Extended Static Output Feedback: An $\mathcal{H}_2$-$\mathcal{H}_\infty$ Control Setting

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Abstract: This contribution aims the problem about controller synthesis in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ for linear time invariant systems (LTI), by means of the extended static feedback of the output. The method consists of designing feedback gains for the injection of the output and its derivatives, which corresponds to the control signal. Conditions for the existence of such controllers are established. The stabilization problem is formulated in the context of linear matrix inequalities (LMI). Multiobjective Performance Indices are also considered

Key–Words: Static Output Feedback, Robust Control, LMI, Polytopical Uncertainty.

1 Introduction

The main goal of all control systems design is its practical implantation. When it is treated the control systems synthesis for the practical application, always it is looked for the greater flexibility and simplifying for implementation, [15]. From which the output feedback control has been a topic of much interest of investigation.

As it is known, one of the controllers who offer those characteristics are the constructed from the state feedback, which have the disadvantage that not always it is possible to have all the states. On the other hand, the static output feedback (SOF) allows all the kindness for an implementation without many exigencies. The major inconvenient for the static output feedback control synthesis is their conditions of existence of such controllers [18]. The SOF problem is referred as: given a LTI system, it is desired to find an output feedback static gain that in closed loop the controlled system exhibits particular behavior characteristics; or in its defect, to determine the nonexistence of such gain [5]. Although several theoretical conditions for the existence of SOF controllers are known, there exists the disadvantage of practical algorithms for the solution without majors' requirements [16].

Notations: In what follows, $\mathbb{I}$ is an identity matrix with an appropriate dimension, M^T denotes the transpose of the matrix M, $M > 0$ means that the matrix M is positive definite, $M < 0$ means that the matrix M is negative definite. In the partitions of symmetrical matrices $\star$ denotes each of its symmetrical blocks.

1.1 Problem Formulation

Consider the LTI system defined by

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad y(t) = Cx(t) \quad (1)$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, and $C \in \Re^{p\times n}$. The problem is described as: given the system (1), with (A, B) stabilizable, it is desired to find a control of the form

$$u(t) = \mathcal{K}y(t) \quad (2)$$

where $\mathcal{K} \in \Re^{m\times p}$ is the static feedback gain for constructing, in such a way that the system in closed loop is stable. This means that if it exists $\mathcal{K}$, then the dynamic matrix of the closed loop $A + B\mathcal{K}C$ must be stable.

Problem 1: *Given the system (1), with (A, B) stabilizable. It is desired to find $\mathcal{K}$ for the control (2) in such a way that the matrix $A + B\mathcal{K}C$ has all eigenvalues in the stable semiplane*

The choice of the gain $\mathcal{K}$ must allow satisfying the performance requirements according to the control system design objectives. The first detail to define is the existence of such static feedback gain. For it several results have been presented: [4, 5, 18, 3, 8]. The established conditions do not indicate directly, the solution algorithms. Thus, several methods for the algorithmic solution of the SOF problem have appeared: [2, 20, 7, 10]. All these solutions continue maintaining the problem of exigent requirements from the computational point of view, limiting the practical application of the techniques of SOF in industrial processes.

In order to maintain the idea of the SOF problem, we consider a numerical example from the following dynamic model:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t); \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) \end{aligned} \tag{3}$$

According to Problem 1, given the structural conditions of the model, it is necessary to find $\mathcal{K}$ to stabilize the system in closed loop. For this particular case, it is not possible to stabilize this system by SOF.

1.2 Preliminaries

In this section some preliminary facts are given, in order to determine additional performance conditions for linear systems. Consider the following continuous time linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{4}$$

where, $x(t) \in \Re^n$ are states, $u(t) \in \Re^m$ are control signals and $y(t) \in \Re^p$ are measurements. Matrices A, B, C, D are well known and with proper dimensions.

The $\mathcal{H}_2$-$\mathcal{H}_\infty$ norms for that system can be described as LMIs. It is known that more conservative results are obtained (for example, in the case of polytopic uncertainty), when there exist relationships between system matrix and the Lyapunov matrix. For to solve that problem, a decoupling of the Lyapunov matrix and the system dynamic matrix must be obtained. Thus, there exist some modifications to classical results of robust control theory like improved versions to Bounded Real Lemma [17, 9], or for $\mathcal{H}_2$ performance [1]. Next, some of these approaches are presented as a basis for developments which will be presented later in this paper.

