Stabilization of Linear Time-varying Uncertain Delay Systems with Double Triangular Configuration

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Abstract: This paper investigates the feedback stabilization problem of linear time-varying uncertain delay systems using linear memoryless state feedback control. Each uncertain parameter and each delay under consideration may take arbitrarily large values. In such a situation, the locations of uncertain entries in the system matrices play an important role. It has been shown that if a system has a particular configuration called a triangular configuration, then the system is stabilizable irrespectively of the given bounds of uncertain variations. In this paper, so far obtained stabilizability conditions are developed into the ones called a double triangular configuration. The objective of this paper is to show that if a system has such an extended configuration, then the system is also stabilizable independently of both the bounds of uncertain parameters and time delays. An illustrative example is presented to verify the effectiveness of the obtained result.

Key–Words: Stabilization, Linear systems, Time delay, Uncertain systems, M-matrix

1 Introduction

The robust stabilization problem of uncertain systems has attracted increasing interest [1]-[3]. This is mainly due to the fact that many dynamical systems contain a certain amount of uncertainties. Differential equations with time delay arise in many areas of applied mathematics [4]-[6], because in most instances physical, chemical, biological or economical phenomena naturally depend not only on the present state but also on some past occurrences. Then, this paper examines the stabilization problem of linear time-varying uncertain delay systems by means of linear memoryless state feedback control.

The systems under consideration contain uncertain entries in the system matrices and uncertain delays in the state variables. Each value of uncertain entries and delays may vary with time independently in an arbitrarily large bound. Under this situation, the locations of uncertain entries in the system matrices play an important role. This paper presents investigation of the permissible locations of uncertain entries, which are allowed to take unlimited large values, for the stabilization using linear state feedback control.

It is useful to classify the existing results on the stabilization of uncertain systems into two categories. The first category includes several results [7]-[10] which provide the stabilizability conditions depending on the bounds of uncertain parameters. The results in the second category [11]-[14] provide the stabilizability conditions that are independent of the bounds of uncertain parameters but which depend on their locations. This paper specifically addresses the second category.

For uncertain systems with delays, the Lyapunov stability approach with the Krasovskii-based or Razumikhin-based method is a commonly used tool. The stabilization problem has been reduced to solving linear matrix inequalities (LMI) [7]-[9]. However, LMI conditions fall into the first category; for this reason, they are often used to determine the permissible bounds of uncertain parameters for the stabilization. When the bounds of uncertain parameter values exceed a certain value, LMI solver becomes infeasible. In such cases, guidelines for redesigning the controller are usually lacking.

On the other hand, the stabilizability conditions in the second category can be verified easily merely by examining the uncertainty locations in given system matrices. Once a system satisfies the stabilizability conditions, a stabilizing controller can be constructed, irrespective of the given bounds of uncertain variations. We can redesign the controller for improving robustness merely by modifying the design parameter when the uncertain parameters exceed the upper bounds given beforehand.

In the second category, the stabilization problem of linear time-varying uncertain systems without delays was studied by Wei [11]. The stabilizability conditions have a particular geometric configuration with respect to the permissible locations of uncertain entries. Using the concept of antisymmetric stepwise configuration (ASC) [11], Wei proved that a linear time-varying uncertain system is stabilizable independently of the given bounds of uncertain variations using linear state feedback control if and only if the system has an ASC. Wei derived the successful result on the stabilization problem of systems without delays, however, his method [11] is inapplicable to systems that contain delays in the state variables.

On the one hand, based on the properties of an M-matrix, Amemiya [12] developed the conditions for the stabilization of linear time-varying uncertain systems with time-varying delays using linear memoryless state feedback control. The conditions obtained in [12] show a similar configuration to an ASC, but the allowable uncertainty locations are fewer than in an ASC by one step.

The aforementioned results presume that all state variables are accessible for designing a controller. However, it is usual that the state variables of the systems are measured through the outputs and hence only limited parts of them can be used directly. The output feedback stabilization of linear uncertain delay systems with limited measurable state variables has been investigated in [13] [14]. The conditions so far obtained show that if a system has a particular configuration called a triangular configuration, then the system is stabilizable independently of the given bounds of uncertain variations. The triangular configuration consists of two kinds of triangular forms, which can be classified into up-low form and left-right form. Both triangular forms are shown below. In this paper, the notation $*$ is always used to denote the permissible location of an uncertain entry.

$$\begin{bmatrix} 0 & 0 & * & * & * & * & * & * & * & 0 \\ 0 & 0 & 0 & * & * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & * & * & * & * & 0 \\ 0 & * & * & * & * & * & * & * & * & * \end{bmatrix} \qquad (1)$$

Triangular configuration with up-low form

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ * & * & * & 0 & 0 & 0 & 0 & 0 & * & * \\ * & * & * & * & 0 & 0 & 0 & * & * & * \\ * & * & * & * & * & 0 & * & * & * & * \\ * & * & * & * & 0 & 0 & 0 & * & * & * \\ * & * & * & 0 & 0 & 0 & 0 & 0 & * & * \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (2)$$

Triangular configuration with left-right form

The up-low form was derived based on the assumption that a system has one-input and two-outputs, while the left-right form was derived under the assumption that a system has two-inputs and one-output.

In this paper, we develop so far obtained stabilizability conditions by introducing a novel configuration. The objective of this paper is to show if a system has such an extended configuration, then the system is stabilizable irrespectively of the given bounds of uncertain variations. The configuration derived here consists of both triangular configurations with up-low form and left-right form. To achieve our objective here, we assume that a system has two-inputs and two-outputs. The stabilization problem discussed here can be reduced to finding the proper variable transformation such that the transformed system satisfies an M-matrix stability criterion. Finding transformations that lead to the M-matrix structure is a very difficult task for us because it requires troublesome hand-written calculations. Nevertheless, this paper provides the successful manner of constructing such a proper variable transformation.

This paper is organized as follows. Some notations and terminology are given in Sec. 2. The system considered here is defined in Sec. 3. In Sec. 4, some preliminary results are introduced to state the present problem. The main result is provided in Sec. 5. The illustrative examples are shown in Sec. 6. Finally, some concluding remarks are presented in Sec. 7.

