Controlled jump Markov processes with local transitions and their fluid approximation

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Abstract: Stochastic jump processes, especially birth-and-death processes, are widely used in the queuing theory, computer networks and information transmission. The state of such process describes the instant length of the queues (numbers of packets at different edges to be transmitted through the net). If the birth and death rates are big, trajectories of such processes are close to the trajectories of deterministic dynamic systems. Therefore, if we consider the related optimal control problems, we expect that the optimal control strategy in the deterministic ('fluid') model will be nearly optimal in the underlying stochastic model. In the current paper, a new technique for calculating the accuracy of this approximation is described. In a nutshell, instead of the study of trajectories, we investigate the corresponding dynamic programming equations. It should be emphasized that we deal also with multiple-dimensional lattices, so that the results are applicable to complex communicating systems of queues. Other areas of application are population dynamics, mathematical epidemiology, and inventory systems.

Key–Words: Birth-and-death process, Continuous time Markov chain, Fluid model, Optimal control, Queuing system, Dynamic programming, μC-rule, Inventory

1 Introduction

The fluid approximation is a powerful tool for investigating Markovian models with local transitions, like birth-and-death processes. Justification of the fluid approximation in different settings can be found in [8, 10, 23, 24, 28]. It should be emphasized that birth-and-death processes are widely used in the queuing theory. The corresponding fluid approximations are helpful for establishing stability of complex queuing networks [8, 9, 11]. Another area of applications is manufacturing: see e.g. [5, 22, 26, 35], where the fluid approximation was used for the analysis of supply-demand or production-inventory systems. Complicated optimal control problems for stochastic processes can be satisfactorily solved after one replaces the model with its fluid limit. This approach was demonstrated in [2, 4, 7, 12, 13, 30], where meaningful examples on controlled queuing networks and epidemics can be found. Note that in [12], the authors considered a discrete time Markov chain; the underlying lattice was multiple-dimensional in [4, 12, 28]. We emphasize that very often the fluid model is used, without justification, for the study of real life processes that are of the birth-and-death type. In principle, that justification could be based on the trajectory-wise convergence proved in [8, 10, 23, 24]. This method allows to prove the convergence of the Bellman functions: see [28], where a specific discounted multiple-dimensional model was investigated. Another approach based on algebraic equations of the dynamic programming type was demostrated in [31]. Since in the controlled framework we are interested in the performance functional, the latter approach seems more appropriate, and we develop it in the present paper. The main advantage is that it makes possible to provide the accuracy of the approximation, in terms of the objective functional. There is a number of recent papers dealing with the connection of the fluid optimization problem and the underlying stochastic model: see [4] and references therein. In the current article, we present a new promising approach to this question.

In Section 2, the ideas are demonstrated in the framework of one-dimensional process. Section 3 is devoted to the special multiple-dimensional problem of optimal scheduling of a multiclass queue, similar to [15, 20, 21, 27, 33]. For such models, the μC-rule is known to be optimal for many versions of the problem: discounted case, fixed time horizon, long-run average loss and so on. (See [27].) The asymptotic optimality of the μC-strategy under the 'heavy traffic' conditions was established in [25]. Note that the exact formulae for the Bellman function were usually not presented. Investigation undertaken in [15, 20, 21] shows that such expressions are very cumbersome. In this connection, formula (17) below gives a simple approximation to the Bellman function, along with the estimate of the error which goes to zero when the scaling parameter n increases. Finally, in Section 4, we apply the theory developed to inventory problems.

2 One-dimensional lattice

Following the standard practice (see e.g. [16]), the controlled birth-and-death process Y_t is defined by the following elements:
$S = \{0, 1, \dots\}$ is the state space,
A is the action space (arbitrary Borel),
$Q = [q_{i,j}(a)]$ is the tri-diagonal matrix of transition rates; $q_{i,j} = 0$ if $|i - j| > 1$;

$$q_{i,i+1}(a) = \begin{cases} \Lambda_i(a) \geq 0, & \text{if } i > 0; \\ 0, & \text{if } i = 0; \end{cases}$$

$$q_{i,i-1}(a) = \begin{cases} M_i(a) > 0, & \text{if } i > 0; \\ 0, & \text{if } i = 0; \end{cases}$$

$$q_{ii}(a) = -q_{i,i+1}(a) - q_{i,i-1}(a);$$

$G(i, a)$ is the (real, measurable) loss rate; $G(0, a) \equiv 0$.

A control strategy Φ is a mapping from S to A. We restrict ourselves to stationary nonrandomised strategies because, under rather general conditions, they are sufficient for solving optimisation problems.

The optimal control problem under consideration looks like follows:

$$W^\Phi(i) = E_i^\Phi \left[\int_0^\infty G(Y_s, \Phi(Y_s)) ds \right] \to \inf_\Phi . \quad (1)$$

Here E_i^Φ is the expectation on the space of trajectories $\{Y_s\}_{s \geq 0}$ starting from $Y_0 = i$ and absorbing at zero, wrt probability measure generated by strategy Φ. The rigorous mathematical constructions can be found in [19, 29]. The Bellman equation for problem (1) looks like follows:

$$\inf_{a \in A} \{ G(i, a) + \Lambda_i(a) V(i + 1) + M_i(a) V(i - 1)$$

$$-[\Lambda_i(a) + M_i(a)] V(i) \} = 0, \text{ if } i > 0; \quad V(0) = 0.$$

In fact, we shall consider a sequence of described models, so that all their parameters ${}^n\Lambda$, ${}^nM$ and ${}^nG$, as well as objective ${}^nW^\Phi$ will be indexed with $n = 1, 2, \dots$ Namely, we assume that measurable functions $\lambda(y, a), \mu(y, a)$, and $g(y, a)$ are fixed for $y > 0$, $a \in A$, such that $\mu(y, a) > \lambda(y, a)$; $\lambda(0, a) = \mu(0, a) = g(0, a) \equiv 0$; and

$$\begin{gathered} {}^n\Lambda_i(a) = n\lambda(i/n, a), \quad {}^nM_i(a) = n\mu(i/n, a), \\ {}^nG(i, a) = g(i/n, a). \end{gathered} \quad (2)$$

This is the standard fluid scaling: see [10, 23, 24, 28]. Now one can introduce the (fluid) absorbing optimal control problem

$$\frac{dy}{d\tau} = \lambda(y, a) - \mu(y, a); \qquad \int_0^\infty g(y, a) d\tau \to \inf_{a(\cdot)}, \quad (3)$$

where the infimum is taken over all control strategies $a(\cdot)$. Usually, the class of (measurable) feedback strategies $a(t) = \varphi(y(t))$, $y(t) > 0$, is sufficient, and the (nearly) optimal strategy can be obtained using the dynamic programming approach, that is, after solving Bellman equation

$$\begin{gathered} \inf_{a \in A} \left\{ \tfrac{dv}{dy} [\lambda(y, a) - \mu(y, a)] + g(y, a) \right\} = 0, \\ v(0) = 0. \end{gathered} \quad (4)$$

We aim to show that if a feedback strategy $\varphi^*(y)$ is (nearly) optimal for problem (3), then strategy $\Phi^*(i) = \varphi^*(i/n)$ will be nearly optimal for (1), if n is big.

In what follows, we assume that the processes Y_t and $y(\tau)$ have a positive trend to zero, functions λ and μ grow not too fast with y, and the loss rate g grows not too fast with y, comparing with λ and μ. Namely, we impose the following conditions on the data:

Conditions 1 (a) For all $y > 0$ and $a \in A$,

$$\lambda(y, a) \geq 0, \quad \mu(y, a) > 0,$$

$$\inf_{a \in A, y > 0} \frac{\mu(y, a)}{\lambda(y, a)} \geq \tilde{\eta} \in (1, \infty).$$

(b) There exist constants $\eta_1 \in (1, \tilde{\eta})$ and $\eta_2 \in (1, \frac{\tilde{\eta}}{\eta_1})$ such that

$$\sup_{a \in A, y > 0} \frac{|g(y, a)|}{[\lambda(y, a) + \mu(y, a)]\eta_1^y} \leq C_1 < \infty;$$

$$\sup_{a \in A, y > 0} \frac{\lambda(y, a) + \mu(y, a)}{\eta_2^y} \leq C_2 < \infty.$$

Under Conditions 1, the controlled process Y_s is regular (non-explosive) for any control strategy.

Definition 1 A feedback control strategy $\varphi(y)$ in the fluid problem (3) will be called normal if there exist finite intervals $(y_0 = 0, y_1), (y_1, y_2), \dots$ with

$\lim_{j\to\infty} y_j = \infty$, *such that, on each such interval, function*

$$\frac{g(y,\varphi(y))}{\mu(y,\varphi(y))-\lambda(y,\varphi(y))} \qquad (5)$$

is Lipschitz continuous.