Lemma 1.1 (Relaxed $\mathcal{H}_2$ performance) *Consider system* (4) *with $D = 0$. The following statements, with $P = P^T > 0$ are equivalent*

i) A is stable and $\left\|C(s\mathbb{I} - A)^{-1}B\right\|_2^2 < \mu$.

ii) There exist P and Z, such that

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & -\mu\mathbb{I} \end{bmatrix} < 0,$$
$$\begin{bmatrix} P & C^T \\ C & Z \end{bmatrix} > 0, \qquad \mathrm{tr}(Z) < 1 \tag{5}$$

iii) There exist P, Z and G such that

$$\begin{bmatrix} -(G+G^T) & G^TA + P & G^TB & G^T \\ A^TG + P & -P & 0 & 0 \\ B^TG & 0 & -\mu\mathbb{I} & 0 \\ G & 0 & 0 & -P \end{bmatrix} < 0,$$
$$\begin{bmatrix} P & C^T \\ C & Z \end{bmatrix} > 0, \qquad \mathrm{tr}(Z) < 1 \tag{6}$$

iv) There exist P, Z and G, such that

$$\begin{bmatrix} -(G+G^T) & G^TA + P + G^T & G^TB \\ A^TG + P + G & -2P & 0 \\ B^TG & 0 & -\mu\mathbb{I} \end{bmatrix} < 0,$$
$$\begin{bmatrix} P & C^T \\ C & Z \end{bmatrix} > 0, \qquad \mathrm{tr}(Z) < 1 \tag{7}$$

Proof
The equivalence between the three first statements has been shown in theorem 3.3 of [1], it is based on projection lemma and its reciprocal version. The equivalence between 2 and 4 is shown in [19]. ■

Remark 1.1 *It is known that more conservative results are obtained (for example, in the case of polytopic uncertainty), when there exist relationships between system matrix and the Lyapunov matrix [12]. This result solves that problem, decoupling the Lyapunov matrix and the system dynamic matrix. Additionally, statement 4 in Lemma 1.1, provides a smaller representation of the $\mathcal{H}_2$ performance condition given by [1]. Further details on this lemma can be found in [19].*

In the same fashion as in the case of $\mathcal{H}_2$ performance condition given before, there are some attempts for the improvement of $\mathcal{H}_\infty$ performance, next one of them is shown.

Lemma 1.2 (Relaxed $\mathcal{H}_\infty$ performance) *Consider system* (4). *The following statements, with $P = P^T > 0$ and matrix G are equivalent*

i) A is stable and $\left\| C(s\mathbb{I} - A)^{-1}B + D \right\|_\infty < \gamma$.

ii) There exist P, such that

$$\begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma^2\mathbb{I} & D^T \\ C & D & -\mathbb{I} \end{bmatrix} < 0. \tag{8}$$

iii) There exist P and G such that, for $\tau \gg 1$

$$\begin{bmatrix} -(G+G^T) & G^TA + P + \tau G^T & 0 & G^TB \\ A^TG + P + \tau G & -2\tau P & C^T & 0 \\ 0 & C & -\mathbb{I} & D \\ B^TG & 0 & D^T & -\gamma^2\mathbb{I} \end{bmatrix} < 0. \tag{9}$$

Proof
Conditions 1 and 2 are the well known *Bounded Real Lemma*. Equivalence between 2 and 3 can be seen in [19]. ■

2 Static feedback control of the Extended Output

Consider the system (1). An extended output is defined by

$$\begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} Cx(t) \\ CAx(t) + CBu(t) \end{pmatrix} \tag{10}$$

Thus, the control law by extended SOF is:

$$u(t) = \begin{pmatrix} \mathcal{K}_0 & \mathcal{K}_1 \end{pmatrix} \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} \tag{11}$$

where $\mathcal{K}_0$ and $\mathcal{K}_1$ are the feedback gains, to determine, for the output and its derivatives. In this case, the derivative the output is used in the same context of the derivative action in the PID-type controller. Thus, the control will be given by

$$u(t) = (\mathbb{I} - \mathcal{K}_1CB)^{-1} (\mathcal{K}_0C + \mathcal{K}_1CA) x(t), \tag{12}$$

where $\mathbb{I}$ is the identity matrix. As it is possible to be observed, the existence of the control depends on the invertibility of the matrix $\mathbb{I} - \mathcal{K}_1CB$, which is a condition “less hard” than the established in the Problem 1 of SOF. Thus, for the control by static feedback of the extended output the following problem can be established:

Problem 2: *Given the system (1), with (A, B) stabilizable. It is desired to find $\mathcal{K}_0$ y $\mathcal{K}_1$ for the control (12) in such a way that the matrix $A + B(\mathbb{I} - \mathcal{K}_1CB)^{-1}(\mathcal{K}_0C + \mathcal{K}_1CA)$ has all eigenvalues in the stable semi-plane.*

Lemma 2.1 *Be $\mathbb{M} = \mathbb{I} - \mathcal{K}_1CB$. There exists a control by static feedback of the extended output of the form*

$$u(t) = \mathbb{M}^{\ddagger} (\mathcal{K}_0C + \mathcal{K}_1CA) x(t), \tag{13}$$

if and only if $\mathbb{M}$ has generalized inverse (pseudoinverse Moore-Penrose), given by $\mathbb{M}^{\ddagger}$.