2 Notations and Terminology

First, some notations and terminology used in the subsequent description are given. For $a, b \in \mathbb{R}^m$ or $A, B \in \mathbb{R}^{n\times m}$, every inequality between a and b or A and B such as $a > b$ or $A > B$ indicates that it is satisfied componentwise by a and b or A and B. If $A \in \mathbb{R}^{n\times m}$ satisfies $A \geq 0$, A is called a nonnegative matrix. The determinant and the transpose of $A \in \mathbb{R}^{n\times n}$ are denoted by $\det(A)$ and A', respectively. For $a = (a_1, ..., a_m)' \in \mathbb{R}^m$, $|a| \in \mathbb{R}^m$

is defined as $|a| = (|a_1|, .., |a_m|)'$. Also for $A = (a_{ij}) \in \mathbb{R}^{n\times m}$, $|A|$ denotes a matrix with $|a_{ij}|$ as its (i, j) entries. Let $diag\{\cdots\}$ denote a diagonal matrix. Let $[a, b]$, a,$b \in \mathbb{R}$ be an interval in $\mathbb{R}$. The set of all continuous or piecewise continuous functions with domain $[a, b]$ and range $\mathbb{R}^n$ is denoted by $\mathcal{C}^n[a, b]$ or $\mathcal{D}^n[a, b]$, respectively. We denote it simply by $\mathcal{C}^n$ or $\mathcal{D}^n$ if the domain is $\mathbb{R}$.

The notation for a class of functions is introduced below. Let $\xi(\mu) \in \mathcal{C}^1$ and let $m \in \mathbb{R}$ be a constant. If $\xi(\mu)$ satisfies the conditions

$$\begin{aligned} &\limsup_{|\mu|\to\infty} \left|\frac{\xi(\mu)}{\mu^m}\right| < \infty, \\ &\limsup_{|\mu|\to\infty} \left|\frac{\xi(\mu)}{\mu^{m-a}}\right| = \infty \end{aligned} \tag{3}$$

for any positive scalar $a \in \mathbb{R}$, then $\xi(\mu)$ is called a function of order m, and we denote this as follows:

$$Ord(\xi(\mu)) = m. \tag{4}$$

The set of all $\mathcal{C}^1$ functions of order m is denoted by $O(m)$,

$$O(m) = \left\{\xi(\mu)|\xi(\mu) \in \mathcal{C}^1,\ Ord(\xi(\mu)) = m\right\}. \tag{5}$$

Also, it is worth to note that m can be a negative number and that the following relations between $\xi_1(\mu) \in O(m_1)$ and $\xi_2(\mu) \in O(m_2)$ hold:

$$Ord(\xi_1(\mu) \pm \xi_2(\mu)) = \max\{m_1, m_2\}, \tag{6}$$
$$Ord(\xi_1(\mu) \times \xi_2(\mu)) = m_1 + m_2, \tag{7}$$
$$Ord(\xi_1(\mu)/\xi_2(\mu)) = m_1 - m_2. \tag{8}$$

A real square matrix all of whose off-diagonal entries are non-positive is called an M-matrix if it is nonsingular and its inverse matrix is non-negative. The set of all M-matrices is denoted by $\mathcal{M}$.

3 System Description

Let n be a fixed positive integer. The system considered here is given by a delay differential equation defined on $x \in \mathbb{R}^n$ for $t \in [t_0, \infty)$ as follows:

$$\begin{aligned} \dot{x}(t) &= A^0x(t) + \Delta A^1(t)x(t) \\ &\quad + \sum_{i=1}^{r} \Delta A^{2i}(t)x(t - \tau_i(t)) + Bu(t), \\ y(t) &= C'x(t), \end{aligned} \tag{9}$$

with an initial curve $\phi \in \mathcal{D}^n[t_0 - \tau_0, t_0]$. Here, A^0, $\Delta A^1(t)$, $\Delta A^{2i}(t)$ $(i = 1, ..., r)$ are all real $n \times n$ matrices, where r is a fixed positive integer; also, A^0 is a known constant matrix. Furthermore, $\Delta A^1(t)$ and $\Delta A^{2i}(t)$ $(i = 1, ..., r)$ are uncertain coefficient matrices and may vary with $t \in [t_0, \infty)$. Other variables are as follows: $u(t) \in \mathbb{R}^2$ is a control variable and $B \in \mathbb{R}^{n\times 2}$ is a known constant matrix. $y(t) \in \mathbb{R}^2$ is an output variable and $C \in \mathbb{R}^{n\times 2}$ is a known constant matrix.

In addition, all $\tau_i(t)$ $(i = 1, ..., r)$ are piecewise continuous functions and are uniformly bounded, i.e., for a non-negative constant τ_0 they satisfy

$$0 \le \tau_i(t) \le \tau_0 \quad (i = 1, ..., r) \tag{10}$$

for all $t \ge t_0$. The upper bound τ_0 can be arbitrarily large and is not necessarily assumed to be known. It is assumed that all the entries of $\Delta A^1(t)$ and $\Delta A^{2i}(t)$ are piecewise continuous functions and are uniformly bounded, i.e., for non-negative constant matrices ΔA^{10} and $\Delta A^{2i0} \in \mathbb{R}^{n\times n}$, they satisfy

$$\left|\Delta A^1(t)\right| \le \Delta A^{10}, \quad \left|\Delta A^{2i}(t)\right| \le \Delta A^{2i0}, \tag{11}$$

for all $t \ge t_0$. The upper bound of each entry can independently take an arbitrarily large value, but each is assumed to be known.

Assumption 1 *Because the system must be controllable and observable, we assume that A^0, $B = (b_1, b_2) \in \mathbb{R}^{n\times 2}$ and $C = (c_1, c_2) \in \mathbb{R}^{n\times 2}$ are given as follows:*

$$\begin{aligned} &A^0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, b_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \\ &b_2 = (0, ..., 0, 1, 0, ..., 0)', \\ &c_1 = (1, 0, ..., 0)', \\ &c_2 = (0, ..., 0, 1, 0, ..., 0)', \end{aligned} \tag{12}$$

where all the entries of B are equal to zero except that the first entry and the kth entry of b_1 and b_2 are equal to 1, respectively. Likewise, all the entries of C are equal to zero except that the first entry and the $(n - l + 1)$th entry of c_1 and c_2 are equal to 1, respectively. k and $(n - l + 1)$ have strong relations to the configuration of uncertain entries and they are defined in the subsequent discussion.

It is seen from [15] that, for a certain value of uncertain entry, the system might lose the controllability or the observability without Assumption 1.

Next, we consider the following system:

$$\dot{z}(t) = (A^0 - LC')z(t) + Ly(t) + Bu(t), \tag{13}$$

where $z(t) \in \mathbb{R}^n$ is an auxiliary state variable and $L \in \mathbb{R}^{n\times 2}$ is a constant matrix. This is an observer in the most basic sense. Our objective is to find a controller for stabilizing the overall $2n$-dimensional system consisting of (9) and (13). Let $e(t)$ be defined by

$$e(t) = z(t) - x(t). \tag{14}$$

Let $u(t)$ be given by

$$u(t) = G'z(t) = G'e(t) + G'x(t), \tag{15}$$

where $G \in \mathbb{R}^{n\times 2}$ is a constant matrix.