Conditions 2 There exist finite intervals $(y'_0 = 0, y'_1), (y'_1, y'_2), \ldots$ *with* $\lim_{j\to\infty} y'_j = \infty$, *such that, on each such interval,* $\inf_{y\in(y'_j,y'_{j+1}),a\in A}[\mu(y,a)+\lambda(y,a)] \ge \delta > 0$, *and functions* μ, λ *and* g *are Lipschitz continuous wrt* y *for each fixed* $a \in A$, *and the Lipschitz constants are* a*-independent (but can be different for different intervals).*

Theorem 1 Suppose Conditions 1 and 2 are satisfied. Then $\forall \hat{y} > 0$, $\forall \varepsilon > 0$ *there exists a normal feedback control strategy* φ^* *in problem (3) on interval* $(0,\hat{y}]$, *such that, for all* $y \in (0,\hat{y}]$, *inequality*

$$v(y) \le v^{\varphi^*}(y) \le v(y) + \varepsilon\hat{y} \qquad (6)$$

holds, i.e. strategy φ^* *is* $\varepsilon\hat{y}$*-optimal.*

For stationary nonrandomised strategy $\Phi^*(i) = \varphi^*(i/n) : S \to A$, *functions* ${}^nW^{\Phi^*}$ *and* v^{φ^*} *are close to each other:*

$$\sup_{0\le i\le \hat{y}n} \left| {}^nW^{\Phi^*}(i) - v^{\varphi^*}(i/n) \right| \le \hat{\varepsilon}(n).$$

Here

$$\hat{\varepsilon}(n) = \frac{K_1}{n} + \frac{K_2}{\tilde{\eta}^n} + K_3(\eta_1^{1/n} - 1),$$

$$K_1 = \frac{\tilde{\eta}+1}{\tilde{\eta}-1}[D(\hat{y}+1) + 3C_1L\eta_1^{\hat{y}+1}];$$

$$K_2 = \frac{\tilde{\eta}+1}{\tilde{\eta}-1}C_1\left[1 + \frac{2(\tilde{\eta}+1)}{(\tilde{\eta}-1)\ln\eta_1}\right]\frac{\eta_1^{\hat{y}+1}\tilde{\eta}^2}{\tilde{\eta}-\eta_1};$$

$$K_3 = \left(\frac{\tilde{\eta}+1}{\tilde{\eta}-1}\right)^2 \frac{3C_1L\eta_1^{\hat{y}+1}}{\ln\eta_1}.$$

The values of D *and* L *come from the strategy* φ^**; namely* L *is such that* $y_L < \hat{y}+1 \le y_L + 1$ *(see Definition 1) and* D *is the common Lipschitz constant of function (5) with* $\varphi = \varphi^*$ *on all intervals* (y_j, y_{j+1}), $j = 0,1,\ldots,L$.

For all large enough n, *strategy* Φ^* *is nearly optimal for all initial states* $Y_0 \in [0,\hat{y}n]$ *in the stochastic problem (1). Namely,*

$$\sup_{0\le i\le \hat{y}n} \left| {}^nW^{\Phi^*}(i) - \inf_{\Phi} {}^nW^{\Phi}(i) \right| \le \delta + 2\hat{\varepsilon}(n), \quad (7)$$

where

$$\delta = \frac{\varepsilon(\hat{y}+1)(\tilde{\eta}+1)}{\tilde{\eta}-1}.$$

The proof will be published in [32].

Example. Consider the M/M/1 queueing system with the controlled input stream $n\lambda(j/n,a) = n(d_0 + d_1a)$, $d_0, d_1 > 0$, $d_0 + d_1 < 1$, $a \in A = [0,1]$; the service intensity $n\mu(j/n,a) = n$ is constant. As usual, n is a fixed large enough parameter. The initial state is $i > 0$, and we observe the trajectory up to the absorption at zero. One can consider this model as a verstion of call admission control. The server always accepts the jobs from one stream with intensity nd_0, but can chose any probability a of accepting jobs from another stream, intensity nd_1. Suppose we are interested in the total expected throughput (to be maximised), as well as the total expected queue length (to be minimised). Therefore, ${}^nG(i,a) = i/n - Ra$, where $R > 0$ is a given constant (Lagrange multiplier), and one has to solve problem (1). The corresponding fluid model is defined by

$$\lambda(y,a) = d_0 + d_1a, \;\; \mu(y,a) = 1, \;\; g(y,a) = y - Ra.$$

Note that Conditions 1 and 2 are satisfied.

The Bellman equation (4) can be explicitly solved:

$$v(y) = \begin{cases} \frac{Ry - y^2/2}{d_0+d_1-1}, & \text{if } 0 \le y \le y^*; \\ \\ \frac{y^2-(y^*)^2}{2(1-d_0)} - \frac{Ry^* - (y^*)^2/2}{1-d_0-d_1}, & \text{if } y > y^*, \end{cases}$$

where

$$y^* \triangleq \frac{R(1-d_0)}{d_1}.$$

Feedback control strategy

$$\varphi^*(y) \triangleq \begin{cases} 1, & \text{if } 0 < y \le y^*; \\ 0, & \text{if } y > y^* \end{cases}$$

is normal and optimal for the problem (3), and $\varepsilon = 0$ in formula (6).

Let us fix $d_0 = 0.25$; $d_1 = 0.5$; $R = 1$. One can take $\tilde{\eta} = 4/3$, $\eta_1 = 7/6$, $\eta_2 = 15/14$, $C_1 = 2.23$, $C_2 = 1.75$. For the normal strategy φ^*, we have $y^* = 1.5$, $L = 1$, $D = 4$. Now, after choosing $\hat{y} = 5$, we have $K_1 = 286$, $K_2 = 38500$, $K_3 = 5360$, and for $n = 100,000$ we obtain $\hat{\varepsilon} = 0.022$ in Theorem 1, which is small enough being compared with $v(y) \in [0,14]$ when $y \in [0,\hat{y}]$. It should be emphasized that this estimate of the accuracy of the fluid approximation is very rough.

3 Multiple-dimensional lattice: μC-rule

Suppose there are $m > 1$ types of jobs to be served by a single server. Arrival and service rates for type

j equal $n\lambda_j$ and $n\mu_j$, where, like previously, n is the scaling parameter. We shall consider the Markovian case when all the service and inter-arrival times are exponential; we assume also that there is infinite space for waiting. The holding cost of one type j job equals C_j/n per time unit. At any moment, the server should chose a job for service from the queue: that is, the action $a = j \in \{1, 2, \ldots, m\}$ means that a type j job is under service. Note that we consider the preemptive service times. The goal is to minimize the total holding cost up to the absorption at the zero state (empty queue).

Mathematical problem looks as follows.

$S = \{(i_1, i_2, \ldots, i_m)\}$ is the state space; $i_j \geq 0$ equals the number of jobs of type j in the system.

$A = \{1, 2, \ldots, m\}$ is the action space.

If $Y = (Y_1, Y_2, \ldots, Y_m)$ is the current state, then only the following transitions can occur:

- one job of type j arrives; transition rate to the new state $Y' = (Y_1, Y_2, \ldots, Y_j + 1, \ldots, Y_m)$ equals $n\lambda_j$; $j = 1, 2, \ldots m$;
- if $Y_j > 0$ and $a = j$ then the service can be completed; transition rate to the new state $Y' = (Y_1, Y_2, \ldots, Y_j - 1, \ldots, Y_m)$ equals $n\mu_j$.

State $Y = 0$ is absorbing.

In what follows, we accept that $a \neq j$ in case $Y_j = 0$: there is no reason to serve a dummy job.

Finally, $G(Y, a) = \frac{1}{n}\sum_{j=1}^{m} C_j Y_j$, and we study problem (1). Of course, initial state $(i_1, i_2, \ldots, i_m)$ and the process Y_s are now multiple-dimensional; the objective to be minimized is denoted as ${}^{n}W^{\Phi}(i_1, i_2, \ldots, i_m)$.

Conditions 3 $\sum_{j=1}^{m} \frac{\lambda_j}{\mu_j} < 1$*, i.e. this queueing system is stable.*

The corresponding fluid model is described by the following equations

$$\frac{dy}{d\tau} = f(y, a), \tag{8}$$

where $y \in \mathbb{R}_+^m$, $a \in A$, and, for $y \neq 0$,

$$f_j(y, a) = \begin{cases} \lambda_j, & \text{if } j \neq a, \text{ or if } y_j = 0, \\ \lambda_j - \mu_j, & \text{if } j = a \text{ and } y_j > 0. \end{cases}$$

If $y = 0$ then $f(y, a) = 0$.

Performance functional:

$$F = \int_0^\infty g(y)d\tau \to \inf_{a(\cdot)}, \tag{9}$$

where $g(y) = \sum_{j=1}^{m} C_j y_j$.

