36\cdot10^{5}$	$1.29\cdot10^{5}$	$1.20\cdot10^{5}$
	CR-PSO v2			GA		
	mean	median	std. dev.	mean	median	std. dev.
Dixon-Price	$2.27\cdot10^{-1}$	$4.60\cdot10^{-9}$	$3.15\cdot10^{-1}$	$1.06\cdot10^{3}$	$2.94\cdot10^{2}$	$1.710\cdot10^{3}$
Rosenbrock	$1.34\cdot10^{4}$	$1.42\cdot10^{3}$	$2.34\cdot10^{4}$	$1.81\cdot10^{3}$	$2.29\cdot10^{2}$	$3.77\cdot10^{3}$
Zakharov	$1.59\cdot10^{5}$	$6.11\cdot10^{4}$	$2.46\cdot10^{5}$	$1.76\cdot10^{1}$	2.86	$4.46\cdot10^{1}$
Griewank	$8.47\cdot10^{-2}$	$7.26\cdot10^{-2}$	$5.82\cdot10^{-2}$	$8.94\cdot10^{-2}$	$7.23\cdot10^{-2}$	$6.46\cdot10^{-2}$
Rastrigin	3.07	2.98	1.90	7.39	7.27	2.82
Ackley	$2.00\cdot10^{1}$	$2.00\cdot10^{1}$	$4.98\cdot10^{-2}$	$2.00\cdot10^{1}$	$2.00\cdot10^{1}$	$5.40\cdot10^{-3}$
Michalewitz	2.32	2.34	$3.68\cdot10^{-1}$	$9.94\cdot10^{-1}$	$9.65\cdot10^{-1}$	$2.38\cdot10^{-1}$
Spherical	$2.78\cdot10^{-13}$	$5.97\cdot10^{-14}$	$8.09\cdot10^{-13}$	$1.22\cdot10^{-1}$	$9.85\cdot10^{-2}$	$1.18\cdot10^{-1}$
Perm	$7.24\cdot10^{21}$	$4.31\cdot10^{1}$	$5.07\cdot10^{22}$	$2.46\cdot10^{12}$	$2.78\cdot10^{3}$	$1.96\cdot10^{13}$
Levy	$1.22\cdot10^{5}$	$1.11\cdot10^{5}$	$1.16\cdot10^{5}$	$4.95\cdot10^{2}$	$4.10\cdot10^{2}$	$5.02\cdot10^{2}$

Table 3. *The results of the second experiment (initial population centered around global optimum).*

