Robust Performance Characterization of PID Controllers in the Frequency Domain

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Abstract: - In this paper a method is introduced for finding all proportional integral derivative (PID) controllers that satisfy a robust performance constraint for a given single-input-single-output transfer function of any order with time-delay. This problem can be solved by finding all achievable PID controllers that simultaneously stabilize the closed-loop characteristic polynomial and satisfy constraints defined by a set of related complex polynomials. A key advantage of this procedure is that it only depends on the frequency response of the system and does not require the plant transfer function coefficients. If the plant transfer function is given, the procedure is still appropriate. The ability to include the time-delay in the nominal model of the system will often allow for designs with reduced conservativeness in the plant uncertainty and an increase in size of the set of all PID controllers that robustly meet the performance requirements.

Key-Words: - Robust performance, frequency response, PID controllers, and time-delay

1 Introduction

Because of the extensive use of proportional integral derivative (PID) controllers in industrial and bioengineering applications, there has been a significant effort to determine the set of all PID controllers that meet specific design goals. As the target of this research is to develop design methods that can be applied in industry, these methods should have several key elements. First, they should be applicable to a broad set of plants. In order for the methods to be applicable in the process control industry, it is particularly important that they handle time-delays. Ideally, the design methods should be simple to understand and easy to implement. Methods that depend only on the frequency response of the system eliminate the need for a plant model, which may not be available in some applications. In biological systems, for example, rational transfer functions models often do not exist.

Not surprisingly, most of the early work in this area sought to find all PID controllers that stabilized the nominal plant model. Bhattacharyya and colleagues did much of early work in this area, where knowledge of a system's rational transfer function model was assumed [1], [2]. Many of these results depend on generalizations of the Hermite-Biehler theorem [3]. They developed results based on theorems by Pontryagin and a generalized Nyquist criterion [4]. The method introduced by Tan in [5] broke the numerator and denominator of the plant transfer function into even and odd parts. In [6], [7], and [8] a new method, which did not involve complex mathematical derivations, was used to solve the problem of stabilizing an arbitrary order transfer function when only the frequency response of the plant transfer function was known. This work was extended in [9] to a unified approach involving delta operators that found the stability region for discrete-time and continuous-time PID controllers. In [8], Saeki introduced a method for finding the number of unstable poles across the boundary of PID controllers.

Beyond stability, investigators have also looked at performance and robustness. The authors in [5], [6], [7], and [9] found regions where the controllers were guaranteed to meet certain gain and phase margin requirements. PID controllers that also satisfy gain crossover, phase crossover, and bandwidth requirements for double integrator systems with delay were found in [10]. In [11], Shafiei and Shenton found all PID controllers that placed the closed-loop poles in certain D-partitions. In [12] and [13], the parameters of PID controller were determined using a metaheuristic algorithm. In [12], the metaheuristic algorithm was used to adjust the PID parameters to meet the performance requirement for a pouring task. In [14], the authors used a fractional PID controller to meet the performance requirement for an active magnetic bearing system. In this paper, an adaptive genetic algorithm was used to determine the PID controller parameters that optimized a multi-objective cost function. In [15], constrained pole assignment was used for design of PD controllers for a double integrator plant model with time delays or time constant.

As these controllers must be implemented on real systems, design methods that deal with robustness are of particular importance. In [16], [17], and [18], Saeki and colleagues looked at different methods for H_∞ controller design of PID controllers. Ho used a generalization of the Hermite-Biehler theorem for H_∞ PID design [19]. Tantaris, Keel, and Bhattacharyya looked at a similar problem for first-order controllers [20]. In [21], Keel and Bhattacharyya looked at PID design given a weighted sensitivity and weighted complementary sensitivity constraint for plants with no poles or zero on the $j\omega$ axis. In [22] Ho and Lin looked at PID controller design for robust performance for a plant that describe by a rational transfer function. Unfortunately, none of these methods that deal with robustness work directly with time-delays, which are prevalent in the process control industry. In [23], Keel and Bhattacharyya do allow for time-delays in the nominal model when they investigate the weighted sensitivity and robust stability problems. However, they do not consider the robust performance problem.