Without loss of generality, further we assume that

$$\mu_1 C_1 \geq \mu_2 C_2 \geq \ldots \geq \mu_m C_m$$

and introduce function

$$v(y_1, y_2, \ldots, y_m) \stackrel{\triangle}{=} v_m(y_1, y_2, \ldots, y_m)$$

defined by the (recursive) formulae

$$v_1(y_1) = \frac{C_1 y_1^2}{2(\mu_1 - \lambda_1)};$$
$$T_1(y_1) = \frac{y_1}{\mu_1 - \lambda_1};$$

$$\begin{aligned} v_{k+1}(y_1, y_2, \ldots, y_{k+1}) &= v_k(y_1, y_2, \ldots, y_k) \\ &+ C_{k+1}\left[y_{k+1} T_k(y_1, y_2, \ldots, y_k) \right. \\ &+ \frac{\lambda_{k+1} T_k^2(y_1, y_2, \ldots, y_k)}{2} \\ &\left. + \frac{(y_{k+1} + \lambda_{k+1} T_k(y_1, y_2, \ldots, y_k))^2}{2(\eta_{k+1} - \lambda_{k+1})} \right]; \end{aligned}$$

$$\begin{aligned} T_{k+1}(y_1, y_2, \ldots, y_{k+1}) &= T_k(y_1, y_2, \ldots, y_k) \\ &+ \frac{y_{k+1} + \lambda_{k+1} T_k(y_1, y_2, \ldots, y_k)}{\eta_{k+1} - \lambda_{k+1}}. \end{aligned} \tag{10}$$

Here

$$\eta_{k+1} \stackrel{\triangle}{=} \mu_{k+1}\left(1 - \frac{\lambda_1}{\mu_1} - \ldots - \frac{\lambda_k}{\mu_k}\right)$$

is the effective service rate of $k + 1$ type jobs under the feedback μC-strategy

$$\begin{aligned} a = \varphi^*(y) &\stackrel{\triangle}{=} (k + 1) \\ \times I\{y_1 = 0, y_2 &= 0, \ldots, y_k = 0, y_{k+1} > 0\}, \end{aligned} \tag{11}$$

where I stands for the indicator function. This strategy allows to serve type $k + 1$ jobs only if there are no jobs of types $1, 2, \ldots, k$. Function v_k coincides with the Bellman function in case there are only the first k types of jobs, T_k is the corresponding time until absorbtion at zero.

One can check, using the induction argument, that

$$\begin{aligned} T_k &= \sum_{i=1}^{k} \frac{y_i}{(\eta_k - \lambda_k)} \cdot \prod_{j=i}^{k-1} \frac{\eta_{j+1}}{\eta_j - \lambda_j} \\ &= \sum_{i=1}^{k} \frac{y_i}{\mu_i\left(1 - \sum_{j=1}^{k} \frac{\lambda_j}{\mu_j}\right)}. \end{aligned} \tag{12}$$

Actually, the μC-strategy is optimal in the stochastic problem (1): this follows from Proposition 3.1 in [27]. The goal of the current section is to study the fluid model (8),(9) and to compare it with problem (1): see Theorem 2.

Lemma 1 Under Condition 3, function v satisfies the dynamic programming equation

$$\min_{a\in A}\left\{g(y)+\sum_{j=1}^{m}\frac{\partial v}{\partial y_j}f_j(y,a)\right\}$$
$$= g(y)+\sum_{j=1}^{m}\frac{\partial v}{\partial y_j}f_j(y,\varphi^*(y))=0; \qquad (13)$$
$$v(0)=0,$$

and hence the μC-strategy φ^* is optimal in the fluid model (8), (9).

Remark 1 Lemma 1 is consistent with [3], where the case of a finite fixed time horizon was studied. Note that the total time until absorption, T_m does not depend on the priorities allocation. For the fixed time interval T_m, the minimal value of the objective $F_{T_m}=\int_0^{T_m} g(y)d\tau$ is provided by the μC-strategy [3]. Hence it is also optimal in terms of functional (9).

Function v (called Bellman function) coincides with the objective functional (9) if the process (8) starts from state y and is governed by the μC-strategy (11). We know the exact formula for v, but it can be derived also from equation (13):

$$g(y)+\sum_{j=1}^{m}\frac{\partial v}{\partial y_j}f_j(y,\varphi^*(y))=0, \quad v(0)=0. \quad (14)$$

From the latter point of view, dynamics (8) is inconvenient: if $y_1=0$ then this component remains zero, but the dynamics on that hyper-plane is non-standard (called sliding mode):

$$f_1(y)=\lambda_1>0, \text{ if } y_1=0,$$

and

$$f_1(y)=\lambda_1-\mu_1<0 \text{ if } y_1>0.$$

It would be better to describe this dynamics in another way, so that solution to (14) does not change and can be built using standard methods of partial differential equations.

When looking for continuously differentiable solutions to (14), we notice that (14) holds also in the limiting case when $y_1\to 0$, so that, e.g. if $y_2>0$ we have two equations when $y_1=0$:

$$C_2y_2+\ldots+C_my_m+\frac{\partial v}{\partial y_1}\lambda_1+\frac{\partial v}{\partial y_2}(\lambda_2-\mu_2)$$
$$+\frac{\partial v}{\partial y_3}\lambda_3+\ldots+\frac{\partial v}{\partial y_m}\lambda_m=0$$

and

$$C_2y_2+\ldots+C_my_m+\frac{\partial v}{\partial y_1}(\lambda_1-\mu_1)+\frac{\partial v}{\partial y_2}\lambda_2$$
$$+\frac{\partial v}{\partial y_3}\lambda_3+\ldots+\frac{\partial v}{\partial y_m}\lambda_m=0.$$

It is convenient to have $\frac{dy_1}{dt}=0$ (coefficient at $\frac{\partial v}{\partial y_1}$) when $y_1=0$, so that we write down the corresponding convex combination:

$$C_2y_2+\ldots+C_my_m$$
$$+\frac{\partial v}{\partial y_1}\left[\left(1-\frac{\lambda_1}{\mu_1}\right)\lambda_1+\frac{\lambda_1}{\mu_1}(\lambda_1-\mu_1)\right]$$
$$+\frac{\partial v}{\partial y_2}\left[\left(1-\frac{\lambda_1}{\mu_1}\right)(\lambda_2-\mu_2)+\frac{\lambda_1}{\mu_1}\cdot\lambda_2\right]$$
$$+\frac{\partial v}{\partial y_3}\lambda_3+\ldots+\frac{\partial v}{\partial y_m}\lambda_m=0.$$

As a result, we obtain the effective service rate for type 2 jobs: $\eta_2=\mu_2\left(1-\frac{\lambda_1}{\mu_1}\right)$. This reasoning applies to types $3,\ldots,m$, so that functions f_j are replaced with

$$\tilde f_j(y,\varphi^*(y))$$
$$=\begin{cases}0, & \text{if } j=1 \text{ and } y_1=0;\\ \lambda_1-\mu_1, & \text{if } j=1 \text{ and } y_1>0;\\ \lambda_j-\mu_j\left(1-\frac{\lambda_1}{\mu_1}\right), & \text{if } \varphi^*(y)=j>1;\\ \lambda_j, & \text{if } j>1 \text{ and } \varphi^*(j)\neq j.\end{cases}$$

If we know function v on the plane $y_1=0$, then we can calculate it (analytically or numerically) for all $y\geq 0$ using the characteristics method for equation

$$g(y)+\sum_{j=1}^{m}\frac{\partial v}{\partial y_j}\tilde f_j(y,\varphi^*(y))=0, \quad v(0)=0.$$

In order to build function v on the plane $y_1=0$, we perform the same operation for the second component y_2 and so on. Eventually, we deal with equations

$$\frac{dy_j}{d\tau}$$
$$=\begin{cases}0, & \text{if } y_1=\ldots=y_j=0;\\ \lambda_j-\eta_j, & \text{if } y_j>0, \text{ but } y_1=\ldots=y_{j-1}=0;\\ \lambda_j, & \text{otherwise.}\end{cases}$$

Now one can build function v firstly on the line $y_1=\ldots=y_{m-1}=0$, having initial condition $v(0)=0$. After that, function v is constructed sequentially on hyper-planes $\{y_1=\ldots=y_{m-2}=0\}$, $\{y_1=\ldots=y_{m-3}=0\}$, $\ldots$, $\{y_1=0\}$, and finally for all $y_1>0$.

Of course, in this particular model we will finish with the same function $v(\cdot)$ given above. But the reasoning presented can be useful in other situations, if one has to compute the performance functional for a given control strategy leading to a sliding mode.

Lemma 2 In the stochastic model described above, under control strategy

$$\Phi^*(Y) \stackrel{\triangle}{=} \varphi^*(Y/n), \tag{15}$$

starting from the initial state $(i_1, i_2, \dots, i_m)$, the expected time to absorbtion at state $Y = 0$ equals

$$^nU(i_1, i_2, \dots, i_m) = \sum_{j=1}^{m} \frac{i_j}{n(\eta_j - \lambda_j)} \prod_{l=j+1}^{m} \frac{\eta_l}{\eta_l - \lambda_l}. \tag{16}$$

We see that $^nU(i_1, \dots, i_m) = T_m\left(\frac{i_1}{n}, \dots, \frac{i_m}{n}\right)$. Actually, T_m is the fluid approximation to ^nU, and, under very general conditions, the accuracy of that approximation is proportional to the maximum of the second derivative of the fluid functional. (See [31], where the one-dimensional case was investigated in depth.) Thus, the coincidence $^nU(\cdot) = T_m(\cdot)$ is not surprising because function T_m is linear.