Definition 2 *System (9) is said to be robustly stabilizable if there exists a linear memoryless state feedback control $u(t) = G'z(t)$ such that the equilibrium point $x = 0$ of the resulting closed-loop system is uniformly and asymptotically stable for all admissible uncertain delays and uncertain parameters.*

4 Preliminaries

The $2n$-dimensional system consisting of $x(t) \in \mathbb{R}^n$ and $e(t) \in \mathbb{R}^n$ is written as follows:

$$\dot{w}(t) = \begin{bmatrix} A^0 - LC' & -\Delta A^1(t) \\ BG' & A^0 + BG' + \Delta A^1(t) \end{bmatrix} w(t) + \sum_{i=1}^{r} \begin{bmatrix} 0 & -\Delta A^{2i}(t) \\ 0 & \Delta A^{2i}(t) \end{bmatrix} w(t - \tau_i(t)), \tag{16}$$

where $w(t) = (e'(t), x'(t))' \in \mathbb{R}^{2n}$.

Because of Assumption 1, it is possible to choose $G \in \mathbb{R}^{n\times 2}$ such that all the eigenvalues of $(A^0 + BG')$ are real, negative and distinct. Likewise, it is also possible to choose $L \in \mathbb{R}^{n\times 2}$ such that all the eigenvalues of $(A^0 - LC')$ are real, negative and distinct. Let G and L be chosen in such a way. In addition, let $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $\sigma_1, \sigma_2, \cdots, \sigma_n$ be such eigenvalues of $(A^0 + BG')$ and $(A^0 - LC')$, respectively. Let T and S be Vandermonde matrices constructed from λ_i and σ_i, respectively, as follows:

$$T = \begin{bmatrix} T^1 & 0 \\ 0 & T^2 \end{bmatrix}, \quad S = \begin{bmatrix} S^1 & 0 \\ 0 & S^2 \end{bmatrix}, \tag{17}$$

where $T^1 \in \mathbb{R}^{k\times k}$, $T^2 \in \mathbb{R}^{(n-k)\times(n-k)}$, $S^1 \in \mathbb{R}^{(n-l)\times(n-l)}$ and $S^2 \in \mathbb{R}^{l\times l}$ are given by

$$T^1 = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_k^2 \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \dots & \lambda_k^{k-1} \end{bmatrix}, \tag{18}$$

$$T^2 = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_{k+1} & \lambda_{k+2} & \dots & \lambda_n \\ \lambda_{k+1}^2 & \lambda_{k+2}^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \dots & \vdots \\ \lambda_{k+1}^{n-k-1} & \lambda_{k+2}^{n-k-1} & \dots & \lambda_n^{n-k-1} \end{bmatrix}, \tag{19}$$

$$S^1 = \begin{bmatrix} \sigma_1^{n-l-1} & \sigma_1^{n-l-2} & \dots & \sigma_1 & 1 \\ \sigma_2^{n-l-1} & \sigma_2^{n-l-2} & \dots & \sigma_2 & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \sigma_{n-l}^{n-l-1} & \sigma_{n-l}^{n-l-2} & \dots & \sigma_{n-l} & 1 \end{bmatrix}, \tag{20}$$

$$S^2 = \begin{bmatrix} \sigma_{n-l+1}^{l-1} & \sigma_{n-l+1}^{l-2} & \dots & \sigma_{n-l+1} & 1 \\ \sigma_{n-l+2}^{l-1} & \sigma_{n-l+2}^{l-2} & \dots & \sigma_{n-l+2} & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \sigma_n^{l-1} & \sigma_n^{l-2} & \dots & \sigma_n & 1 \end{bmatrix}. \tag{21}$$

T and S are well known to be nonsingular in view of the above assumptions. Here, let v be defined by

$$v = \begin{bmatrix} Se \\ T^{-1}x \end{bmatrix}. \tag{22}$$

Then, system (16) can be transformed into

$$\begin{aligned} \dot{v}(t) &= \begin{bmatrix} S(A^0 - LC')S^{-1} & 0 \\ 0 & T^{-1}(A^0 + BG')T \end{bmatrix} v(t) \\ &+ H^1 v(t) + H^2 v(t - \tau_i(t)), \end{aligned} \tag{23}$$

where H^1 and H^2 are given as follows:

$$H^1 = \begin{bmatrix} 0 & -S\Delta A^1(t)T \\ T^{-1}BG'S^{-1} & T^{-1}\Delta A^1(t)T \end{bmatrix}, \tag{24}$$

$$H^2 = \sum_{i=1}^{r} \begin{bmatrix} 0 & -S\Delta A^{2i}(t)T \\ 0 & T^{-1}\Delta A^{2i}(t)T \end{bmatrix}. \tag{25}$$

Define Λ and Σ as follows:

$$\begin{aligned} \Lambda &= T^{-1}(A^0 + BG')T = \text{diag.}(\lambda_1, \lambda_2, \cdots, \lambda_n), \\ \Sigma &= S(A^0 - LC')S^{-1} = \text{diag.}(\sigma_1, \sigma_2, \cdots, \sigma_n). \end{aligned} \tag{26}$$

Let P^1 and P^2 be defined as follows:

$$P^1 = \begin{bmatrix} \Sigma & 0 \\ 0 & \Lambda \end{bmatrix}, \tag{27}$$

$$P^2 = \begin{bmatrix} 0 & |S|\,\Delta A^{30}\,|T| \\ |T^{-1}|\,|BG'|\,|S^{-1}| & |T^{-1}|\,\Delta A^{30}\,|T| \end{bmatrix}, \tag{28}$$

where ΔA^{30} is given by

$$\Delta A^{30} = \Delta A^{10} + \sum\nolimits_{i=1}^{r} \Delta A^{2i0}. \tag{29}$$

In addition, let P be defined by

$$P = -P^1 - P^2. \tag{30}$$

Since $|A + B| \le |A| + |B|$ for any $A, B \in \mathbb{R}^{n\times n}$, we see that the following relation holds.

$$|H^1| + |H^2| \le P^2. \tag{31}$$

Now, we introduce the fundamental lemma which plays a crucial role to lead the main result.

Lemma 3 ([13]) *If there exist T and S which assure*

$$P \in \mathcal{M}, \tag{32}$$

then system (9) is robustly stabilizable.

Note that our problem has been reduced to finding T and S that enable P to satisfy condition (32). In the subsequent discussion, we consider the possibility of choosing T and S that assure $P \in \mathcal{M}$.

5 Main Results

In this section, we first introduce a novel configuration with respect to the permissible locations of uncertain entries. That means the stabilizability conditions in [14] are extended with respect to the allowable locations of uncertain entries. Next, the main result is given, which shows that if such developed conditions are satisfied, then system (9) is robustly stabilizable.