Indeed, if $\mathbb{M}^T$ is the transpose of $\mathbb{M}$ matrix, which is assumed with complete rank by columns, then the pseudoinverse Moore-Penrose matrix is $\mathbb{M}^{\ddagger} = \left(\mathbb{M}^T\mathbb{M}\right)^{-1}\mathbb{M}^T$, which, if it exists, allows to calculate the control law. This condition debilitates and generalizes the established ones in [11], allowing also to obtain a solution to the SOF problem.

2.1 Example

In order to verify the effectiveness of the technique, it will be considered the dynamic model given in (3). There, it is possible to stabilize the system in closed loop by generalized SOF since $\mathbb{I} - \mathcal{K}_1CB$ has inverse, then

$$u(t) = \begin{pmatrix} k_0 + k_1 & k_1 \end{pmatrix} x(t),$$

and the dynamic matrix of closed loop is

$$A_c = \begin{pmatrix} 0 & 1 \\ 2 + k_0 + k_1 & 3 + k_1 \end{pmatrix}$$

in such a way that with an appropriate selection of k_0 and k_1 this system becomes stabilized. Unlike the basic SOF (Problem 1), in this case the stabilization can be obtained only using the extended output, that corresponds to the derivative output.

3 LMI Formulation

Consider the problem of stabilization by feedback of the extended output in the context of LMIs. That is, the system (1), to find $\mathcal{K}_0$ and $\mathcal{K}_1$ in such a way that the system in closed loop is stable in the sense of Lyapunov, which signifies that the following BMI (bilinear matrix inequalities) are satisfied:

$$\mathcal{A}_c^T \mathbb{P} + \mathbb{P}\mathcal{A}_c \prec 0, \qquad \mathbb{P} \succ 0; \tag{14}$$

where A_c is the dynamic matrix of closed loop.

Theorem 3.1 *Lets consider the system given by (1) with the pair (A, B) stablizable. A control by SOF extended of the form (13) exists, that stabilizes the system in closed loop, if there exists $\mathbb{M}^{\ddagger}$ and the symmetrical matrix $P \succ 0$, and the matrices Y, Z such that the following LMI is satisfied*

$$PA^T + AP + BY + BZ + Y^T B^T + Z^T B^T \prec 0 \tag{15}$$

for which, the feedback gains are obtained from

$$\begin{aligned} \mathcal{K}_0 &= \mathbb{V}^{\ddagger} Y P^{-1} C^T \left(CC^T\right)^{-1} & (16)\\ \mathcal{K}_1 &= \mathbb{V}^{\ddagger} Z P^{-1} A^T C^T \left(CAA^T C^T\right)^{-1} & (17)\end{aligned}$$

where

$$\mathbb{V} = \mathbb{I} + ZP^{-1}A^T C^T \left(CAA^T C^T\right)^{-1} B.$$

Proof
Indeed, if $u(t)$ is of the form presented in (13), then the dynamic matrix of the closed loop corresponds to

$$A_c = A + B\mathbb{M}^{\ddagger}\left(\mathcal{K}_0 C + \mathcal{K}_1 CA\right),$$

then from the expression (14), with $P = \mathbb{P}^{-1}$, the following matrix inequality is obtained

$$\begin{aligned} &PA^T + AP + PC^T\mathcal{K}_0^T(\mathbb{M}^{\ddagger})^T B^T +\\ &PA^T C^T \mathcal{K}_1^T (\mathbb{M}^{\ddagger})^T B^T + B\mathbb{M}^{\ddagger}\mathcal{K}_0 CP +\\ &B\mathbb{M}^{\ddagger}\mathcal{K}_1 CAP \prec 0.\end{aligned}$$

If

$$\begin{aligned} Y &= \mathbb{M}^{\ddagger}\mathcal{K}_0 CP\\ Z &= \mathbb{M}^{\ddagger}\mathcal{K}_1 CAP,\end{aligned}$$

then

$$PA^T + AP + (Y^T + Z^T)B^T + B(Y + Z) \prec 0,$$

that it is expressed in the LMI (15). Since

$$\begin{aligned} \mathbb{M}^{\ddagger}\mathcal{K}_0 &= YP^{-1}C^T(CC^T)^{-1}\\ \mathbb{M}^{\ddagger}\mathcal{K}_1 &= ZP^{-1}A^T C^T (CAA^T C^T)^{-1},\end{aligned}$$

and $\mathbb{M} = \mathbb{I} - \mathcal{K}_1 CB$ then

$$\mathbb{M}^{\ddagger} = \mathbb{I} + ZP^{-1}A^T C^T (CAA^T C^T)^{-1} B = \mathbb{V}$$

therefore $\mathcal{K}_0$ and $\mathcal{K}_1$ are obtained from equations (16) and (17), respectively. ■

Remark 3.1 *It is important to indicate that the conditions for calculating $\mathcal{K}_0$ and $\mathcal{K}_1$ are equivalent to the established in [11], but in this case they are less restrictive for the types of systems since that the stabilization can be obtained, by means of the appropriated calculation of only $\mathcal{K}_1$, whose design condition enlarges the class of systems, still more when are combined with the synthesis of $\mathcal{K}_0$.*

3.1 $\mathcal{H}_2$ Control

Consider the LTI system defined by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) + B_\omega \omega(t)\\ y(t) &= Cx(t)\end{aligned} \tag{18}$$

where $\omega(t)$ is a unknown disturbance.