Theorem 2 For the feedback μC-strategy Φ^*, for any vector $\hat{y} \in \mathbb{R}^m_+$, the following inequality holds:

$$\begin{aligned} \sup_{0 \le (i_1, i_2, \dots, i_m) \le \hat{y}n} & \Big| \, ^nW^{\Phi^*}(i_1, i_2, \dots, i_m) \\ & - v(i_1/n, i_2/n, \dots, i_m/n) \Big| \\ \le \frac{mD}{n} & \left(\max_{1 \le j \le m} \max\{\lambda_j, \mu_j - \lambda_j\} \right) \\ & \times \sum_{j=1}^{m} \frac{\hat{y}_j}{\eta_j - \lambda_j} \prod_{l=j+1}^{m} \frac{\eta_l}{\eta_l - \lambda_l}, \end{aligned} \tag{17}$$

where $D = \max_{1 \le j \le m} \left| \frac{\partial^2 v}{\partial y_j^2} \right|$ is a constant since function v is quadratic; vector inequalities are component-wise.

As was already mentioned, the μC-strategy (11) is optimal in problem (1); it is also optimal for the stochastic discounted problem and, consequently, for the long-run average loss. (Consider the limit as the discount factor goes to zero.) One can study the fluid discounted model and provide the asymptotic formula for the accuracy, similar to (17). The following example shows that the μC-strategy is not necessarily optimal if there is finite space for waiting. Take $m = 2$ and suppose there cannot be more than one job of each type in the system. Assume for simplicity that $\mu_1 = \mu_2 = \mu$. There are only four states in this stochastic system: $(i_1, i_2) \in \{(0,0), (0,1), (1,0), (1,1)\}$, and the decision should be made only in state $(1,1)$. The Bellman equation looks like follows

$$\begin{aligned} \min\{& C_1 + C_2 + \mu V(0,1) - \mu V(1,1); \\ \{& C_1 + C_2 + \mu V(1,0) - \mu V(1,1)\} = 0. \end{aligned} \tag{18}$$

The first (second) line corresponds to the priority given to the first (second) type. Additional equations:

$$C_1 + \lambda_2 V(1,1) + \mu V(0,0) - (\lambda_2 + \mu) V(1,0) = 0;$$

$$C_2 + \lambda_1 V(1,1) + \mu V(0,0) - (\lambda_1 + \mu) V(0,1) = 0;$$

$$V(0,0) = 0.$$

As usual, we assume that $\mu C_1 \ge \mu C_2$.

One can check that if $C_1 - C_2 \ge (C_1 + C_2)\frac{\lambda_1 - \lambda_2}{\mu}$ then the μC-strategy is optimal: the first expression in (18) is smaller and

$$V(0,1) = \frac{\mu C_2 + \lambda_1 (C_1 + C_2)}{\mu^2};$$

$$V(1,0) = \frac{\mu^2 C_1 + 2\mu\lambda_2 C_2 + \mu\lambda_2 C_1 + \lambda_1\lambda_2 (C_1 + C_2)}{\mu^2(\lambda_2 + \mu)};$$

$$V(1,1) = \frac{\lambda_1 (C_1 + C_2) + 2\mu C_2 + \mu C_1}{\mu^2}.$$

But if $C_1 - C_2 < (C_1 + C_2)\frac{\lambda_1 - \lambda_2}{\mu}$ then the μC-strategy is not optimal: the second expression in (18) is smaller and

$$V(0,1) = \frac{\mu^2 C_2 + 2\mu\lambda_1 C_1 + \mu\lambda_1 C_2 + \lambda_1\lambda_2 (C_1 + C_2)}{\mu^2(\lambda_1 + \mu)};$$

$$V(1,0) = \frac{\mu C_1 + \lambda_2 (C_1 + C_2)}{\mu^2};$$

$$V(1,1) = \frac{\lambda_2 (C_1 + C_2) + 2\mu C_1 + \mu C_2}{\mu^2}.$$

In case $\lambda_1 \gg \lambda_2$, it is worth serving the second type job (even if $C_1\mu > C_2\mu$) because after that, while serving the first type, it is unlikely that a job of type 2 will arrive, and we can quickly reach the absorbing zero state.

Let us have a quick look at the following long-run average modification of the stochastic model. After reaching the zero state, the Y_t process can jump to states $(1, 0, \dots, 0)$, $(0, 1, \dots, 0)$,...,$(0, 0, \dots, 1)$ with probabilities $\lambda_1 / \left(\sum_{j=1}^m \lambda_j\right)$, $\lambda_2 / \left(\sum_{j=1}^m \lambda_j\right)$,...,

$\lambda_m / \left(\sum_{j=1}^m \lambda_j\right)$. Therefore, under a stationary control strategy Φ, the expected total loss, between the two consecutive visits to the zero state, equals

$$[\lambda_1 \, {}^nW^\Phi(1,0,\ldots,0) + \lambda_2 \, {}^nW^\Phi(0,1,\ldots,0) + \ldots + \lambda_m \, {}^nW^\Phi(0,0,\ldots,1)] \Big/ \sum_{j=1}^m \lambda_j. \tag{19}$$

Denoting ${}^nU^\Phi(i_1, i_2, \ldots, i_m)$ the expected time to hit zero starting from state $(i_1, i_2, \ldots, i_m) \neq 0$, we see that the expected time interval, between the two consecutive visits to zero, equals

$$[\lambda_1 \, {}^nU^\Phi(1,0,\ldots,0) + \lambda_2 \, {}^nU^\Phi(0,1,\ldots,0) + \ldots + \lambda_m \, {}^nU^\Phi(0,0,\ldots,1)] \Big/ \sum_{j=1}^m \lambda_j. \tag{20}$$

Lemma 3 For any stationary control strategy Φ expression (20) coincides with

$$\frac{\sum_{j=1}^m \frac{\lambda_j}{\mu_j}}{n\left(1 - \sum_{j=1}^m \frac{\lambda_j}{\mu_j}\right) \sum_{j=1}^m \lambda_j}. \tag{21}$$

The process Y_t is regenerative [34]. Therefore, the long-run average loss coincides with the ratio of (19) to the regeneration period

$$T = \frac{\sum_{j=1}^m \frac{\lambda_j}{\mu_j}}{n\left(1 - \sum_{j=1}^m \frac{\lambda_j}{\mu_j}\right) \sum_{j=1}^m \lambda_j} + \frac{1}{n \sum_{j=1}^m \lambda_k}$$
$$= \frac{1}{n\left(\sum_{j=1}^m \lambda_k\right)\left(1 - \sum_{j=1}^m \frac{\lambda_j}{\mu_j}\right)}.$$

The second summand is the expected length of the idle period. Therefore, a stationary control strategy is optimal in the absorbing case if and only if it is optimal in the model with long-run average loss. Note, we deal with the unichain model, so that stationary strategies form a sufficient class, and a strategy is optimal iff it is optimal for any initial condition. Since the μC-strategy is optimal in problem (1), it provides also the minimum to the long-run average loss.

Controlled fluid models of general communication networks with linear holding cost $\sum_{j=1}^m C_j Y_j$ were studied in [13]. Optimality of the μC-strategy established in the above Lemma 1 is consistent with the more general result [13]: there exists a finite collection of polyhedral cones, covering the total state space $\mathbb{R}_+^m$, such that the value of the optimal feedback control strategy is constant inside each of those cones. At the same time, the optimal fluid strategy, translated back to the underlying stochastic network using formula (15), can be far not optimal: see [14], where a simple example of a tandem queue was discussed. In such queues, the μC-strategy can also be not optimal, as it was shown in [17].

4 Applications to the inventory theory

Optimization methods are widely used for solving real life problems in reliability [18], inventory theory [6] and so on. We start with a rather general fluid model of an inventory system, which is a particular case of the model studied in [5]. If the inventory level is $y \geq 0$ then the demand rate is $\mu(y) > 0$, so that

$$\frac{dy}{d\tau} = -\mu(y).$$

At the moment τ^* when $y(\tau^*)$ reaches zero, the cycle is over and $y(\tau^* + 0) = z > 0$, i.e. the replenishment is instantaneous, the set-up cost being K. Holding y units results in the cost $g(y)$ per time unit. If we take into account also the profit, then one has to adjust function g by subtracting $c(y)\mu(y)$, where $c(y)$ is the profit from selling one unit. We are interested in minimizing the (long-run) total cost per unit time (tcu):

$$tcu(z) = \lim_{T\to\infty} \frac{1}{T}\left\{\int_0^T g(y(\tau))d\tau + K\left[\frac{T}{T_c}\right]\right\}$$
$$= \left[\int_0^z \frac{g(y)}{\mu(y)} dy + K\right] \Big/ T_c \to \inf_{z>0}.$$

Here $\left[\frac{T}{T_c}\right]$ is the integer part, T_c is the length of the cycle, that is the solution to equation

$$\int_0^{T_c} \mu(y(\tau))d\tau = z, \text{ i.e. } T_c = \int_0^z \frac{dy}{\mu(y)}.$$

The best possible value z^* is called economic order quantity (eoq).