In [24], [25], [26] and [27], the authors of this paper developed techniques for finding all achievable PID controllers that simultaneously stabilized the closed-loop system and satisfied an H_∞ sensitivity, complementary sensitivity, weighted sensitivity, or robust stability constraint, respectively. In this paper, we consider the robust performance problem for systems with multiplicative uncertainty. The technique is applicable to proper single-input single-output (SISO) systems of arbitrary order with time-delay. This method does not require the rational plant transfer function model, but depends on the frequency response of the system. If the plant transfer function is known, we can apply the same procedure by first computing the frequency response.

The remainder of this paper is organized as follows. In Section 2, the design methodology is introduced. In Section 3, this method is applied to a numerical example. Finally, the results of this paper are summarized in Section 4.

2 Design Methodology

Consider the SISO system shown in Fig. 1, where $G_\Delta(s)$ represents the perturbed plant, $G_p(s)$ is the nominal plant, and $G_c(s)$ is the PID controller. The reference input and the error signals are $R(s)$ and $Z(s)$, respectively. W_S is the sensitivity function weight, W_T is the multiplicative uncertainty weight, and $\left|\Delta_I(j\omega)\right| \le 1$ is the uncertain perturbation. The nominal plant transfer function can be written as

$$G_p(s) = G_o(s)\, e^{-\bar{\tau} s}, \tag{1}$$

where $G_o(s)$ is an arbitrary-order transfer function, and $\bar{\tau}$ is the nominal time-delay. The ability to include the time-delay in the nominal model allows the designer to find much tighter uncertainty bounds in systems with known delays than would be possible otherwise [28]. The PID controller is defined as

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s, \tag{2}$$

where K_p, K_i, and K_d are the proportional, integral, and derivative gains, respectively.

The transfer functions in Fig. 1 can all be expressed in the frequency domain. The plant transfer function can be written in terms of its real and imaginary parts as

$$G_p(j\omega) = R_e(\omega) + j I_m(\omega)\,. \tag{3}$$

The PID controller is defined in the frequency domain as

$$G_c(j\omega) = K_p + \frac{K_i}{j\omega} + K_d j\omega. \tag{4}$$

The sensitivity function weight W_S and the multiplicative uncertainty weight W_T are defined in terms of their real and imaginary parts as

$$W_S(j\omega) = A_S(\omega) + jB_S(\omega), \tag{5}$$

and

$$W_T(j\omega) = A_T(\omega) + jB_T(\omega). \tag{6}$$

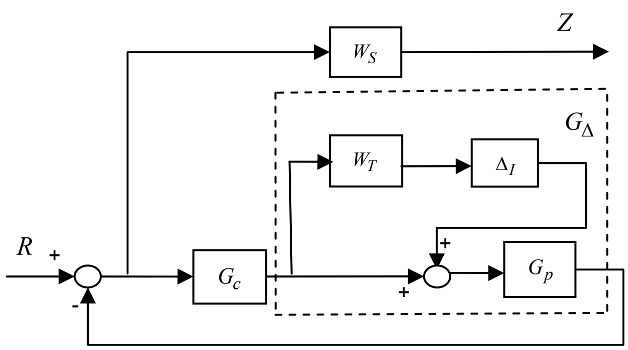

Fig. 1 Block diagram of the system with multiplicative uncertainty

The deterministic values of K_p, K_i, and K_d for which the closed-loop characteristic polynomial is Hurwitz stable have been found in [6] and [7]. In this paper, the problem is to find all PID controllers that satisfy the robust performance constraint of the feedback system in Fig. 1 for all $\left|\Delta_I(j\omega)\right| \le 1$. The robust performance constraint for a SISO system can be written as

$$\left|W_S(j\omega)S(j\omega)\right| + \left|W_T(j\omega)T(j\omega)\right| \le \gamma \qquad \forall\omega, \tag{7}$$

where $S(j\omega) = \dfrac{1}{1+G_p(j\omega)G_c(j\omega)}$ is the sensitivity function, $T(j\omega) = \dfrac{G_p(j\omega)G_c(j\omega)}{1+G_p(j\omega)G_c(j\omega)}$ is the complementary sensitivity function, and γ is the robust performance constraint and ideally is one [16]. The complex functions in (7) can be written in terms of their magnitudes and phase angles as

$$\left(\begin{array}{l}\left|\left|W_S(j\omega)S(j\omega)\right| e^{j\angle W_S(j\omega)S(j\omega)}\right| + \\ \left|\left|W_T(j\omega)T(j\omega)\right| e^{j\angle W_T(j\omega)T(j\omega)}\right|\end{array}\right) \le \gamma, \qquad \forall\omega. \tag{8}$$