Benchmark	TVAC - PSO			CR-PSO v1		
	mean	median	std. dev.	mean	median	std. dev.
Dixon-Price	$1.77\cdot10^{-1}$	$6.36\cdot10^{-6}$	$2.94\cdot10^{-1}$	$2.26\cdot10^{-1}$	$6.35\cdot10^{-10}$	$3.17\cdot10^{-1}$
Rosenbrock	$1.61\cdot10^{1}$	1.19	$7.85\cdot10^{1}$	4.30	$6.09\cdot10^{-1}$	$2.43\cdot10^{1}$
Zakharov	$2.62\cdot10^{-6}$	$4.15\cdot10^{-7}$	$6.48\cdot10^{-6}$	$1.27\cdot10^{-10}$	$2.89\cdot10^{-11}$	$2.79\cdot10^{-10}$
Griewank	$1.02\cdot10^{-1}$	$9.40\cdot10^{-2}$	$4.90\cdot10^{-2}$	$4.49\cdot10^{-2}$	$3.44\cdot10^{-2}$	$3.18\cdot10^{-2}$
Rastrigin	4.21	3.02	2.96	2.62	1.98	1.89
Ackley	$4.95\cdot10^{-2}$	$1.11\cdot10^{-4}$	$2.82\cdot10^{-1}$	$8.30\cdot10^{-7}$	$5.42\cdot10^{-7}$	$1.10\cdot10^{-6}$
Michalewitz	1.42	1.42	$5.97\cdot10^{-1}$	1.97	2.03	$6.10\cdot10^{-1}$
Spherical	$7.72\cdot10^{-9}$	$2.88\cdot10^{-9}$	$1.51\cdot10^{-8}$	$1.02\cdot10^{-12}$	$1.29\cdot10^{-13}$	$6.54\cdot10^{-12}$
Perm	$2.70\cdot10^{2}$	$5.92\cdot10^{1}$	$4.61\cdot10^{2}$	$1.30\cdot10^{1}$	4.29	$2.82\cdot10^{1}$
Levy	$4.25\cdot10^{-2}$	$3.34\cdot10^{-9}$	$2.19\cdot10^{-1}$	$2.71\cdot10^{-13}$	$7.46\cdot10^{-14}$	$5.49\cdot10^{-13}$
	CR-PSO v2			GA		
	mean	median	std. dev.	mean	median	std. dev.
Dixon-Price	$2.191\cdot10^{-1}$	$1.76\cdot10^{-11}$	$3.14\cdot10^{-1}$	2.61	1.09	4.05
Rosenbrock	3.87	$7.93\cdot10^{-1}$	$1.67\cdot10^{1}$	$3.96\cdot10^{1}$	$1.99\cdot10^{1}$	$5.50\cdot10^{1}$
Zakharov	$2.87\cdot10^{-12}$	$4.22\cdot10^{-13}$	$7.24\cdot10^{-12}$	1.77	1.07	2.11
Griewank	$4.48\cdot10^{-2}$	$3.49\cdot10^{-2}$	$2.95\cdot10^{-2}$	$4.59\cdot10^{-2}$	$4.02\cdot10^{-2}$	$2.35\cdot10^{-2}$
Rastrigin	2.40	1.98	1.66	5.94	5.61	2.59
Ackley	$5.41\cdot10^{-8}$	$4.15\cdot10^{-8}$	$4.37\cdot10^{-8}$	1.13	1.06	$5.74\cdot10^{-1}$
Michalewitz	1.98	2.07	$5.82\cdot10^{-1}$	$9.70\cdot10^{-1}$	$9.48\cdot10^{-1}$	$2.52\cdot10^{-1}$
Spherical	$1.79\cdot10^{-15}$	$4.10\cdot10^{-16}$	$5.99\cdot10^{-15}$	$8.20\cdot10^{-2}$	$6.12\cdot10^{-2}$	$7.04\cdot10^{-2}$
Perm	$3.16\cdot10^{1}$	9.60	$6.99\cdot10^{1}$	$7.61\cdot10^{3}$	$2.85\cdot10^{3}$	$1.62\cdot10^{4}$
Levy	$1.45\cdot10^{-15}$	$3.36\cdot10^{-16}$	$3.09\cdot10^{-15}$	$3.66\cdot10^{-2}$	$2.74\cdot10^{-2}$	$3.20\cdot10^{-2}$

References:

[1] Bäck, T., Hammel, U., Schwefel, H.-P.: Evolutionary Computation: Comments on the History and Current State. *IEEE Transactions on Evolutionary Computation*, 1, 1997, 3-17

[2] Michalewicz, Z.: *Genetic Algorithms + Data Structures = Evolution Programming*, 3rd ed., Springer-Verlag, 1999

[3] Kirkpatrick, S., Gellat, C.D., Vecchi, M.P.: Optimization by Simulated Annealing. *Science*, 220, 1983, 671-680

[4] Dorigo, M., Blum, C.: Ant colony optimization theory: A survey. *Theoretical Computer Science* 344, 2005, 243-278

[5] Socha, K., Dorigo, M.: Ant colony optimization for continuous domains. *European Journal of Operational Research*, 185, 2008, 1155-1173

[6] Jovanović, R., Tuba, M., Simian D.: Analysis of Parallel Implementations of the Ant Colony Optimization Applied to the Minimum Weight Vertex Cover Problem. *Proceedings of 9th WSEAS Int. Conf. on Simulation, Modeling and Optimization (SMO '09)*, Budapest, Hungary, 2009.

[7] Karaboga, D., Basturk, B.: A powerful and efficient algorithm for numerical function optimization: artificial bee colony (ABC) algorithm. *Journal of Global Optimization* 39, 2007, 459-471

[8] Liu, Y., Passino, K.M.: Biomimicry of Social Foraging Bacteria for Distributed Optimization: Models, Principles, and Emergent Behaviors. *Journal of Optimization Theory and Applications* 115 (3), 2002, 603–628

[9] Kennedy, J, Eberhart, R.C.: Particle Swarm Optimization. *Proceedings of IEEE International Conference on Neural Networks*, Perth, Australia 1995, 1942-1948

[10] Dong, et al, An Application of Swarm Optimization to Nonlinear Programming. *Computers and Mathematics with Applications*, 49, 2005, 1655-1668

[11] Ratnaweera, A., Saman, K.H., Watson, H.C.: Self-Organizing Hierarchical Particle Swarm Optimizer with Time-Varying Acceleration Coefficients. *IEEE Transactions on Evolutionary Computation* 8 (3), 2004, 240-255

[12] Schutte, J.C., Groenwold, A.A.: A Study of Global Optimization Using Particle Swarms. *Journal of Global Optimization* 31, 2005, 93-108

[13] Y. Shi, R.C. Eberhart, Empirical study of particle swarm optimization, *Proc. IEEE Int. Congr. Evolutionary Computation*, 3, 1999, 101-106.