All the introduced integrals are well defined under the following conditions.

Conditions 4 (a) There exist constants C_1, C_2, and δ such that

$$\delta \leq \mu(y) \leq C_1, \quad |g(y)| \leq C_2.$$

(b) Functions $g(y)$ and $\mu(y)$ are Lipschitz continuous with Lipschitz constants C_3 and C_4 correspondingly.

The corresponding stochastic model looks like follows. The product is measured and demanded in (small) units, so that the state space $S = \{0, 1, 2, \dots, Z = [nz]\}$; the square brackets stay for the integer part; n is the (big) scaling parameter. The random process Y_s under consideration represents the number of units of product in stock at time moment s. Transition rates equal $q_{i,i-1} = n\mu(i/n)$, if $i \geq 1$, and $q_{0,Z} = n\lambda$: the lead time is exponential with parameter $n\lambda$. All the remainder values $q_{i,j\neq i}$ are zero. The loss rate is

$$^nG(i) = g(i/n) + K\, n\mu(1/n) I\{i = 1\}.$$

(The last term corresponds to the expected set-up cost at the moment when $Y_s = 0$.) Performance functional:

$$^nTCU(Z) = \lim_{T\to\infty} \frac{1}{T} E_i \left[\int_0^T {}^nG(Y_s) ds \right] \to \inf_{Z>0}. \quad (22)$$

Like previously, we shall estimate the difference $|\,^nTCU([nz]) - tcu(z)|$ and prove that eoq z^* provides a nearly optimal value $[nz^*]$ to Z.

Theorem 3 Suppose the order size $z > 0$ is fixed and Conditions 4 hold for $y \in [0, z]$. Then

$$|\,^nTCU([nz]) - tcu(z)|$$

$$\leq \left[\left(C_2 + \frac{K\delta}{z} + \frac{C_2\delta}{zn\lambda} \right) \left(zC_4 + \delta + \frac{\delta^2}{\lambda} \right) + z(C_2C_4 + C_1C_3) + C_2\delta + \frac{\delta^2}{\lambda} \right] \times \frac{C_1^2}{n(z - 1/n)\delta^3}. \quad (23)$$

We see that the righthand side of (23) decreases when z grows up. So, it is convenient to estimate the lower boundaries for eoq z^* and for EOQ Z^*, solution to (22). Further, we assume that function g is non-negative. Clearly, if it is bounded from below, we can increase it by an appropriate constant, so that the values of z^* and Z^* will not change.

The derivative

$$\frac{d\, tcu(z)}{dz} < \frac{1}{T_c^2} \left\{ \frac{C_2}{\delta^2} z - \frac{K}{C_1} \right\}$$

is negative for all $z < \frac{\delta^2 K}{C_1 C_2}$. Similarly, the difference

$$\left(\sum_{i=1}^{Z+1} \frac{g(i/n)}{n\mu(i/n)} + K + \frac{g(0)}{n\lambda} \right) \Big/ \left(\sum_{i=1}^{Z+1} \frac{1}{n\mu(i/n)} + \frac{1}{n\lambda} \right)$$

$$- \left(\sum_{i=1}^{Z} \frac{g(i/n)}{n\mu(i/n)} + K + \frac{g(0)}{n\lambda} \right) \Big/ \left(\sum_{i=1}^{Z} \frac{1}{n\mu(i/n)} + \frac{1}{n\lambda} \right)$$

$$< \frac{C_2(Z + \delta/\lambda)}{\delta^2} - \frac{Kn}{C_1}$$

is negative if $\frac{Z}{n} < \frac{\delta^2 K}{C_1 C_2} - \frac{\delta}{n\lambda}$. Therefore, we can omit from consideration the values of z and Z/n below $\frac{\delta^2 K}{C_1 C_2} - \frac{\delta}{n\lambda}$ and formulate the following statement.

Corollary 1 If $g(z) > 0$ and N is such a number that $\frac{\delta^2 K}{C_1 C_2} - \frac{\delta}{N\lambda} - \frac{1}{N} > 0$, then, for all $n \geq N$, for all $z \geq \frac{\delta^2 K}{C_1 C_2} - \frac{\delta}{n\lambda}$, the following inequality holds

$$|\,^nTCU([nz]) - tcu(z)| \leq \frac{E}{n}$$

where

$$E = \frac{C_1^3 C_2 N\lambda}{(\delta^2 KN\lambda - \delta C_1 C_2 - C_1 C_2 \lambda)\delta^3} \times \left\{ \left(C_2 + \frac{K\delta C_1 C_2 N\lambda}{\delta^2 KN\lambda - \delta C_1 C_2} \right) \times \left(C_4 \cdot \frac{\delta^2 KN\lambda - \delta C_1 C_2}{C_1 C_2 N\lambda} + \delta + \frac{\delta^2}{\lambda} \right) + \frac{\delta^2 KN\lambda - \delta C_1 C_2}{C_1 C_2 N\lambda}(C_2C_4 + C_1C_3) + C_2\delta + \frac{\delta^2}{\lambda} \right\}$$

We see that, if we find the eoq z^* then, for $n \geq N$,

$$\left|\,^nTCU([nz^*]) - \inf_Z {}^nTCU(Z) \right| \leq \frac{2E}{n}.$$

Note that it is usually much easier to find z^* than the EOQ Z^* for the stochastic model, and if we accept the value of $[nz^*]$ then, for big n, $^nTCU([nz^*])$ will be close to the best possible.

In the classical case, when $\mu(y) \equiv D$ and $g(y) = hy$, we obtain:

$$tcu(z) = \frac{KD}{z} + \frac{hz}{2}$$

and

$$^nTCU(Z) = \frac{\lambda(hZ^2 + hz + 2n^2 DK)}{2(\lambda n Z + nD)}.$$

Under fixed n and Z, formulae

$$P(0) = \frac{1}{n\lambda} \Big/ \left(\sum_{j=1}^{Z} \frac{1}{n\mu(j/n)} + \frac{1}{n\lambda} \right)$$

$$P(i) = \frac{1}{n\mu(i/n)} \Big/ \left(\sum_{j=1}^{Z} \frac{1}{n\mu(j/n)} + \frac{1}{n\lambda} \right),$$
$$i = 1, 2, \ldots, Z$$

provide the stationary distribution for the jump random process Y_s.

At the same time, in the fluid model the invariant probability density is given by expression

$$p(y) = \frac{1}{\mu(y)} \Big/ \int_0^z \frac{du}{\mu(u)}.$$

We see that, if $Z = [nz]$, then

$$|P(0)| \leq \frac{C_1}{n(z - 1/n)\lambda}$$

and, for $i = 1, 2, \ldots, [nz]$,

$$\left| P(i) - \int_{(i-1)/n}^{i/n} p(u) du \right|$$

$$\leq \frac{1}{\left(\sum_{j=1}^{[nz]} \frac{1}{n\mu(j/n)} + \frac{1}{n\lambda} \right) \left(\int_0^z \frac{du}{\mu(u)} \right)}$$

$$\times \left\{ \frac{1}{n\mu(i/n)} \left| \int_0^z \frac{du}{\mu(u)} - \sum_{j=1}^{[nz]} \frac{1}{n\mu(j/n)} - \frac{1}{n\lambda} \right| \right.$$

$$\left. + \left(\sum_{j=1}^{[nz]} \frac{1}{n\mu(j/n)} + \frac{1}{n\lambda} \right) \left| \frac{1}{n\mu(i/n)} - \int_{(i-1)/n}^{i/n} \frac{du}{\mu(u)} \right| \right\}$$

$$\leq \frac{C_1^2 \left(2zC_4 + \delta + \frac{\delta^2}{\lambda} \right)}{z(z - 1/n)n^2\delta^3}.$$

Therefore, again, if n is large, one can very precisely estimate the distribution P based on the density p for the fluid model.

5 Conclusion

The convergance of trajectories of jump processes with local transitions to those of the corresponding 'fluid' dynamic systems was previously proved based on the Law of Large Numbers (see e.g. [23]). In the present work, we provide the rate of that convergence in terms of the objective functionals, in the framework of controlled models, and present the explicit formulae for the error term, based only on the initial data. Meaningful examples show that the theory developed can be applied to many real life situations. Another field of applications is population dynamics and mathematical epidemiology, which is not touched in the current article.

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Appendix

Proof of Lemma 1. According to Remark 1, the φ^* strategy is optimal in problem (8), (9), but the form of the Bellman function was not provided in [3]. Thus we present the brief plan how to prove equation (13).