If (8) holds, then for each value of ω

$$W_S(j\omega)S(j\omega)e^{j\theta_S} + W_T(j\omega)T(j\omega)e^{j\theta_T} \le \gamma \quad \forall\omega, \tag{9}$$

where $\theta_S = -\angle W_S(j\omega)S(j\omega)$ for some $\theta_S \in [0, 2\pi)$, and $\theta_T = -\angle W_T(j\omega)T(j\omega)$ for some $\theta_T \in [0, 2\pi)$. Consequently, all PID controllers that satisfy (7) must lie at the intersection of all controllers that satisfy (9) for some $\theta_S \in [0, 2\pi)$ and $\theta_T \in [0, 2\pi)$.

To find this region, for each value of $\theta_S \in [0, 2\pi)$ and $\theta_T \in [0, 2\pi)$ we will find all PID controllers on the boundary of (9). It is easy to show from (9) that PID controllers on the boundary must satisfy

$$P(\omega, \theta_S, \theta_T, \gamma) = 0, \tag{10}$$

where,

$$P(\omega, \theta_S, \theta_T, \gamma) = \left(\begin{array}{l}1 + G_p(j\omega)G_c(j\omega) - \dfrac{1}{\gamma}W_S(j\omega)e^{j\theta_S} - \\ \dfrac{1}{\gamma}W_T(j\omega)G_p(j\omega)G_c(j\omega)e^{j\theta_T}\end{array}\right).$$

By substituting (3), (4), (5), (6), $e^{j\theta_S} = \cos\theta_S + j\sin\theta_S$, and $e^{j\theta_T} = \cos\theta_T + j\sin\theta_T$ into (10), the frequency response of this "modified" characteristic polynomial can be rewritten

$$P(\omega,\theta_S,\theta_T,\gamma)= 1+\left(\left(R_e(\omega)+jI_m(\omega)\right)\left(K_p+\frac{K_i}{j\omega}+K_d j\omega\right)\right)- \frac{1}{\gamma}\left(\left(A_S(\omega)+jB_S(\omega)\right)\left(\cos\theta_S+j\sin\theta_S\right)\right)- \frac{1}{\gamma}\left(\begin{array}{l}\left(A_T(\omega)+jB_T(\omega)\right)\left(R_e(\omega)+jI_m(\omega)\right)\\ \left(K_p+\frac{K_i}{j\omega}+K_d j\omega\right)\left(\cos\theta_T+j\sin\theta_T\right)\end{array}\right). \tag{11}$$

Note that (11) reduces to the frequency response of the standard closed-loop characteristic polynomial as $\gamma\to\infty$. Expanding (11) in terms of its real and imaginary parts yields

$$X_{Rp}K_p+X_{Ri}K_i+X_{Rd}K_d=Y_R, \tag{12}$$

and

$$X_{Ip}K_p+X_{Ii}K_i+X_{Id}K_d=Y_I, \tag{13}$$

where

$$\begin{aligned}
X_{Rp}(\omega)&=\omega\left(\alpha_T R_e(\omega)+\beta_T I_m(\omega)\right),\\
X_{Ri}(\omega)&=-\beta_T R_e(\omega)+\alpha_T I_m(\omega),\\
X_{Rd}(\omega)&=\omega^2\left(\beta_T R_e(\omega)-\alpha_T I_m(\omega)\right),\\
Y_R(\omega)&=-\omega\alpha_S,\\
X_{Ip}(\omega)&=\omega\left(-\beta_T R_e(\omega)+\alpha_T I_m(\omega)\right),\\
X_{Ii}(\omega)&=-\alpha_T R_e(\omega)-\beta_T I_m(\omega),\\
X_{Id}(\omega)&=\omega^2\left(\alpha_T R_e(\omega)+\beta_T I_m(\omega)\right),\\
Y_I(\omega)&=\omega\beta_S,\\
\alpha_T&=\frac{1}{\gamma}\left(-A_T(\omega)\cos\theta_T+B_T(\omega)\sin\theta_T\right)+1,\\
\beta_T&=\frac{1}{\gamma}\left(A_T(\omega)\sin\theta_T+B_T(\omega)\cos\theta_T\right),\\
\alpha_S&=\frac{1}{\gamma}\left(-A_S(\omega)\cos\theta_S+B_S(\omega)\sin\theta_S\right)+1,\\
\beta_S&=\frac{1}{\gamma}\left(A_S(\omega)\sin\theta_S+B_S(\omega)\cos\theta_S\right).
\end{aligned}$$