Problem 3: *Given the system (18), with (A, B_u) stabilizable. Is is desired to find $\mathcal{K}_0$ and $\mathcal{K}_1$ for the control (12) in such a way that the closed loop dynamic matrix $\mathcal{A}_c$ is stable and that the norm-2 of the transfer function from the disturbance to output is minimum, that is $\parallel H_{y\omega}(s) \parallel_2 < \gamma$.*

Applying the control given by (12), then the system in closed loop will be

$$\begin{array}{rcl} \dot{x}(t) & = & \mathcal{A}_c x(t) + \mathcal{B}_c \omega(t) \\ y(t) & = & \mathcal{C}_c x(t) \end{array} \tag{19}$$

$$\begin{array}{rcl} \mathcal{A}_c & = & A + B_u \mathbb{M}^{\ddagger} \left(\mathcal{K}_0 C + \mathcal{K}_1 C A \right) \\ \mathcal{B}_c & = & B_\omega + B_u \mathbb{M}^{\ddagger} \mathcal{K}_1 C B_\omega, \qquad \mathcal{C}_c = C. \end{array}$$

Thus, $H_{y\omega}(s)$ corresponds to

$$H_{y\omega}(s) = \left[\begin{array}{c|c} \mathcal{A}_c & \mathcal{B}_c \\ \hline \mathcal{C}_c & 0 \end{array} \right] = \mathcal{C}_c (s\mathbb{I} - \mathcal{A}_c)^{-1} \mathcal{B}_c \tag{20}$$

Formulating the norm-2 as LMI, the following result can be obtained [6]:

Lemma 3.1 *Inequality* $\| H_{y\omega}(s) \|_2 < \gamma$ *is fulfilled if, and only if, there exists a symmetrical matrix* $\mathbb{X}$, $\mathbb{X} \succ 0$, *and a matrix* $\mathbb{Z}$ *such that*

$$\begin{bmatrix} \mathcal{A}_c^T \mathbb{X} + \mathbb{X} \mathcal{A}_c & \mathbb{X} \mathcal{B}_c \\ (\circ)^T & -\gamma \mathbb{I} \end{bmatrix} < 0.$$

$$\begin{bmatrix} \mathbb{X} & \mathcal{C}_c^T \\ (\circ)^T & \mathbb{Z} \end{bmatrix} > 0, \qquad tr(\mathbb{Z}) > 1. \tag{21}$$

Theorem 3.2 *Let's consider the system given by (18) with pair* (A, B_u) *stablizable. A control by SOF extended of the form (13) exists, that stabilizes the system in closed loop and* $\| H_{y\omega}(s) \|_2 < \gamma$, *if there exists* $\mathbb{M}^{\ddagger}$ *and the symmetrical matrix* $P \succ 0$, *the matrix* $\mathbb{Z}$, *and the matrices* X, Y, Z *such that the following LMIs are satisfied*

$$\begin{bmatrix} PA^T + AP + Y^T B_u^T + Z^T B_u^T + B_u Y + B_u Z & B_\omega + X B_\omega \\ B_\omega^T + B_\omega^T X^T & -\gamma \mathbb{I} \end{bmatrix} \prec 0 \tag{22}$$

$$\begin{bmatrix} P & PC^T \\ CP & \mathbb{Z} \end{bmatrix} > 0, \qquad tr(\mathbb{Z}) > 1. \tag{23}$$

for which, the feedback gains are obtained from

$$\mathcal{K}_0 = \mathbb{V}^{\ddagger} Y P^{-1} C^T \left(CC^T \right)^{-1} \tag{24}$$

$$\mathcal{K}_1 = \mathbb{V}^{\ddagger} Z P^{-1} A^T C^T \left(CAA^T C^T \right)^{-1} \tag{25}$$

where

$$\mathbb{V} = \mathbb{I} + XCB_u.$$

Proof

The demonstration is based on the application of the norm-∞ as LMI, from which a congruent transformation is used

$$\begin{pmatrix} \mathbb{X}^{-1} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$$

with $P = \mathbb{X}^{-1}$; and the following change of variables [13]:

$$\begin{array}{rcl} X & = & \mathbb{M}^{\ddagger} \mathcal{K}_1 \\ Y & = & \mathbb{M}^{\ddagger} \mathcal{K}_0 C P \\ Z & = & \mathbb{M}^{\ddagger} \mathcal{K}_1 C A P. \end{array}$$

Since $\mathbb{M} = \mathbb{I} - \mathcal{K}_1 C B_u$, is deduced that

$$\begin{array}{rcl} \mathbb{M}^{\ddagger} & = & \mathbb{I} + \mathbb{M}^{\ddagger} \mathcal{K}_1 C B_u \\ & = & \mathbb{I} + XCB_u = \mathbb{V}. \end{array}$$

Then, $\mathcal{K}_0$ and $\mathcal{K}_1$ are obtained from the expressions for Y and Z. ■

In this case, the conditions are similar to the presented in the solution of Problem 2 via LMI. On the other hand, the systems to control by SOF are extended.