In the proofs below, we repeatidly apply the induction method and use formulae (10), along with equalities

$$\frac{\partial T_m}{\partial y_m} = \frac{1}{\eta_m - \lambda_m};$$
$$\frac{\partial v_m}{\partial y_m} = C_m T_{m-1} + \frac{C_m(y_m + \lambda_m T_{m-1})}{\eta_m - \lambda_m}; \qquad (24)$$

$$\mu_i \eta_{i+1} - \mu_{i+1}\eta_i + \mu_{i+1}\lambda_i = 0, \quad i = 1, 2, \ldots, m. \qquad (25)$$

(Here $T_0 \triangleq 0$.)

(a) Firstly, we prove inequality

$$\Delta = \mu_m \frac{\partial v_{m+1}}{\partial y_m} - \mu_{m+1} \frac{\partial v_{m+1}}{\partial y_{m+1}} \geq 0 \qquad (26)$$

for an arbitrary $m \geq 1$. As a result, we can be sure that action $a = m$ (in the model with $m + 1$ types of jobs) is better than $a = m+1$ if $y_m > 0$. In particular, if $m = 1$, this is the proof of the left equation in (13).

One can show that

$$\Delta = \mu_m C_m T_{m-1} + \frac{\mu_m C_m (y_m + \lambda_m T_{m-1})}{\eta_m - \lambda_m}$$

$$-C_{m+1}\mu_{m+1}T_m + \frac{\mu_m C_{m+1}\lambda_{m+1}T_m}{\eta_m - \lambda_m}$$

$$+\frac{\mu_m C_{m+1}\lambda_{m+1}^2 T_m}{(\eta_{m+1} - \lambda_{m+1})(\eta_m - \lambda_m)}$$

$$-\frac{\mu_{m+1}C_{m+1}\lambda_{m+1}T_m}{\eta_{m+1} - \lambda_{m+1}} = B_1 y_m + B_2 T_{m-1}$$

and $B_1, B_2 \geq 0$.

(b) Secondly, we prove that, for each $i = 1, 2, \ldots, m-1$,

$$\mu_i \frac{\partial v_{m+1}}{\partial y_i} - \mu_{i+1} \frac{\partial v_{m+1}}{\partial y_{i+1}} = \mu_i \frac{\partial v_m}{\partial y_i} - \mu_{i+1} \frac{\partial v_m}{\partial y_{i+1}} \tag{27}$$

(for an arbitrary $m > 1$). As a result, we can be sure that if action $a = i$ was better than $a = i+1$ (if $y_i > 0$) in the model with m types of jobs, then it remains better in the model with $(m+1)$ types of jobs. According to (a), we will establish the left equality in (13) for any value of $m \geq 2$.

Clearly, the difference between the lefthand and the righthand sides of (27) equals

$$\Big\{ C_{m+1} y_{m+1} + C_{m+1} \lambda_{m+1} T_m + \frac{C_{m+1}\lambda_{m+1}}{\eta_{m+1} - \lambda_{m+1}} (y_{m+1} + \lambda_{m+1} T_m) \Big\} \times \left\{ \mu_i \frac{\partial T_m}{\partial y_i} - \mu_{i+1} \frac{\partial T_m}{\partial y_{i+1}} \right\}.$$

But

$$\mu_i \frac{\partial T_m}{\partial y_i} - \mu_{i+1} \frac{\partial T_m}{\partial y_{i+1}} = 0.$$

(c) Before proceeding to the proof of the righthand equality (13), we compare $v_m(0, \ldots, 0, y_{j+1}, \ldots, y_m)$ and $v_{m-j}(y_{j+1}, \ldots, y_m)$. Starting with with $j = 1$ and using the induction argument (wrt m), one can see that

$$v_m(0, y_2, \ldots, y_m) = v^1_{m-1}(y_2, \ldots, y_m), \tag{28}$$

where function $v^1_{m-1}(\cdot)$ is the same as $v_{m-1}(\cdot)$ if we replace its parameters $\mu_2, \ldots, \mu_m$ with $\mu_2 - \frac{\lambda_1}{\mu_1}\mu_2, \ldots, \mu_m - \frac{\lambda_1}{\mu_1}\mu_m$ correspondingly. As a result, we have

$$v_m(0, \ldots, 0, y_{j+1}, \ldots, y_m) = v^j_{m-j}(y_{j+1}, \ldots, y_m), \tag{29}$$

where function $v^j_{m-j}(\cdot)$ is the same as $v_{m-j}(\cdot)$ if we replace its parameters $\mu_{j+1}, \ldots, \mu_m$ with the effective service rates

$$\mu_{j+1} - \frac{\lambda_1}{\mu_1}\mu_{j+1} - \ldots - \frac{\lambda_j}{\mu_j}\mu_{j+1}; \quad \ldots;$$

$$\mu_m - \frac{\lambda_1}{\mu_1}\mu_m - \ldots - \frac{\lambda_j}{\mu_j}\mu_m.$$

(d) Next, one can prove equality

$$\sum_{j=1}^m \frac{\partial T_m}{\partial y_j} f_j(y, \varphi^*(y)) = -1 \tag{30}$$

by induction (if $y \neq 0$). In fact, (30) holds for any priorities allocation, not only for the φ^* strategy.

(e) To establish the righthand equality in (13), we start with the case $y_1 > 0$ and prove (by induction wrt m and using (30)) that

$$g(y) + \sum_{j=1}^m \frac{\partial v_m}{\partial y_j} f_j(y, \varphi^*(y)) = 0.$$

(f) Now suppose $y_1 = \ldots = y_i = 0, y_{i+1} > 0$ $(i \geq 1)$. According to (29) and item (e),

$$\frac{\partial v_m(0, \ldots, 0, y_{i+1}, \ldots, y_m)}{\partial y_{i+1}} \Big[\lambda_{i+1} - \Big(\mu_{i+1} - \frac{\lambda_1}{\mu_1}\mu_{i+1} - \ldots - \frac{\lambda_i}{\mu_i}\mu_{i+1} \Big) \Big] + \sum_{j=i+1}^m \frac{\partial v_m(0, \ldots, 0, y_{i+1}, \ldots, y_m)}{\partial y_j} \lambda_j = 0. \tag{31}$$

We intend to prove that $\forall m \geq 2\ \forall i < m\ \forall k \leq i$

$$\left.\frac{\partial v_m}{\partial y_k}\right|_{y_1 = \ldots = y_i = 0} \mu_k = \left.\frac{\partial v_m}{\partial y_{i+1}}\right|_{y_1 = \ldots = y_i = 0} \mu_{i+1}. \tag{32}$$

Using induction (wrt m) and (27), we see that (32) holds for $i = k = 1$.

Suppose $1 < i < m$.

If $k = i$ then, using (29), we see that

$$\left.\frac{\partial v_m}{\partial y_i}\right|_{y_1 = \ldots = y_i = 0} \mu_i \left(1 - \frac{\lambda_1}{\mu_1} - \ldots - \frac{\lambda_{i-1}}{\mu_{i-1}} \right)$$

$$= \left.\frac{\partial v^{i-1}_{m-i+1}}{\partial y_i}\right|_{y_i = 0} \mu_i \left(1 - \frac{\lambda_1}{\mu_1} - \ldots - \frac{\lambda_{i-1}}{\mu_{i-1}} \right)$$

$$= \left.\frac{\partial v^{i-1}_{m-i+1}}{\partial y_{i+1}}\right|_{y_i = 0} \mu_{i+1} \left(1 - \frac{\lambda_1}{\mu_1} - \ldots - \frac{\lambda_{i-1}}{\mu_{i-1}} \right);$$

thus (32) holds for $k = i$.

Exactly in the same way we consider $k = i - 1$ and conclude that

$$\left.\frac{\partial v_m}{\partial y_{i-1}}\right|_{y_1 = \ldots = y_{i-1} = 0} \mu_{i-1} \left(1 - \frac{\lambda_1}{\mu_1} - \ldots - \frac{\lambda_{i-2}}{\mu_{i-2}} \right)$$

$$= \left.\frac{\partial v_m}{\partial y_i}\right|_{y_1 = \ldots = y_{i-1} = 0} \mu_i \left(1 - \frac{\lambda_1}{\mu_1} - \ldots - \frac{\lambda_{i-2}}{\mu_{i-2}} \right).$$

In particular, if $y_i = 0$ then

$$\left.\frac{\partial v_m}{\partial y_{i-1}}\right|_{y_1 = \ldots = y_i = 0} \mu_{i-1} = \left.\frac{\partial v_m}{\partial y_i}\right|_{y_1 = \ldots = y_i = 0} \mu_i$$

$$= \left.\frac{\partial v_m}{\partial y_{i+1}}\right|_{y_1 = \ldots = y_i = 0} \mu_{i+1}.$$

Arguing similarly, we see that (32) holds for any $k = i, i-1, \ldots, 1$.