[14] F. van den Bergh: *An analysis of particle swarm optimizers*, PhD Thesis, University of Pretoria, 2001.

[15] Dong, Y., Tang, J., Xu, B., Wang, D.: An Application of Swarm Optimization to Nonlinear Programming, *Computers and Mathematics with Applications*, 49, 2005, 1655-1668.

[16] Dimopoulos, G.G.: Mixed-variable engineering optimization based on evolutionary and social metaphors. *Computer methods in applied mathematics and engineering* , 196, 2007, 803-817.

[17] He, Q., Wang, L.: An effective co-evolutionary particle swarm optimization for constrained engineering design problems. *Engineering Applications of Artificial Intelligence*, 20, 2007, 89-99.

[18] Liu, L., Liu, W., Cartes, D.A.: Particle swarm optimization based-parameter identification applied to permanent magnet synchronous motors, *Engineering Applications of Artificial Intelligence,* 21, 2008, 1092–1100

[19] Ž. Kanović, M. Rapaić, An application of PSO algotrithm to optimization of water dostribution system pump operation, (in serbian), *Proceedings of LII ETRAN*, Igalo, Montenegro, 2007

[20] Li, L.D., Zhou, J., Yu, Y.: Performance based unit loading optimization using particle swarm optimization approach. *Proceedings of the 5th WSEAS Int. Conf. on Computational Intelligence, Man-Machine Systems and Cybernetics*, Venice, Italy, 2006.

[21] Jatmiko, W., Mursanto, P., Kusumoputro, B., Sekiyama, K., Fukuda, T.: Modified PSO Algorithm Based on Flow of Wind for Odor Source Localization Problems in Dynamic Environments.

WSEAS Transactions on Systems, 2(7), 2008, 106-113

[22] Reis, C., Tenreiro Machado, J.A., Cunha, J.B.: Evolutionary Techniques in Circuit Design and Optimization. *WSEAS Transactions on Power Systems,* 1 (7), 2006, 1337-1342

[23] Thanushkodi, K., Pandian, S.M.V., Dhivyapragash, R.S., Jothikumar, M., Sriramnivas, S., Vinodh, K.: An Efficient Particle Swarm Optimization for Economic Dispatch Problems with Non-Smooth cost Functions. *WSEAS Transactions on Power Systems*, 4 (3) 2008, 257-266

[24] Ozcan, E., Mohan, C.K.: Analysis of Simple Particle Swarm Optimizer. *IEEE International Conference on Evolutionary Computation*, Anchorage, Alaska, USA, 1998

[25] Ozcan, E., Mohan, C.K.: Particle Swarm Optimization: Surfing the Waves. *Proc. Congres on Evolutionary Computation*, Washington DC, USA, 1999, 1939-1944

[26] Clerc, M., Kennedy, J.: The Particle Swarm - Explosion, Stability and Convergence in a Multidimensional Complex Space. *IEEE Transactions on Evolutionary Computation*, 6 (1), 2002, 58-73.

[27] Jiang, M., Luo, Y.P., Yang, S.Y.: Stochastic convergence analysis and parameter selection of standard particle swarm optimization algorithm. *Information Processing Letters*, 102, 2007, 8-16.

[28] Ž. Kanović, M. Rapaić, An application of PSO algotrithm to optimization of water dostribution system pump operation, (in serbian), Proceedings of *LI ETRAN*, Igalo, Montenegro, 2007.

[29] Rapaić, M.R., Kanović, Ž., Jeličić Z.D., Discrete particle swarm optimization algorithm for solving optimal sensor deployment problem, *Journal of Automatic Control*, 18 (1), 2008, 9-14.

[30] Rapaić, M.R., Kanović, Ž.: Time-varying PSO - convergence analysis, convergence related parameterization and new parameter adjustment schemes. *Information Processing Leters*, 109, 2009, 548-552.

[31] DaCunha, J.J.: Stability for time varying linear dynamic systems on time scales. *Journal of Computational and Applied Mathematics*, 176 (2) 2005, 381-410

[32] Elaydi, S.: *An Introduction to Difference Equations*, 3rd ed. Springer-Verlag, 2005.

[33] Åström, K.J., Wittenmark, B.: *Computer Controlled Systems – Theory and Design*, 3rd ed. Prentice Hall, 1997.