This is a three-dimensional system in terms of the controller parameters K_p, K_i, and K_d. First, the boundary of (11) will be found in the ($K_{p,}$ K_i) plane for a fixed value of K_d. After setting K_d to the fixed value $\tilde{K}_d$, (12) and (13) can be rewritten as

$$\begin{bmatrix}X_{Rp} & X_{Ri}\\ X_{Ip} & X_{Ii}\end{bmatrix}\begin{bmatrix}K_p\\ K_i\end{bmatrix}=\begin{bmatrix}Y_R-X_{Rd}\tilde{K}_d\\ Y_I-X_{Id}\tilde{K}_d\end{bmatrix}. \tag{14}$$

Solving (14) for all $\omega\neq 0$, $\theta_S\in[0,2\pi)$, and $\theta_T\in[0,2\pi)$ gives the following equations:

$$K_p(\omega,\theta_S,\theta_T,\gamma)= \frac{\left(\begin{array}{l} -R_e(\omega)\left(\begin{array}{l}1+\frac{1}{\gamma}\left(\begin{array}{l}-A_S(\omega)\cos\theta_S-A_T(\omega)\cos\theta_T+\\ B_S(\omega)\sin\theta_S+B_T(\omega)\sin\theta_T\end{array}\right)\\ +\frac{1}{\gamma^2}\left(\begin{array}{l}\left(\begin{array}{l}A_S(\omega)A_T(\omega)+\\ B_S(\omega)B_T(\omega)\end{array}\right)\cos\left(\theta_S-\theta_T\right)+\\ \left(\begin{array}{l}-B_S(\omega)A_T(\omega)+\\ A_S(\omega)B_T(\omega)\end{array}\right)\sin\left(\theta_S-\theta_T\right)\end{array}\right)\end{array}\right)-\\ I_m(\omega)\left(\begin{array}{l}\frac{1}{\gamma}\left(\begin{array}{l}-B_S(\omega)\cos\theta_S+B_T(\omega)\cos\theta_T-\\ A_S(\omega)\sin\theta_S+A_T(\omega)\sin\theta_T\end{array}\right)\\ +\frac{1}{\gamma^2}\left(\begin{array}{l}\left(\begin{array}{l}B_S(\omega)A_T(\omega)-\\ A_S(\omega)B_T(\omega)\end{array}\right)\cos\left(\theta_S-\theta_T\right)+\\ \left(\begin{array}{l}A_S(\omega)A_T(\omega)+\\ B_S(\omega)B_T(\omega)\end{array}\right)\sin\left(\theta_S-\theta_T\right)\end{array}\right)\end{array}\right)\end{array}\right)}{D(\omega)}, \tag{15}$$

$$K_i(\omega,\theta_S,\theta_T,\gamma)=\omega^2\tilde{K}_d+\frac{\left(\begin{array}{l}\omega R_e(\omega)\left(\begin{array}{l}\frac{1}{\gamma}\left(\begin{array}{l}-B_S(\omega)\cos\theta_S+B_T(\omega)\cos\theta_T-\\ A_S(\omega)\sin\theta_S+A_T(\omega)\sin\theta_T\end{array}\right)\\ +\frac{1}{\gamma^2}\left(\begin{array}{l}\left(\begin{array}{l}B_S(\omega)A_T(\omega)-\\ A_S(\omega)B_T(\omega)\end{array}\right)\cos\left(\theta_S-\theta_T\right)+\\ \left(\begin{array}{l}A_S(\omega)A_T(\omega)+\\ B_S(\omega)B_T(\omega)\end{array}\right)\sin\left(\theta_S-\theta_T\right)\end{array}\right)\end{array}\right)+\\ \omega I_m(\omega)\left(\begin{array}{l}-1+\frac{1}{\gamma}\left(\begin{array}{l}A_S(\omega)\cos\theta_S+A_T(\omega)\cos\theta_T-\\ B_S(\omega)\sin\theta_S-B_T(\omega)\sin\theta_T\end{array}\right)\\ +\frac{1}{\gamma^2}\left(\begin{array}{l}\left(\begin{array}{l}-A_S(\omega)A_T(\omega)-\\ B_S(\omega)B_T(\omega)\end{array}\right)\cos\left(\theta_S-\theta_T\right)+\\ +\left(\begin{array}{l}B_S(\omega)A_T(\omega)-\\ A_S(\omega)B_T(\omega)\end{array}\right)\sin\left(\theta_S-\theta_T\right)\end{array}\right)\end{array}\right)\end{array}\right)}{D(\omega)}. \tag{16}$$