Now we can replace

$$\frac{\partial v_m(0,\ldots,0,y_{i+1},\ldots,y_m)}{\partial y_{i+1}}\left(\frac{\lambda_k}{\mu_k}\mu_{i+1}\right)$$

with $\dfrac{\partial v_m}{\partial y_k}\lambda_k$, so that (31) transforms to equation

$$\frac{\partial v_m}{\partial y_1}\lambda_1+\ldots+\frac{\partial v_m}{\partial y_i}\lambda_i+\frac{\partial v_m}{\partial y_{i+1}}(\lambda_{i+1}-\mu_{i+1})$$
$$+\sum_{j=i+2}^{m}\frac{\partial v_m}{\partial y_j}\lambda_j=0$$

which is valid if $y_1=\ldots=y_i=0,\ y_{i+1}>0$.

The righthand equality in (13) is proved. ■

Proof of Lemma 2. Function ${}^nU(\cdot)$ coincides with the minimal non-negative solution to equation

$$\begin{aligned}&\left(\mu_{\Phi^*}+\sum_{j=1}^{m}\lambda_j\right)\ {}^nU(i_1,\ldots,i_m)=\frac{1}{n}\\&+\mu_{\Phi^*}\ {}^nU(i_1,\ldots,i_{\Phi^*}-1,\ldots,i_m)\\&+\sum_{j=1}^{m}\lambda_j\ {}^nU(i_1,\ldots,i_j+1,\ldots,i_m),\end{aligned}\tag{33}$$

where $\Phi^*=\Phi^*(i_1,\ldots,i_m)$.

Note that function on the righthand side of (16) is linear wrt $(i_1,\ldots,i_m)$ and coincides with $T_m\left(\frac{i_1}{n},\ldots,\frac{i_m}{n}\right)$. If we substitute it in the righthand side of (33), we obtain

$$\frac{1}{n}+\left(\mu_{\Phi^*}+\sum_{j=1}^{m}\lambda_j\right)T_m\left(\frac{i_1}{n},\ldots,\frac{i_m}{n}\right)$$
$$+\frac{1}{n}\left[\sum_{j=1}^{m}\frac{\partial T_m}{\partial y_j}f_j\left(\frac{i_1}{n},\ldots,\frac{i_m}{n},\Phi^*\right)\right].$$

Note that $f_j\left(\frac{i_1}{n},\ldots,\frac{i_m}{n},\Phi^*\right)=f_j(y,\varphi^*(y))$ for $y_1=\frac{i_1}{n},\ldots,y_m=\frac{i_m}{n}$. Hence, according to (30), the expression obtained coincides with the lefthand side of (33).

Equation (33) has other non-negative solutions, but all of them have the form ${}^nU(\cdot)+F(i_1,\ldots,i_m)$, where $F(\cdot)$ solves the homogeneous version of (33) and hence $F(\cdot)$ is either a (non-negative) constant, or grows exponentially with some of its arguments $i_1,\ldots,i_m$. ■

Proof of Theorem 2. According to the Dynkin formula

$${}^nW^{\Phi^*}(i_1,i_2,\ldots,y_m)$$
$$=E^{\Phi^*}_{(i_1,i_2,\ldots,i_m)}\left[\int_0^\infty G(Y_t,\Phi^*(Y_t/n))dt\right]$$
$$=v(i_1/n,i_2/n,\ldots,i_m/n)$$
$$+E^{\Phi^*}_{(i_1,i_2,\ldots,i_m)}\left[\int_0^\infty\left\{\sum_{j=1}^{m}C_j\frac{Y_{tj}}{n}\right.\right.$$
$$+\sum_{j=1}^{m}n\lambda_j v(Y_{t1}/n,Y_{t2}/n,\ldots,(Y_{tj}+1)/n,\ldots,Y_{tm}/n)$$
$$+n\mu_{\Phi^*(Y_t)}v(Y_{t1}/n,Y_{t2}/n,\ldots,(Y_{t\Phi^*(Y_t)}-1)/n,\ldots,Y_{tm}/n)$$
$$-\left[\sum_{j=1}^{m}n\lambda_j+n\mu_{\Phi^*(Y_t)}\right]$$
$$\left.\left.\times\ v(Y_{t1}/n,Y_{t2}/n,\ldots,Y_{tm}/n)\right\}I\{Y_t\neq 0\}dt\right].$$

Therefore,

$${}^nW^{\Phi^*}(i_1,i_2,\ldots,y_m)\le v(i_1/n,i_2/n,\ldots,i_m/n)$$
$$+E^{\Phi^*}_{(i_1,i_2,\ldots,i_m)}\left[\int_0^\infty I\{Y_t\neq 0\}dt\left\{g(Y_t/n)\right.\right.$$
$$\left.\left.+\sum_{j=1}^{m}\frac{\partial v}{\partial y_j}(Y_t/n)f_j(Y_t/n,\varphi^*(Y_t/n))\right\}\right]$$
$$+E^{\Phi^*}_{(i_1,i_2,\ldots,i_m)}\left[\int_0^\infty\frac{mD}{n}I\{Y_t\neq 0\}dt\right.$$
$$\left.\times\left(\max_{1\le j\le m}f_j(Y_t/n,\varphi^*(Y_t/n))\right)\right]$$
$$=v(i_1/n,i_2/n,\ldots,i_m/n)$$
$$+\frac{mD}{n}\left(\max_{1\le j\le m}\max\{\lambda_j,\mu_j-\lambda_j\}\right)\ {}^nU(i_1,i_2,\ldots,i_m).$$

Similarly,

$${}^nW^{\Phi^*}(i_1,i_2,\ldots,y_m)\ge v(i_1/n,i_2/n,\ldots,i_m/n)$$
$$-\frac{mD}{n}\left(\max_{1\le j\le m}\max\{\lambda_j,\mu_j-\lambda_j\}\right)\ {}^nU(i_1,i_2,\ldots,i_m).$$

■

Proof of Lemma 3. Suppose, control strategy Φ^* given by (15) is applied. Using (16), we see that expression in square brackets in (20) equals

$$\frac{1}{n}\left[\frac{\lambda_1}{\mu_1-\lambda_1}\prod_{l=2}^{m}\frac{\eta_l}{\eta_l-\lambda_l}+\frac{\lambda_2}{\eta_2-\lambda_2}\prod_{l=3}^{m}\frac{\eta_l}{\eta_l-\lambda_l}+\right.$$
$$\left.\ldots+\frac{\lambda_m}{\eta_m-\lambda_m}\right]=F_m.$$

Clearly, $F_1 = \frac{\lambda_1}{n(\mu_1-\lambda_1)} = \frac{\frac{\lambda_1}{\mu_1}}{n\left(1-\frac{\lambda_1}{\mu_1}\right)}$. Suppose

$$F_m = \frac{\sum_{j=1}^m \frac{\lambda_j}{\mu_j}}{n\left(1-\sum_{j=1}^m \frac{\lambda_j}{\mu_j}\right)}$$

and consider $m+1$:

$$F_{m+1} = F_m \frac{\eta_{m+1}}{\eta_{m+1}-\lambda_{m+1}} + \frac{\lambda_{m+1}}{n(\eta_{m+1}-\lambda_{m+1})} = \frac{1}{n}$$

$$\times\left[\frac{\sum_{j=1}^m \frac{\lambda_j}{\mu_j}}{1-\sum_{j=1}^m \frac{\lambda_j}{\mu_j}} \cdot \frac{\mu_{m+1}\left(1-\sum_{j=1}^m \frac{\lambda_j}{\mu_j}\right)}{\mu_{m+1}\left(1-\sum_{j=1}^m \frac{\lambda_j}{\mu_j}\right)-\lambda_{m+1}}\right.$$

$$\left.+\frac{\lambda_{m+1}}{\mu_{m+1}\left(1-\sum_{j=1}^m \frac{\lambda_j}{\mu_j}\right)-\lambda_{m+1}}\right]$$

$$= \frac{1}{n}\cdot\frac{\mu_{m+1}\sum_{j=1}^m \frac{\lambda_j}{\mu_j}+\lambda_{m+1}}{\mu_{m+1}\left(1-\sum_{j=1}^m \frac{\lambda_j}{\mu_j}\right)-\lambda_{m+1}}$$

$$= \frac{\sum_{j=1}^{m+1} \frac{\lambda_j}{\mu_j}}{n\left(1-\sum_{j=1}^{m+1} \frac{\lambda_j}{\mu_j}\right)}.$$

We have proved Lemma for control strategy Φ^*. Since (21) is independent of the order of the jobs types, it holds for any priorities system, i.e. for any stationary non-randomized strategy, and what is important, it does not depend on the strategy. Hence it is valid also for any stationary control strategy Φ. ■

Proof of Theorem 3. We already have the explicit formula for $tcu(z)$. It is well known that $nTCU(Z)$ can be obtained by solving the following equations wrt $nTCU(Z)$ and function V [36]:

$$\begin{aligned}
nTCU(Z) &= g(0)+n\lambda[V(Z)-V(0)];\\
nTCU(Z) &= g(1/n)+K\,n\mu(1/n)\\
&\quad +n\mu(1/n)[V(0)-V(1)];\\
nTCU(Z) &= g(2/n)+n\mu(2/n)[V(1)-V(2)];\\
\dots &= \dots;\\
nTCU(Z) &= g(Z/n)\\
&\quad +n\mu(Z/n)[V(Z-1)-V(Z)].
\end{aligned}$$

Now it is clear that

$$^nTCU(Z) = \frac{\sum_{i=1}^Z \frac{g(i/n)}{n\mu(i/n)}+K+\frac{g(0)}{n\lambda}}{\sum_{i=1}^Z \frac{1}{n\mu(i/n)}+\frac{1}{n\lambda}}.$$

Below, we regularly use the obvious inequality

$$\left|\frac{a}{b}-\frac{c}{d}\right| \le \frac{|a||d-b|+|b||a-c|}{|bd|}.$$

For instance, function $g(y)/\mu(y)$ is Lipschitz with constant $\frac{C_2C_4+C_1C_3}{\delta^2}$, function $1/\mu(y)$ is Lipschitz with constant $\frac{C_4}{\delta^2}$.