First, we introduce a set of matrices $\Omega(k, l) \in \mathbb{R}^{n\times n}$ as follows:

Definition 4 *Let k and l be fixed integers such that $0 < k < n/2$, $0 < l < n/2$ and $n - 2k - 2l + 3 > 0$. For these k and l, let $\Omega(k, l) = \{D = (d_{ij}) \in \mathbb{R}^{n\times n}\}$ be a set of matrices with the following properties:*
1. If $1 \le i \le k$, then $d_{ij} = 0$, for $i + 1 \le j \le 2k - i + 1$.
2. If $k + 1 \le i \le 2k - 1$, then $d_{ij} = 0$, for $2k - i + 1 \le j \le i + 1$.
3. If $2k \le i \le n$, then $d_{ij} = 0$, for $1 \le j \le 2k - i + 1$.
4. If $2k - i + 2 \le j \le n - l + 1$, then $d_{ij} = 0$, for $j - 1 \le i \le 2n - 2l - j + 1$.
5. If $n - l + 2 \le j \le n$, then $d_{ij} = 0$, for $2n - 2l - j + 1 \le i \le j - 1$.

Note that the conditions in Definition 4 determine the permissible locations of uncertain entries in the system matrices. Now, we state the main result.

Theorem 5 *If for fixed k and l,*

$$\Delta A^{30} \in \Omega(k, l) \tag{33}$$

then system (9) is robustly stabilizable.

System (9) is said to have a double triangular configuration if the system satisfies condition (33). A schematic view of the system having a double triangular configuration is shown below. In fact, k and $(n - l + 1)$ indicate the positions of the apexes of a double triangular configuration. We see from (34) that a double triangular configuration is obtained by combining an up-low triangular configuration (1) with a left-right triangular configuration (2). A comparison with the stabilizability conditions in [14] shows that the allowable uncertainty locations of a double triangular configuration are more numerous than those of a triangular configuration.

$$\begin{array}{c}
\begin{array}{ccccccccccccc}
 & & & \text{k} & & & & & & n-l+1 & & & \\
 & & & \downarrow & & & & & & \downarrow & & &
\end{array}\\
\left[\begin{array}{cccccccc|ccccc}
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\
* & * & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * \\
* & * & * & 0 & 0 & 0 & * & * & * & * & * & * & * \\
* & * & * & * & 0 & * & * & * & * & * & * & * & * \\
* & * & * & 0 & 0 & 0 & * & * & * & * & * & * & * \\
* & * & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * \\
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & *
\end{array}\right]
\end{array} \tag{34}$$

Proof of Theorem 5: According to Lemma 3, the existence of T and S which assure $P \in \mathcal{M}$ is investigated in the rest of this section. On evaluating the existence of T and S, it is important how to choose the eigenvalues $\lambda_i (i = 1, ..., n)$ of $(A^0 + BG')$ and the eigenvalues $\sigma_i (i = 1, ..., n)$ of $(A^0 - LC')$.

Here, let μ be a positive number and let $\alpha_i (i = 1, ..., n)$ be all negative numbers that are different from one another. Likewise, let $\beta_i (i = 1, ..., n)$ be all negative numbers that are different from one another. Let μ be chosen much larger than all the entries of ΔA^{30}. That means μ is much larger than all the upper bounds of uncertain entries. $\alpha_i (i = 1, ..., n)$ are used for distinguishing eigenvalues λ_i from one another. Likewise, $\beta_i (i = 1, ..., n)$ are used for distinguishing eigenvalues σ_i from one another. Proper way of choosing λ_i and σ_i are shown below.

$$\begin{aligned}
&\lambda_i = \alpha_i \mu^1 \in O(1) \ \ (i = 1, \ldots, k),\\
&\lambda_i = \alpha_i \mu^{-1} \in O(-1) \ \ (i = k + 1, \ldots, n - l),\\
&\lambda_i = \alpha_i \mu^1 \in O(1) \ \ (i = n - l + 1, \ldots, n),
\end{aligned} \tag{35}$$

$$\sigma_i = \beta_i \mu^1 \in O(1) \ \ (i = 1, \ldots, k),$$
$$\sigma_i = \beta_i \mu^{-1} \in O(-1) \ \ (i = k+1, \ldots, n-l),$$
$$\sigma_i = \beta_i \mu^1 \in O(1) \ \ (i = n-l+1, \ldots, n),$$

To complete the proof of Theorem 5, we should show that if we choose λ_i and σ_i as in (35), then T and S constructed from such λ_i and σ_i assure $P \in \mathcal{M}$.

Now, we can write T and S as follows:

$$T^1 = \begin{bmatrix} \boxed{\quad 0 \quad} \\ \vdots \\ \boxed{\quad i-1 \quad} \\ \vdots \\ \boxed{\quad k-1 \quad} \end{bmatrix}, \tag{36}$$

$$\begin{aligned} T^2 &= \left[\, T^{21} \,\middle|\, T^{22} \,\right], \\ &= \left[\begin{array}{c|c} \boxed{\quad 0 \quad} & \boxed{\quad 0 \quad} \\ \vdots & \vdots \\ \boxed{\quad -i+1 \quad} & \boxed{\quad i-1 \quad} \\ \vdots & \vdots \\ \boxed{\quad k-n+1 \quad} & \boxed{\quad n-k-1 \quad} \end{array}\right], \end{aligned} \tag{37}$$

$$S^1 = \begin{bmatrix} S^{11} \\ S^{12} \end{bmatrix}, \tag{38}$$

$$S^{11} = \begin{bmatrix} \boxed{\begin{matrix} n \\ -l \\ -1 \end{matrix}} \cdots \boxed{\begin{matrix} n \\ -l \\ -j \end{matrix}} \cdots \boxed{\ 0\ } \end{bmatrix}, \tag{39}$$

$$S^{12} = \begin{bmatrix} \boxed{\begin{matrix} l \\ -n \\ +1 \end{matrix}} \cdots \boxed{\begin{matrix} l \\ -n \\ +j \end{matrix}} \cdots \boxed{\ 0\ } \end{bmatrix}, \tag{40}$$

$$S^{2} = \begin{bmatrix} \boxed{\begin{matrix} l \\ -1 \end{matrix}} \cdots \boxed{\begin{matrix} l \\ -j \end{matrix}} \cdots \boxed{\ 0\ } \end{bmatrix}, \tag{41}$$

where T^{21} and T^{22} denote $(n-k) \times (n-k-l)$ and $(n-k) \times l$ matrices, respectively. In addition, S^{11} and S^{12} denote $k \times (n-l)$ and $(n-k-l) \times (n-l)$ matrices, respectively. In the above notation, $\boxed{\quad m \quad}$ and $\boxed{m}$ denote a row vector and a column vector, whose all entries are functions of μ of order m, respectively. For convenience, we adopt such notation for matrices in the subsequent discussion and neglect further explanation when it is clear. The notations of (36) and (37) mean that all the entries of the i-th row of T^1, T^{21} and T^{22} are functions of μ of order $(i-1)$, $(-i+1)$ and $(i-1)$, respectively. The notations of (39), (40) and (41) mean that all the entries of the j-th column of S^{11}, S^{12} and S^2 are functions of μ of order $(n-l-j)$, $(l-n+j)$ and $(l-j)$, respectively. From the relations between the roots and the coefficients of the characteristic equation $\det(A^0 + BG')$, we find that $G' \in \mathbb{R}^{2 \times n}$ has the following structure:

$$G' = \left[\begin{array}{ccc|ccc|ccc} O(k) & \cdots & O(1) & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ \hline 0 & \cdots & 0 & O(2l+k-n) & \cdots & O(l-1) & O(l) & \cdots & O(1) \end{array}\right]. \tag{42}$$