3.2 Relaxed $\mathcal{H}_2$ Control

From Lemma 1.1, the following result can be obtained

Theorem 3.3 *Let's consider the system given by (18) with pair* (A, B_u) *stablizable. A control by SOF extended of the form (13) exists, that stabilizes the system in closed loop and* $\| H_{y\omega}(s) \|_2 < \gamma$, *if there exists* $\mathbb{M}^{\ddagger}$ *and the symmetrical matrix* $P \succ 0$, *the matrix* $\mathbb{Z}$ *and the matrices* Y, Z *such that the following LMIs are satisfied*

$$\begin{bmatrix} -(G + G^T) & G^T A + YC + ZCA + P + G^T & G^T B_\omega + ZCB_\omega \\ AG + C^T Y^T + A^T C^T Z^T + P + G & -2P & 0 \\ B_\omega^T G + B_\omega^T C^T Z^T & 0 & -\gamma \mathbb{I} \end{bmatrix} \prec 0 \tag{26}$$

$$\begin{bmatrix} P & PC^T \\ CP & \mathbb{Z} \end{bmatrix} > 0, \qquad \mathrm{tr}\,(\mathbb{Z}) > 1. \qquad (27)$$

for which, the feedback gains are obtained from

$$\mathcal{K}_0 = \mathbb{V}^{\ddagger} B_u^{\ddagger} (G^T)^{-1} Y \qquad (28)$$

$$\mathcal{K}_1 = \mathbb{V}^{\ddagger} B_u^{\ddagger} (G^T)^{-1} Z \qquad (29)$$

where

$$\mathbb{V} = \mathbb{M}^{\ddagger} = \mathbb{I} + B_u^{\ddagger} (G^T)^{-1} Z C B_u.$$

Proof
The demonstration is analogous to the previous case. ■

3.3 $\mathcal{H}_\infty$ Control

Consider the LTI system defined by (18). Then, the following control problem can be stablished:

Problem 4: *Given the system (18), with (A, B_u) stabilizable. Is is desired to find $\mathcal{K}_0$ and $\mathcal{K}_1$ for the control (12) in such a way that the closed loop dynamic matrix $\mathcal{A}_c$ is stable and that the norm-∞ of the transfer function from the disturbance to output is minimum, that is $\| H_{y\omega}(s) \|_\infty < \mu$.*

Applying the control given by (12), then the system in closed loop will be

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}_c x(t) + \mathcal{B}_c \omega(t) \\ y(t) &= \mathcal{C}_c x(t) \end{aligned} \qquad (30)$$

$$\begin{aligned} \mathcal{A}_c &= A + B_u \mathbb{M}^{\ddagger} (\mathcal{K}_0 C + \mathcal{K}_1 C A) \\ \mathcal{B}_c &= B_\omega + B_u \mathbb{M}^{\ddagger} \mathcal{K}_1 C B_\omega, \qquad \mathcal{C}_c = C. \end{aligned}$$

Thus, $H_{y\omega}(s)$ corresponds to

$$H_{y\omega}(s) = \left[\begin{array}{c|c} \mathcal{A}_c & \mathcal{B}_c \\ \hline \mathcal{C}_c & 0 \end{array} \right] = \mathcal{C}_c (s\mathbb{I} - \mathcal{A}_c)^{-1} \mathcal{B}_c \quad (31)$$

Formulating the norm-∞ as LMI, the following result can be obtained [6]:

Lemma 3.2 *Inequality $\| H_{y\omega}(s) \|_\infty < \mu$ is fulfilled if, and only if a symmetrical matrix $\mathbb{X}$, $\mathbb{X} \succ 0$, exists such that*

$$\begin{bmatrix} \mathcal{A}_c^T \mathbb{X} + \mathbb{X} \mathcal{A}_c & \mathbb{X} \mathcal{B}_c & \mathcal{C}_c^T \\ (\circ)^T & -\mu \mathbb{I} & 0 \\ (\circ)^T & (\circ)^T & -\mu \mathbb{I} \end{bmatrix} < 0. \qquad (32)$$