where,

$$D(\omega)=\left|G_p(j\omega)\right|^2\left(\begin{array}{l}1-\frac{2}{\gamma}\left(\begin{array}{l}A_T(\omega)\cos\theta_T+\\ B_T(\omega)\sin\theta_T\end{array}\right)+\\ \frac{1}{\gamma^2}\left|W_T(j\omega)\right|^2\end{array}\right),$$

$\left|G_p(j\omega)\right|^2=R_e^{\,2}(\omega)+I_m^{\,2}(\omega)$, and

$\left|W_T(j\omega)\right|^2=A_T^{\,2}(\omega)+B_T^{\,2}(\omega)$.

Setting $\omega=0$ in (14) , we obtain

$$\begin{bmatrix}0 & X_{Ri}(0)\\ 0 & X_{Ii}(0)\end{bmatrix}\begin{bmatrix}K_p\\ K_i\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}, \tag{17}$$

and conclude that $K_p(0,\theta_S,\theta_T,\gamma)$ is arbitrary and $K_i(0,\theta_S,\theta_T,\gamma)=0$, unless $I_m(0)=R_e(0)=0$, which holds only when $G_p(s)$ has a zero at the origin.

The procedure can be repeated in the (K_p, K_d) plane. After setting K_i to a fixed value $\tilde{K}_i$, (12) and (13) can be rewritten as

$$\begin{bmatrix}X_{Rp} & X_{Rd}\\ X_{Ip} & X_{Id}\end{bmatrix}\begin{bmatrix}K_p\\ K_d\end{bmatrix}=\begin{bmatrix}Y_R-X_{Ri}\tilde{K}_i\\ Y_I-X_{Ii}\tilde{K}_i\end{bmatrix}. \tag{18}$$

Solving (18) for all $\omega\neq 0$, $\theta_S\in[0,2\pi)$ and $\theta_T\in[0,2\pi)$ gives the same expression as (15) for $K_p(\omega,\theta_S,\theta_T,\gamma)$, and the following equation for $K_d(\omega,\theta_S,\theta_T,\gamma)$:

$$K_d(\omega,\theta_S,\theta_T,\gamma)=\frac{\tilde{K}_i}{\omega^2}+\frac{\left(\begin{array}{l}-R_e(\omega)\left(\begin{array}{l}\frac{1}{\gamma}\left(\begin{array}{l}-B_S(\omega)\cos\theta_S+B_T(\omega)\cos\theta_T-\\ A_S(\omega)\sin\theta_S+A_T(\omega)\sin\theta_T\end{array}\right)\\ +\frac{1}{\gamma^2}\left(\begin{array}{l}\left(\begin{array}{l}B_S(\omega)A_T(\omega)-\\ A_S(\omega)B_T(\omega)\end{array}\right)\cos\left(\theta_S-\theta_T\right)+\\ \left(\begin{array}{l}A_S(\omega)A_T(\omega)+\\ B_S(\omega)B_T(\omega)\end{array}\right)\sin\left(\theta_S-\theta_T\right)\end{array}\right)\end{array}\right)-\\ I_m(\omega)\left(\begin{array}{l}-1+\frac{1}{\gamma}\left(\begin{array}{l}A_S(\omega)\cos\theta_S+A_T(\omega)\cos\theta_T-\\ B_S(\omega)\sin\theta_S-B_T(\omega)\sin\theta_T\end{array}\right)\\ +\frac{1}{\gamma^2}\left(\begin{array}{l}\left(\begin{array}{l}-A_S(\omega)A_T(\omega)-\\ B_S(\omega)B_T(\omega)\end{array}\right)\cos\left(\theta_S-\theta_T\right)+\\ \left(\begin{array}{l}B_S(\omega)A_T(\omega)-\\ A_S(\omega)B_T(\omega)\end{array}\right)\sin\left(\theta_S-\theta_T\right)\end{array}\right)\end{array}\right)\end{array}\right)}{\omega D(\omega)}. \tag{19}$$