Considering such structures of T, S and G', it turns out from the careful calculation that each block matrix in (28) is further decomposed into 9 block matrices as follows:

$$\left|T^{-1}\right| \Delta A^{30} \left|T\right| = \left[\begin{array}{c|c|c} \boxed{0} & \boxed{-k} & \boxed{n-k-1} \\ \hline \boxed{k-2} & \boxed{-2} & \boxed{\begin{matrix} 2n-2k \\ -2l-2 \end{matrix}} \\ \hline \boxed{\begin{matrix} 3k+2l \\ -2n-2 \end{matrix}} & \boxed{2k+2l-2n} & \boxed{0} \end{array}\right]. \tag{43}$$

$$\left|S\right| \Delta A^{30} \left|T\right| = \left[\begin{array}{c|c|c} \boxed{n-l-1} & \boxed{\begin{matrix} n-k \\ -l-1 \end{matrix}} & \boxed{\begin{matrix} 2n-k \\ -l-2 \end{matrix}} \\ \hline \boxed{\begin{matrix} 2k+l \\ -n-1 \end{matrix}} & \boxed{\begin{matrix} k+l \\ -n-1 \end{matrix}} & \boxed{\begin{matrix} n-k \\ -l-1 \end{matrix}} \\ \hline \begin{matrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \boxed{\begin{matrix} 2l+k \\ -n-1 \end{matrix}} & \boxed{n-k-1} \end{array}\right]. \tag{44}$$

$$\left|T^{-1}\right| \left|BG'\right| \left|S^{-1}\right| =$$

$$\left[\begin{array}{c|c|c} \boxed{l-n+2} & \boxed{n-2k-l} & \begin{matrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{matrix} \\ \hline \boxed{\begin{matrix} k+l \\ -n-2 \end{matrix}} & \boxed{\begin{matrix} n-k \\ -l-2 \end{matrix}} & \boxed{n-k-2l} \\ \hline \boxed{\begin{matrix} 3k+3l \\ -3n \end{matrix}} & \boxed{k+l-n} & \boxed{k-n+2} \end{array}\right]. \tag{45}$$

In the above notation, all the entries of each block matrix are functions of μ of the same order. In all these (3×3)-block matrices, all blocks of the first, second, and third row represent matrices with k, $(n-k-l)$, and l rows, respectively, and all blocks of the first, second, and third column represent matrices with k, $(n-k-l)$, and l columns, respectively.

Now, let $P \in \mathbb{R}^{2n \times 2n}$ in (30) be decomposed into 4 block matrices as follows:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \tag{46}$$

where

$$\begin{aligned} P_{11} &= -\Sigma, & (47)\\ P_{12} &= -|S|\Delta A^{30}|T|, & (48)\\ P_{21} &= -|T^{-1}||BG'||S^{-1}|, & (49)\\ P_{22} &= -\Lambda - |T^{-1}|\Delta A^{30}|T|. & (50) \end{aligned}$$

It is apparent that $P \in \mathcal{M}$ if and only if

$$\begin{aligned} &P_{11} \in \mathcal{M}, & (51)\\ &P_{22} \in \mathcal{M}, & (52)\\ &P_{22} - P_{21}P_{11}^{-1}P_{12} \in \mathcal{M}. & (53) \end{aligned}$$

Taking into account the fact that P_{11} is a diagonal matrix whose every entry is positive, we see that $P_{11} \in \mathcal{M}$. It remains to show that $P_{22} \in \mathcal{M}$ and $P_{22} - P_{21}P_{11}^{-1}P_{12} \in \mathcal{M}$ are satisfied.

The following lemma shown in [16] is useful for verification of whether a given matrix is an M-matrix.

Lemma 6 ([16]) *Let $B \in \mathbb{R}^{n \times n}$ be a diagonal matrix whose every entry is positive, and let $C \in \mathbb{R}^{n \times n}$. Let B and C be decomposed into 9 block matrices as follows:*

$$B = \left[\begin{array}{c|c|c} B_{11} & 0 & 0 \\ \hline 0 & B_{22} & 0 \\ \hline 0 & 0 & B_{33} \end{array}\right], \; C = \left[\begin{array}{c|c|c} C_{11} & C_{12} & C_{13} \\ \hline C_{21} & C_{22} & C_{23} \\ \hline C_{31} & C_{32} & C_{33} \end{array}\right], \tag{54}$$

Suppose that all the entries of each block matrix are functions of μ of the same order. Let all the entries of B_{ij} and C_{ij} belong to $O(b_{ij})$ and $O(c_{ij})$, respectively. For sufficiently large μ, if

$$\begin{aligned} &b_{11} > c_{11}, & (55)\\ &b_{22} > c_{22}, & (56)\\ &b_{33} > c_{33}, & (57)\\ &b_{11} > c_{12} - b_{22} + c_{21}, & (58)\\ &b_{11} > c_{13} - b_{33} + c_{31}, & (59)\\ &b_{22} > c_{23} - b_{33} + c_{32}, & (60)\\ &b_{11} > \max\{c_{12}, (c_{13} - b_{33} + c_{32})\} - b_{22} & \\ &\qquad + \max\{c_{21}, (c_{23} - b_{33} + c_{31})\}, & (61) \end{aligned}$$

then the matrix $A = B - |C|$ is an M-matrix.

Using Lemma 6, we can deduce whether the matrix whose entry represents the functional order is an M-matrix. Taking into account the fact that Λ is a diagonal matrix in which all diagonal entries belong to $O(1)$ from the first to the kth entry, $O(-1)$ from the $(k+1)$th to the $(n-l)$th entry or $O(1)$ from the $(n-l+1)$th to the nth entry, we have the following inequalities:

$$\begin{aligned} 1 &> 0, & (62)\\ -1 &> -2, & (63)\\ 1 &> 0, & (64)\\ 1 &> (-k) - (-1) + (k-2) = -1, & (65)\\ 1 &> (n-k-1) - 1 + (3k+2l-2n-2) & \\ &= -n + 2k + 2l - 4, & (66)\\ -1 &> (2n-2k-2l-2) - 1 + (2k+2l-2n) & \\ &= -3, & (67)\\ 1 &> \max\{-k, -n+k+2l-2\} - (-1) & \\ &\quad + \max\{k-2, k-5\} & \\ &= -n + 2k + 2l - 3, & (68) \end{aligned}$$

Using $n - 2k - 2l + 3 > 0$, we see that all the inequalities (55)-(61) are satisfied for P_{22}. Hence, it follows from Lemma 6 that $P_{22} \in \mathcal{M}$.