Theorem 3.4 *Let's consider the system given by (18) with pair (A, B_u) stablizable. A control by SOF extended of the form (13) exists, that stabilizes the system in closed loop and $\| H_{y\omega}(s) \|_\infty < \mu$, if there exists $\mathbb{M}^{\ddagger}$ and the symmetrical matrix $P \succ 0$, and the matrices X, Y, Z such that the following LMI is satisfied*

$$\begin{bmatrix} PA^T + AP + Y^T B_u^T + Z^T B_u^T + B_u Y + B_u Z & B_\omega + X B_\omega & PC^T \\ B_\omega^T + B_\omega^T X^T & -\mu \mathbb{I} & 0 \\ CP & 0 & -\mu \mathbb{I} \end{bmatrix} \prec 0 \qquad (33)$$

for which, the feedback gains are obtained from

$$\mathcal{K}_0 = \mathbb{V}^{\ddagger} Y P^{-1} C^T \left(C C^T \right)^{-1} \qquad (34)$$

$$\mathcal{K}_1 = \mathbb{V}^{\ddagger} Z P^{-1} A^T C^T \left(C A A^T C^T \right)^{-1} \qquad (35)$$

where

$$\mathbb{V} = \mathbb{I} + X C B_u.$$

Proof
The demonstration is based on the application of the norm-∞ as LMI, from which a congruent transformation is used

$$\begin{pmatrix} \mathbb{X}^{-1} & 0 & 0 \\ 0 & \mathbb{I} & 0 \\ 0 & 0 & \mathbb{I} \end{pmatrix}$$

with $P = \mathbb{X}^{-1}$; and the following change of variables [13]:

$$\begin{aligned} X &= \mathbb{M}^{\ddagger} \mathcal{K}_1 \\ Y &= \mathbb{M}^{\ddagger} \mathcal{K}_0 C P \\ Z &= \mathbb{M}^{\ddagger} \mathcal{K}_1 C A P. \end{aligned}$$

Since $\mathbb{M} = \mathbb{I} - \mathcal{K}_1 C B_u$, is deduced that

$$\begin{aligned} \mathbb{M}^{\ddagger} &= \mathbb{I} + \mathbb{M}^{\ddagger} \mathcal{K}_1 C B_u \\ &= \mathbb{I} + X C B_u = \mathbb{V}. \end{aligned}$$

Then, $\mathcal{K}_0$ and $\mathcal{K}_1$ are obtained from the expressions for Y and Z. ■

In this case, the conditions are similar to the presented in the solution of Problem 2 via LMI. On the other hand, the systems to control by SOF are extended.

3.4 Relaxed $\mathcal{H}_\infty$ Control

From Lemma 1.2, the following result can be obtained

Theorem 3.5 *Let's consider the system given by (18) with pair (A, B_u) stablizable. A control by SOF extended of the form (13) exists, that stabilizes the system in closed loop and $\| H_{y\omega}(s) \|_\infty < \mu$, if there exists $\mathbb{M}^\ddagger$, $\tau >> 1$, and the symmetrical matrix $P \succ 0$, and the matrices Y, Z such that the following LMI is satisfied*

$$\begin{bmatrix} -G - G^T & G^T A + YC + ZCA + P + \tau G^T & 0 & G^T B_\omega + ZCB_\omega \\ \star & -2\tau P & C^T & 0 \\ \star & \star & -\mathbb{I} & 0 \\ \star & \star & \star & -\mu^2 \mathbb{I} \end{bmatrix} \prec 0 \qquad (36)$$

for which, the feedback gains are obtained from

$$\mathcal{K}_0 = \mathbb{V}^\ddagger B_u^\ddagger (G^T)^{-1} Y \qquad (37)$$
$$\mathcal{K}_1 = \mathbb{V}^\ddagger B_u^\ddagger (G^T)^{-1} Z \qquad (38)$$

where

$$\mathbb{V} = \mathbb{M}^\ddagger = \mathbb{I} + B_u^\ddagger (G^T)^{-1} ZCB_u.$$

Proof
The demonstration is analogous to the previous case. ■

From the results for $\mathcal{H}_2$ and $\mathcal{H}_\infty$, a control law that allows to satisfier multi-objective indices can be obtained:

Corollary 4 *Let's consider the system given by (18) with pair (A, B_u) stablizable. A control by SOF extended of the form (13) exists, that stabilizes the system in closed loop with $\| H_{y\omega}(s) \|_2 < \gamma$ and $\| H_{y\omega}(s) \|_\infty < \mu$, if there exists $\mathbb{M}^\ddagger$ and the symmetrical matrix $P \succ 0$, the matrix $\mathbb{Z}$, and the matrices X, Y, Z such that the following LMIs are satisfied*

$$\begin{bmatrix} PA^T + AP + Y^T B_u^T + Z^T B_u^T + B_u Y + B_u Z & B_\omega + XB_\omega \\ B_\omega^T + B_\omega^T X^T & -\gamma \mathbb{I} \end{bmatrix} \prec 0 \qquad (39)$$