Now

$$|\,^nTCU([nz])-tcu(z)|$$

$$\le \frac{1}{\left[\sum_{i=1}^{[nz]}\frac{1}{n\mu(i/n)}+\frac{1}{n\lambda}\right]\int_0^z\frac{dy}{\mu(y)}}$$

$$\times\left\{\left|\sum_{i=1}^{[nz]}\frac{g(i/n)}{n\mu(i/n)}+K+\frac{g(0)}{n\lambda}\right|\right.$$

$$\times\left|\int_0^z\frac{dy}{\mu(y)}-\sum_{i=1}^{[nz]}\frac{1}{n\mu(i/n)}-\frac{1}{n\lambda}\right|$$

$$+\left(\sum_{i=1}^{[nz]}\frac{1}{n\mu(i/n)}+\frac{1}{n\lambda}\right)$$

$$\left.\left|\sum_{i=1}^{[nz]}\frac{g(i/n)}{n\mu(i/n)}+\frac{g(0)}{n\lambda}-\int_0^z\frac{g(y)dy}{\mu(y)}\right|\right\}$$

$$\le \frac{C_1^2}{(z-1/n)z}\left\{\left(\frac{zC_2}{\delta}+K+\frac{C_2}{n\lambda}\right)\right.$$

$$\left(\frac{zC_4}{\delta^2 n}+\frac{1}{\delta n}+\frac{1}{n\lambda}\right)$$

$$\left.+\frac{z}{\delta}\left(\frac{z(C_2C_4+C_1C_3)}{\delta^2 n}+\frac{C_2}{\delta n}+\frac{1}{n\lambda}\right)\right\}.$$

■

References:

[1] H.S. Ahn, I. Duenyas and R.Q. Zhang, Optimal control of a flexible server, *Adv. in Appl. Probab.* 36, 2004, pp. 139-170.

[2] K. Avrachenkov, U. Ayesta and A. Piunovskiy, Optimal choice of the buffer size in the Internet routers. *Proc. of CDC'05, 44-th IEEE conf. on Decision and Control. Spain*, 2005, pp.1143-1148.

[3] F. Avram, D. Bertsimas and M. Ricard, Fluid models of sequencing problems in open queueing networks; an optimal control approach. *Stochastic Networks*, IMA Vol. Math. Appl. 71, Springer, NY, 1995, pp.199-234.

[4] N. Bäuerle, Optimal control of queueing networks: an approach via fluid models. *Adv. Appl. Prob.* 34, 2002, pp. 313-328.

[5] O. Berman and D. Perry, An EOQ model with state-dependent demand rate, *Eur. J. of Oper. Res.* 171, 2006, pp. 255-272.

[6] P.G. Bradford and M.N. Katehakis, Constrained inventory allocation and its applications. *WSEAS Trans. Math.* 6, 2007, pp. 263-270.

[7] D. Clancy and A. Piunovskiy, An explicit optimal isolation policy for a deterministic epidemic model. *Applied Mathem. and Computation* 163, 2005, pp. 1109-1121.

[8] J.G. Dai, On positive Harris recurrence of multiclass queueing networks: a unified approach via fluid limit models, *Ann. Appl. Prob.* 5, 1995, pp. 49-77.

[9] J.G. Dai JG and S.P. Meyn, Stability and convergence of moments for multiclass queueing networks via fluid limit models. *IEEE Trans. on Autom. Control* 40, 1995, pp. 1889-1904.

[10] S.N. Ethier and T.G. Kurtz, *Markov Processes: Characterization and Convergence.* Wiley, NY 1986.

[11] S. Foss and A. Kovalevskii, A stability criterion via fluid limits and its application to a Polling system. *Queueing Systems* 32, 1999, pp. 131-168.

[12] A.S, Gajrat, A. Hordijk, V.A. Malyshev and F.M. Spieksma, Fluid approximations of Markov decision chains. *Markov Processes and Related Fields* 3, 1997, pp. 129-150.

[13] A.S. Gajrat and A. Hordijk, On the structure of the optimal server control for fluid networks, *Math. Meth. Oper. Res.* 62, 2005, pp. 55-75.

[14] A.S. Gajrat, A. Hordijk and A. Ridder, Large-deviations analysis of the fluid approximation for a controllable tandem queue, *The Annals of Appl. Probab.* 13, 2003, pp. 1423-1448.

[15] R. Groenevelt, G. Koole G and P. Nain, On the bias vector of a two-class preemptive priority queue. *Math. Meth. Oper. Res.* 55, 2002, pp. 107-120.

[16] X.P. Guo, O. Hernandez-Lerma and T. Prieto-Rumeau, A survey of recent results on continuous-time Markov decision processes. *TOP* 14, 2006, pp. 177-257.

[17] A. Hordijk and G. Koole, The μC-rule is not optimal in the second node of the tandem queue: a counterexample, *Adv. Appl. Prob.* 24, 1992, pp. 234-237.

[18] M.N. Katehakis and K.S. Puranam, On optimal replacement under semi-Markov conditions. *WSEAS Trans. Math.* 6, 2007, pp. 263-270.

[19] M.Yu. Kitaev and V.V. Rykov, *Controlled Queueing Systems.* CRC Press, Boca Raton 1995.

[20] G. Koole and P. Nain, An explicit solution for the value function of a priority queue. *Queueing Systems* 47, 2004, pp. 251-282.

[21] G. Koole and P. Nain, On the value function of a priority queue with an application to a controlled polling model. *Queueing Systems* 34, 2000, pp. 199-214.

[22] C. Maglaras, Revenue management for a multiclass single-server queue via a fluid model analysis. *Operations Research* 54, 2006, pp. 914-932.

[23] A. Mandelbaum and G. Pats, State-dependent queues: approximations and applications, *Stochastic Networks*, F.Kelly and R.Williams (eds), Proc. of the IMA. Springer, NY. 71, 1994, pp. 239-282.

[24] A. Mandelbaum, W.A. Massey and M.I. Reiman, Strong approximations for Markovian service networks, *Queueing Systems* 30, 1998, pp. 149-201.

[25] A. Mandelbaum and A.L. Stolyar, Scheduling flexible servers with convex delay costs: heavy-traffic optimality of the generalized $c\mu$-rule, *Oper. Res.* 52, 2004, pp. 836-855.

[26] D. Mitra, Stochastic theory of a fluid model of producers and consumers coupled by a buffer. *Adv. Appl. Prob.* 20, 1988, pp. 646-676.

[27] P. Nain and D. Towsley, Optimal scheduling in a machine with stochastic varying processing rate. *IEEE Trans. Aut. Control* 39, 1994, pp. 1853-1855.

[28] G. Pang and M.V. Day, Fluid limits of optimally controlled queueing networks, *J. of Applied Math. and Stoch. Anal.* 2007, Article ID 68958, 19 p, doi: 10.1155/2007/68958.

[29] A. Piunovskiy, A controlled jump discounted model with constraints, *Theory Probab. Appl.* 42, 1998, pp. 51-72.

[30] A. Piunovskiy, Multicriteria impulsive control of jump Markov processes. *Math. Meth. Oper. Res.* 60, 2004, pp. 125-144.

[31] A. Piunovskiy, Random walk, birth-and-death process and their fluid approximations: absorbing case, *Math. Meth. Oper. Res.* 2009 to appear, doi: 10.1007/s00186-008-0269-y.

[32] A. Piunovskiy and Y. Zhang, Accuracy of fluid approximation to controlled birth-and-death processes: absorbing case, *Math. Meth. Oper. Res.* (submitted).

[33] S. Podvalny, S. Titov, V. Burkovsky and S. Semynin, Optimal scheduling of queueing networks with switching times using genetic algorithms. *WSEAS Trans. Syst.* 5, 2006, pp. 1060-1065.

[34] Sh.M. Ross, *Introduction to Probability Models.* Academic Press, Amsterdam etc. 2003.

[35] R.D.H. Warburton, An exact analytical solution to the production inventory control problem. *Int. J. of Production Economics* 92, 2004, pp. 81-96.

[36] Q. Zhu and T. Prieto-Rumeau, Bias and overtaking optimality for continuous-time jump Markov decision processes in Polish spaces. *J. App. Prob.* 45, 2008, pp. 417-429.