On the one hand, it is seen from the careful calculation that $P_{22} - P_{21}P_{11}^{-1}P_{12}$ has the following order structure:

$$P_{22} - P_{21}P_{11}^{-1}P_{12} = -\Lambda -$$

$$\left[\begin{array}{c|c|c} \boxed{1} & \boxed{-k} & \boxed{n-k-1} \\ \hline \boxed{k-2} & \boxed{-1} & \boxed{\begin{matrix} 2n-2k \\ -2l-2 \end{matrix}} \\ \hline \boxed{3k+2l-2n} & \boxed{2k+2l-2n} & \boxed{1} \end{array}\right]. \tag{69}$$

Likewise, we have the following inequalities:

$$\begin{aligned} 1 &> 0, && (70) \\ -1 &> -2, && (71) \\ 1 &> 0, && (72) \\ 1 &> (-k)-(-1)+(k-2) = -1, && (73) \\ 1 &> (n-k-1)-1+(3k+2l-2n) \\ &= -n+2k+2l-2, && (74) \\ -1 &> (2n-2k-2l-2)-1+(2k+2l-2n) \\ &= -3, && (75) \\ 1 &> \max\{-k, -n+k+2l-2\}-(-1) \\ &\quad +\max\{k-2, k-3\} \\ &= -n+2k+2l-3, && (76) \end{aligned}$$

Under the assumption of $n-2k-2l+3>0$, we see that all the inequalities (55)-(61) are satisfied for $P_{22}-P_{21}P_{11}^{-1}P_{12}$. Hence, it follows from Lemma 6 that $P_{22}-P_{21}P_{11}^{-1}P_{12} \in \mathcal{M}$.

Consequently, it follows from (51)-(53) that $P \in \mathcal{M}$. Therefore, using Lemma 3, we can conclude that system (9) is robustly stabilizable.

6 Illustrative Examples

6.1 Design of Controller

An illustrative example is given here. Consider the following system:

$$\begin{aligned} \dot{x}(t) &= A^0x(t)+\Delta A^1(t)x(t) \\ &\quad +\Delta A^{21}(t)x(t-\tau_1(t))+Bu(t), \\ y(t) &= C'x(t), \end{aligned} \tag{77}$$

where $\tau_1(t)=0.6(2.1+\sin(t))$,

$$A^0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Delta A^1(t) = \frac{1}{2}\begin{bmatrix} \sin(t) & 0 & \cos(t) & \sin(t) \\ 0 & 0 & 0 & \cos(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin(t) \end{bmatrix},$$

$$\Delta A^{21}(t) = \frac{1}{2}\begin{bmatrix} \sin(3t) & 0 & \cos(3t) & \sin(3t) \\ 0 & 0 & 0 & \cos(3t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin(3t) \end{bmatrix}.$$

For the above system, the locations of the time-varying system parameters satisfy condition (33) of Theorem 5. Consequently, we see that system (77) is robustly stabilizable.

Note that the upper bounds of the time-varying system parameters such that the following inequality holds for all $t \geq t_0$ are given as follows:

$$\left|\Delta A^1(t)\right| + \left|\Delta A^{21}(t)\right| \leq A^{30}, \tag{78}$$

where

$$A^{30} := \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{79}$$

We find that $k=1$ and $l=1$ from the structure of $\Omega(k,l)$ as follows:

$$\Delta A^{30} \in \Omega(1,1) = \begin{array}{c} \begin{array}{cccc} \text{k} & & & n-l+1 \\ \downarrow & & & \downarrow \end{array} \\ \begin{bmatrix} * & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \end{array} \tag{80}$$

Using (35), we can choose the proper eigenvalues λ_i and σ_i so that T and S constructed from such λ_i and σ_i assure $P \in \mathcal{M}$.

Here, we set $\alpha_i (i=1,...,4)$ and $\beta_i (i=1,...,4)$ as negative numbers shown below.

$$\begin{aligned} &\alpha_1=-0.6,\ \alpha_2=-0.2,\ \alpha_3=-0.8,\ \alpha_4=-1.0, \\ &\beta_1=-0.6,\ \beta_2=-0.2,\ \beta_3=-0.8,\ \beta_4=-1.0. \end{aligned} \tag{81}$$

μ is chosen as a positive number that is much larger than all the upper bounds of time-varying parameters:

$$\mu = 20. \tag{82}$$

We obtain the following eigenvalues:

$$\begin{array}{ll} \lambda_1 = \sigma_1 = 12, & \lambda_2 = \sigma_2 = 0.01, \\ \lambda_3 = \sigma_3 = 0.04, & \lambda_4 = \sigma_4 = 20. \end{array} \tag{83}$$

G and L can be found from the relations between the eigenvalues and coefficients of the characteristic equation $\det(A^0 + BG')$ and $\det(A^0 - LC')$, respectively. For the eigenvalues in (83), G and L are given as follows:

$$G = \begin{bmatrix} 0 & -12 \\ -0.008 & -1 \\ -1.0004 & 0 \\ -20.05 & 0 \end{bmatrix}, \tag{84}$$

$$L = \begin{bmatrix} 12.05 & 0 \\ 0.6004 & 0 \\ 0.0048 & 1 \\ 0 & 20 \end{bmatrix}. \tag{85}$$

Let T and S be constructed by $\lambda_i (i = 1, ..., 4)$ and $\sigma_i (i = 1, ..., 4)$ in (83), respectively. Then, we can calculate P and P^{-1} as follows:

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} & p_{27} & p_{28} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p_{37} & p_{38} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} & p_{47} & p_{48} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} & p_{57} & p_{58} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} & p_{67} & p_{68} \\ p_{71} & p_{72} & p_{73} & p_{74} & p_{75} & p_{76} & p_{77} & p_{78} \\ p_{81} & p_{82} & p_{83} & p_{84} & p_{85} & p_{86} & p_{87} & p_{88} \end{bmatrix}, \tag{86}$$

where

$$\begin{aligned}
&p_{11} = 12, p_{12} = -9.49 \times 10^{-13}, p_{13} = -9.51 \times 10^{-13}, \\
&p_{14} = 0, p_{15} = -144, p_{16} = -1.46, p_{17} = -6.01, \\
&p_{18} = -65280, p_{21} = 9.13 \times 10^{-22}, p_{22} = 0.01, \\
&p_{23} = 4.54 \times 10^{-19}, p_{24} = 0, p_{25} = -1.00 \times 10^{-4}, \\
&p_{26} = -2.01 \times 10^{-6}, p_{27} = -2.02 \times 10^{-5}, \\
&p_{28} = -4.04, p_{31} = -1.36 \times 10^{-20}, p_{32} = 0, \\
&p_{33} = 0.04, p_{34} = 0, p_{35} = -0.02, \\
&p_{36} = -2.02 \times 10^{-5}, p_{37} = -1.31 \times 10^{-4}, \\
&p_{38} = -16.7, p_{41} = 0, p_{42} = 0, p_{43} = 0, p_{44} = 20, \\
&p_{45} = 0, p_{46} = -1.00 \times 10^{-4}, p_{47} = -0.02, \\
&p_{48} = -400, p_{51} = -0.08, p_{52} = -66.8 p_{53} = -66.9 \\
&p_{54} = 0, p_{55} = 11, p_{56} = -0.01, p_{57} = -0.04, \\
&p_{58} = -420, p_{61} = -9.30 \times 10^{-6}, p_{62} = -2.67, \\
&p_{63} = -1.00, p_{64} = -33.4, p_{65} = 0, p_{66} = 0.01, \\
&p_{67} = -0.004, p_{68} = -1200, p_{71} = -9.32 \times 10^{-6}, \\
&p_{72} = -2.68, p_{73} = -1.01, p_{74} = -33.5, p_{75} = 0, \\
&p_{76} = -2.00 \times 10^{-4}, p_{77} = 0.04, p_{78} = -801, \\
&p_{81} = -1.40 \times 10^{-8}, p_{82} = -0.004, p_{83} = -0.01, \\
&p_{84} = -0.05, p_{85} = 0, p_{86} = -2.51 \times 10^{-7}, \\
&p_{87} = -4.01 \times 10^{-6}, p_{88} = 18.9,
\end{aligned}$$

$$P^{-1} = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} & q_{15} & q_{16} & q_{17} & q_{18} \\ q_{21} & q_{22} & q_{23} & q_{24} & q_{25} & q_{26} & q_{27} & q_{28} \\ q_{31} & q_{32} & q_{33} & q_{34} & q_{35} & q_{36} & q_{37} & q_{38} \\ q_{41} & q_{42} & q_{43} & q_{44} & q_{45} & q_{46} & q_{47} & q_{48} \\ q_{51} & q_{52} & q_{53} & q_{54} & q_{55} & q_{56} & q_{57} & q_{58} \\ q_{61} & q_{62} & q_{63} & q_{64} & q_{65} & q_{66} & q_{67} & q_{68} \\ q_{71} & q_{72} & q_{73} & q_{74} & q_{75} & q_{76} & q_{77} & q_{78} \\ q_{81} & q_{82} & q_{83} & q_{84} & q_{85} & q_{86} & q_{87} & q_{88} \end{bmatrix}, \tag{87}$$

where

$$\begin{aligned}
&q_{11} = 0.21, q_{12} = 5.49 \times 10^5, q_{13} = 7.04 \times 10^4, \\
&q_{14} = 2.68 \times 10^3, q_{15} = 17.9, q_{16} = 362, \\
&q_{17} = 801, q_{18} = 2.93 \times 10^5, q_{21} = 2.44 \times 10^{-4}, \\
&q_{22} = 1.21 \times 10^3, q_{23} = 140, q_{24} = 5.48, \\
&q_{25} = 0.03, q_{26} = 0.72, q_{27} = 1.64, q_{28} = 611, \\
&q_{31} = 5.62 \times 10^{-4}, q_{32} = 2.47 \times 10^3, q_{33} = 341, \\
&q_{34} = 12.2, q_{35} = 0.08, q_{36} = 1.62, q_{37} = 3.64, \\
&q_{38} = 1.34 \times 10^3, q_{41} = 6.93 \times 10^{-6}, q_{42} = 32.5, \\
&q_{43} = 4.07, q_{44} = 0.212, q_{45} = 9.78 \times 10^{-4}, q_{46} = 0.02, \\
&q_{47} = 0.05, q_{48} = 18.4, q_{51} = 0.01, q_{52} = 2.82 \times 10^4, \\
&q_{53} = 3.66 \times 10^3, q_{54} = 136, q_{55} = 0.969, q_{56} = 18.1, \\
&q_{57} = 40.6, q_{58} = 1.49 \times 10^4, q_{61} = 0.19, \\
&q_{62} = 8.75 \times 10^5, q_{63} = 1.09 \times 10^5, q_{64} = 4.37 \times 10^3, \\
&q_{65} = 26.4, q_{66} = 651, q_{67} = 1.26 \times 10^3, \\
&q_{68} = 4.70 \times 10^5, q_{71} = 0.04, q_{72} = 2.02 \times 10^5, \\
&q_{73} = 2.54 \times 10^4, q_{74} = 1.01 \times 10^3, q_{75} = 6.11, \\
&q_{76} = 128, q_{77} = 315, q_{78} = 1.08 \times 10^5, \\
&q_{81} = 1.26 \times 10^{-7}, q_{82} = 0.59, q_{83} = 0.07, \\
&q_{84} = 0.003, q_{85} = 1.78 \times 10^{-5}, q_{86} = 3.72 \times 10^{-4}, \\
&q_{87} = 8.47 \times 10^{-4}, q_{88} = 0.37.
\end{aligned}$$

From (86) and (87), it is obvious that all off-diagonal entries of P are non-positive and that P^{-1} is a non-negative matrix. Then it follows that $P \in \mathcal{M}$. Therefore, system (77) is robustly stabilizable by constructing the feedback gain G in (84) and the observer gain in (85). In designing a stabilizing feedback gain, the upper bounds of delays are assumed not to be known. For that reason, the controller designed from the given upper bounds of uncertain parameters is applicable for any delays, however large they might be because the stability condition used here is independent of time-varying delays.

6.2 Network Configuration

Designing a network structure of information flow arises in many systems including communication systems, formation moving (flying) systems, molecular biological systems, genetic systems, etc. Consider two network structures shown below. For simplicity, each subject $X_i (i = 1, \cdots, 7)$ is governed by the state equation $\dot{x}_i = v_i$, where x_i is the state variable of X_i and v_i is the input of information flow. Let $X_i \longrightarrow X_j$ denote that $x_i(t)$ is employed in the information input $v_j(t)$ of X_j. Let $X_i \cdots\succ X_j$ denote that $\epsilon_{ji} x_i(t-\tau(t))$ is employed in the information input v_j, where ϵ_{ji} is an uncertain parameter.