$$\begin{bmatrix} P & PC^T \\ CP & \mathbb{Z} \end{bmatrix} > 0, \qquad tr(\mathbb{Z}) > 1, \qquad (40)$$

and

$$\begin{bmatrix} PA^T + AP + Y^T B_u^T + Z^T B_u^T + B_u Y + B_u Z & B_\omega + XB_\omega & PC^T \\ B_\omega^T + B_\omega^T X^T & -\mu \mathbb{I} & 0 \\ CP & 0 & -\mu \mathbb{I} \end{bmatrix} \prec 0 \qquad (41)$$

for which, the feedback gains are obtained from

$$\mathcal{K}_0 = \mathbb{V}^\ddagger Y P^{-1} C^T \left(CC^T\right)^{-1} \qquad (42)$$
$$\mathcal{K}_1 = \mathbb{V}^\ddagger Z P^{-1} A^T C^T \left(CAA^T C^T\right)^{-1} \qquad (43)$$

where

$$\mathbb{V} = \mathbb{I} + XCB_u.$$

Of equal way, in order to reduce the conservatism, an extended version of the results in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ can be presented, in order to satisfy multi-objective performance indices. Thus, the following result can be established:

Corollary 5 *Let's consider the system given by (18) with pair (A, B_u) stablizable. A control by SOF extended of the form (13) exists, that stabilizes the system in closed loop with $\| H_{y\omega}(s) \|_2 < \gamma$ and $\| H_{y\omega}(s) \|_\infty < \mu$, if there exists $\mathbb{M}^\ddagger$ and the symmetrical matrix $P \succ 0$, the matrix $\mathbb{Z}$, and the matrices X, Y, Z such that the following LMIs are satisfied*

$$\begin{bmatrix} -(G + G^T) & G^T A + YC + ZCA + P + G^T & G^T B_\omega + ZCB_\omega \\ AG + C^T Y^T + A^T C^T Z^T + P + G & -2P & 0 \\ B_\omega^T G + B_\omega^T C^T Z^T & 0 & -\gamma \mathbb{I} \end{bmatrix} \prec 0 \qquad (44)$$

$$\begin{bmatrix} P & PC^T \\ CP & \mathbb{Z} \end{bmatrix} > 0, \qquad tr(\mathbb{Z}) > 1. \qquad (45)$$

and

$$\begin{bmatrix} -G-G^T & G^TA+YC+ZCA+P+\tau G^T & 0 & G^TB_\omega+ZCB_\omega \\ \star & -2\tau P & C^T & 0 \\ \star & \star & -\mathbb{I} & 0 \\ \star & \star & \star & -\mu^2\mathbb{I} \end{bmatrix} \prec 0 \qquad (46)$$

for which, the feedback gains are obtained from

$$\mathcal{K}_0 = \mathbb{V}^\ddagger B_u^\ddagger (G^T)^{-1} Y \qquad (47)$$
$$\mathcal{K}_1 = \mathbb{V}^\ddagger B_u^\ddagger (G^T)^{-1} Z \qquad (48)$$

where

$$\mathbb{V} = \mathbb{M}^\ddagger = \mathbb{I} + B_u^\ddagger (G^T)^{-1} ZCB_u.$$

Remark 5.1 *By other hand, it is possible to observe that there are conditions to obtain a total disturbance rejection. Indeed, if $\mathcal{K}_1$ is selected in such a way that*

$$(\mathbb{I} + B_u \mathbb{M}^\ddagger \mathcal{K}_1 C) \in \ker(B_\omega),$$

then the disturbance is rejected. Therefore, once selected $\mathcal{K}_1$, is possible to design$\mathcal{K}_0$ in such a way that the dynamic matrix of closed loop is stable. In addition, it is possible to design those gains in order to satisfy multi-objective indices, [14]

On the other hand, it is possible to obtain tracking or regulation of the output, again with a suitable selection of $\mathcal{K}_0$, $\mathcal{K}_1$ and one gain on the signal of reference to assure a null error between the output and that signal to track.

6 Numerical evaluation

We consider the system given by [10]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega(t)$$
$$y(t) = \begin{bmatrix} 0 & 5 & -1 \\ -1 & -1 & 0 \end{bmatrix} x(t)$$

Applying the results of the Theorem 3.4 then:

$$\mathcal{K}_0 = [-3.1825 \quad 4.5089],$$
$$\mathcal{K}_1 = [0.8959 \quad 10.6448]$$

thus, the dynamic matrix of closed loop is

$$\mathcal{A}_c = \; = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3.8786 & -23.7415 & -2.5659 \end{bmatrix},$$

whose eigenvalues are

$$-0.1662; \quad -1.1999 \pm 4.6801\jmath.$$

The value μ obtained of the minimization of the norm-∞ for the transfer function $H_{y\omega}(s)$ has been < 5. This way the stabilization of the system in closed loop with an attenuation of the disturbance is demonstrated.

For the simulation the diagram of the Figure 1 has been used.

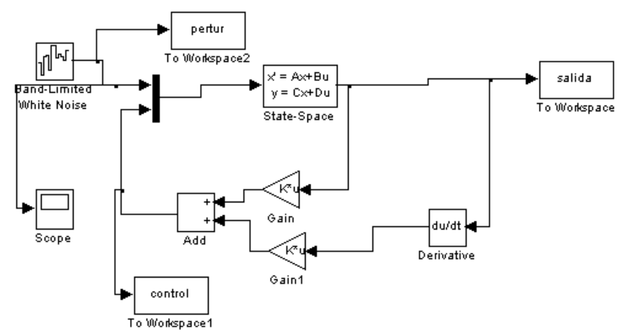

Figure 1: Diagram for simulation.

The results are showed in the Figure 2 and Figure 3, which corresponds to the control and the controlled output. Like it can be noticed, the outputs are stabilized around the zero, only affected by the presence of the disturbance, which is in the Figure 4.

7 Conclusions

From the output and its derivatives, the problem of static output feedback control has been considered and solved, using both signals. The solution allows extending the space of the systems that can be stabilized by the traditional techniques of

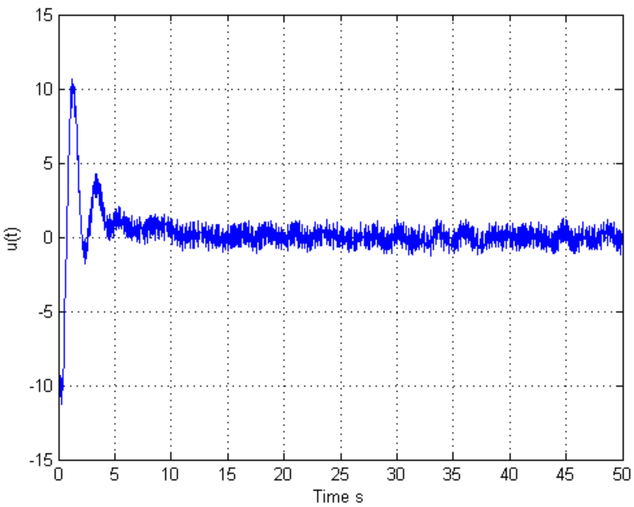

Figure 2: The control signal.

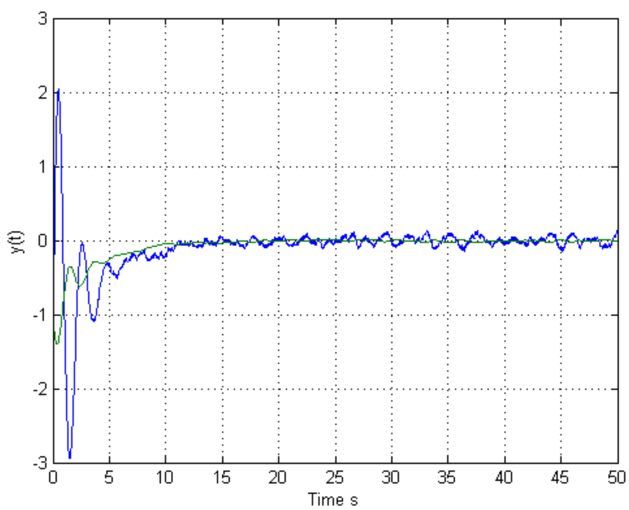

Figure 3: The controlled outputs.

static feedback of the output. The use of the output derivatives is the same context of the derivative action in controller PID.

The conditions for the problem solution have been studied, same that for the control in $\mathcal{H}_2$ and $\mathcal{H}_\infty$, using the description of those norms as linear matrix inequalities. The conditions are analogous to the problems of static feedback of the output already known. Multiobjective performance indices have also been considered, where LMIs extended have been considered, in order to reduce the conservatism.

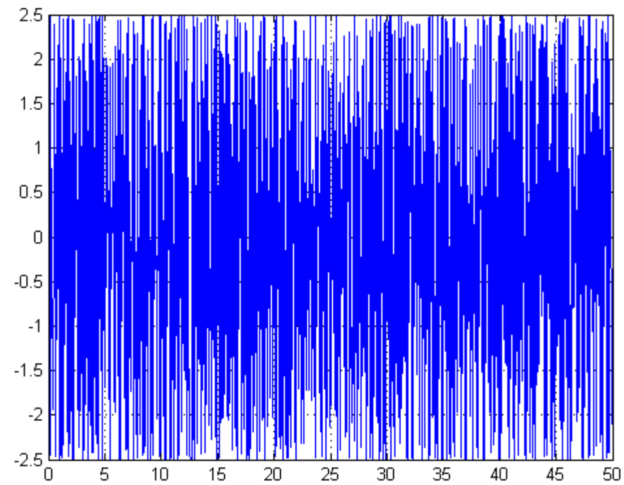

Figure 4: The disturbance signal.

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