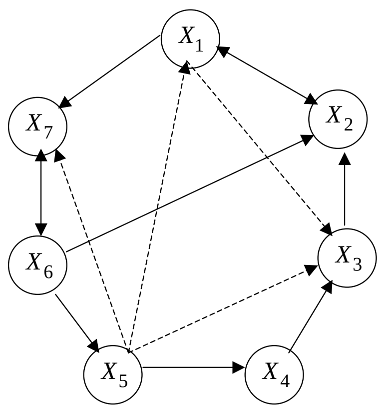

Figure 1: Network1

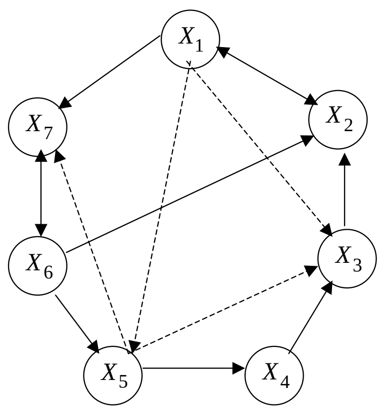

Figure 2: Network2

Network 1 (Figure 1) and Network 2 (Figure 2) can be described by the following state equation:

$$\begin{aligned} \dot{x}(t) &= A^0 x(t) + \Delta A^1 x(t-\tau(t)) + Bu(t), \\ y(t) &= C' x(t), \end{aligned} \tag{88}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \\ x_7(t) \end{bmatrix}, A^0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Therein, for Network 1 and Network 2, ΔA^1 is given by (89) and (90), respectively.

$$\Delta A^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \varepsilon_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{31} & 0 & 0 & 0 & \varepsilon_{35} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{75} & 0 & 0 \end{bmatrix}. \tag{89}$$

$$\Delta A^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{31} & 0 & 0 & 0 & \varepsilon_{35} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_{51} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{75} & 0 & 0 \end{bmatrix}. \tag{90}$$

It is seen from (89) and (90) that

$$\Delta A^1 \text{ in (89)} \in \Omega(2,2), \tag{91}$$

$$\Delta A^1 \text{ in (90)} \notin \Omega(k,l) \quad \text{for any } k,\, l. \tag{92}$$

Therefore, we see from (91) and (92) that Network 1 has a double triangular configuration, while Network 2 doesnot have a double triangular configuration. Consequently, we see from Theorem 5 that Network 1 can be stabilized by means of linear memoryless state feedback control however large the given bounds of uncertain parameters and time delays might be.

7 Conclusion

The stabilization problem of linear uncertain delay systems using linear memoryless state feedback control was investigated in this paper. In particular, we investigated the permissible locations of uncertain entries and delays, both of which are allowed to take unlimited large values for the stabilization. It was shown that if uncertain entries enter the system matrices in a way to form a particular geometric pattern called a double triangular configuration, then the system is stabilizable irrespectively of both the bounds of uncertain parameters and delays. It was found that a double triangular configuration derived here consists of both an up-low triangular form and a right-left triangular form. Hence, it was seen that the allowable uncertainty locations of a double triangular configuration are more numerous than those of a triangular configuration. The triangular configuration shown here has a strong similarity to an antisymmetric stepwise configuration by Wei [11]. To develop the conditions obtained here into the ones of antisymmetric stepwise configurations is a problem to be considered next.

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References:

[1] V. Horga, D. Lucache, M. Ratoi, A. Simion and M. Albu, Robust Control Strategy to Parameter Variations for an Interior PM Synchronous Machine, *WSEAS Trans. on Syst. and Contr.*, vol. 3, pp.453-462, 2008.

[2] J. He, Y. He, S. Guo and M. Fang, Tension Robust Control Strategy Based on Self-Optimizing Algorithm, *WSEAS Trans. on Syst. and Contr.*, vol. 4, pp.151-161, 2009.

[3] P. Nazarian, Control of a Radiative Furnace under Precise and Uncertainty Conditions, *WSEAS Trans. on Syst. and Contr.*, vol. 4, pp.211-220, 2009.

[4] F. N. Koumboulis, N. D. Kouvakas and P. N. Paraskevopoulos, Analytic Modeling and Metaheuristic PID Control of a Neutral Time Delay Test Case Central Heating System, *WSEAS Trans. on Syst. and Contr.*, vol. 3, pp.967-981, 2008.

[5] T. Emami and J. M. Watkins Robust Performance Characterization of PID Controllers in the Frequency Domain, *WSEAS Trans. on Syst. and Contr.*, vol. 4, pp.232-242, 2009.

[6] K. Zakova, One Type of Controller Design for Delayed Double Integrator System, *WSEAS Trans. on Syst. and Contr.*, vol. 3, pp.62-69, 2008.

[7] M. S. Mahmoud and N. F. AI-Muthairi, Quadratic stabilization of continuous time systems with state-delay and norm-bounded time-varying uncertainties, *IEEE Trans. on Automat. Contr.*, vol. 39, pp.2135-2139, 1994.

[8] S.-I. Niculescu, H_∞ Memoryless control with α-stability constraint for time-delay systems: an LMI approach, *IEEE Trans. on Automat. Contr.*, vol. 43, pp.739-743, 1998.

[9] E. Fridman and U. Shaked, Parameter dependent stability and stabilization of uncertain time-delay systems, *IEEE Trans. on Automat. Contr.*, vol. 48, pp.861-866, 2003.

[10] M. Alfidi and A. Hmamed, Robust Stability Analysis for 2-D Continuous-Time Systems Via Parameter-Dependent Lyapunov Functions, *WSEAS Trans. on Syst. and Contr.*, vol. 2, pp.497-503, 2007.

[11] K. Wei, Quadratic stabilizability of linear systems with structural independent time-varying uncertainties, *IEEE Trans. on Automat. Contr.*, vol. 35, pp.268-277, 1990.

[12] T. Amemiya and G. Leitmann, A method for designing a stabilizing control for a class of uncertain linear delay systems, *Dynam. and Contr.*, vol. 4, pp.147-167, 1994.

[13] T. Amemiya, A stabilizing control for a class of uncertain delay systems with limited measurable state variables, *Dynam. and Contr.*, vol. 7, pp.235-262, 1997.

[14] T. Amemiya, Triangular configuration of uncertain systems stabilizable by means of feedback controller, *Appl. Math. and Compu.*, vol. 120, pp.45-54, 2001.

[15] T. Hashimoto and T. Amemiya, Dual Invariance Structure of Controllability and Observability, *Proc. of the 2009 IEEE Conf. on Decis. and Contr.*, Shanghai, (to appear).

[16] T. Hashimoto and T. Amemiya, A Note on a Criterion for M-matrix, *Compu. Math. and Model.*, Vol.20, No.3, pp.318-325, 2009.