At $\omega=0$, $\tilde{K}_i$ must be equal to zero for a solution to exist. Furthermore, as $I_m(0)=0$ for all real plants, $K_d(0,\theta_S,\theta_T,\gamma)$ is arbitrary and

$$K_p(0,\theta_S,\theta_T,\gamma)=\frac{-\left(1-\frac{1}{\gamma}A_S(0)\cos\theta_S+\frac{1}{\gamma}B_S(0)\sin\theta_S\right)}{\left(R_e(0)\left(-\frac{1}{\gamma}A_T(0)\cos\theta_T+\frac{1}{\gamma}B_T(0)\sin\theta_T+1\right)\right)}. \tag{20}$$

Lastly, the solution is found in the (K_i, K_d) plane. After setting K_p to a fixed value of $\tilde{K}_p$, (12) and (13) are rewritten as

$$\begin{bmatrix} X_{Ri} & X_{Rd} \\ X_{Ii} & X_{Id} \end{bmatrix} \begin{bmatrix} K_i \\ K_d \end{bmatrix} = \begin{bmatrix} Y_R - X_{Rp}\tilde{K}_p \\ Y_I - X_{Ip}\tilde{K}_p \end{bmatrix}. \tag{21}$$

Although the coefficient matrix is singular, a solution will exist in two cases. First, at $\omega = 0$ $K_d(0,\theta_S,\theta_T,\gamma)$ is arbitrary and $K_i(0,\theta_S,\theta_T,\gamma) = 0$, unless $I_m(0) = R_e(0) = 0$, which holds only when the plant has a zero at the origin. In such a case, a PID compensator should be avoided as the PID pole cancels the zero at the origin and the system becomes internally unstable. A second set of solutions occurs at any frequency ω_i where $K_p(\omega_i,\theta_S,\theta_T,\gamma)$ (from (15)) is equal to $\tilde{K}_p$. At these frequencies, $K_d(\omega_i,\theta_S,\theta_T,\gamma)$ and $K_i(\omega_i,\theta_S,\theta_T,\gamma)$ must satisfy the following straight line equation

$$K_d(\omega_i,\theta_S,\theta_T,\gamma) = \frac{K_i(\omega_i,\theta_S,\theta_T,\gamma)}{\omega_i^2} + \frac{\left(\begin{array}{l} -R_e(\omega_i)\left(\begin{array}{l} \dfrac{1}{\gamma}\left(\begin{array}{l} -B_S(\omega_i)\cos\theta_S + B_T(\omega_i)\cos\theta_T - \\ A_S(\omega_i)\sin\theta_S + A_T(\omega_i)\sin\theta_T \end{array} \right) \\ + \dfrac{1}{\gamma^2}\left(\begin{array}{l} \left(\begin{array}{l} B_S(\omega_i)A_T(\omega_i) - \\ A_S(\omega_i)B_T(\omega_i) \end{array} \right)\cos\left(\theta_S - \theta_T\right) + \\ \left(\begin{array}{l} A_S(\omega_i)A_T(\omega_i) + \\ B_S(\omega_i)B_T(\omega_i) \end{array} \right)\sin\left(\theta_S - \theta_T\right) \end{array} \right) \end{array} \right) - \\ I_m(\omega_i)\left(\begin{array}{l} -1 + \dfrac{1}{\gamma}\left(\begin{array}{l} A_S(\omega_i)\cos\theta_S + A_T(\omega_i)\cos\theta_T - \\ B_S(\omega_i)\sin\theta_S + B_T(\omega_i)\sin\theta_T \end{array} \right) \\ + \dfrac{1}{\gamma^2}\left(\begin{array}{l} \left(\begin{array}{l} -A_S(\omega_i)A_T(\omega_i) - \\ B_S(\omega_i)B_T(\omega_i) \end{array} \right)\cos\left(\theta_S - \theta_T\right) + \\ \left(\begin{array}{l} B_S(\omega_i)A_T(\omega_i) - \\ A_S(\omega_i)B_T(\omega_i) \end{array} \right)\sin\left(\theta_S - \theta_T\right) \end{array} \right) \end{array} \right) \end{array} \right)}{\omega_i D(\omega_i)}. \tag{22}$$

3 Example

In this section, a numerical example is used to demonstrate the application of this method. Consider the second order plant transfer function from [4] where we assume the feedback loop has an unknown time-delay with a range of 0.1 to 1.1 seconds. The goal is to find all PID controllers that stabilize the system and satisfy the robust performance constraint in (7) where $\gamma = 1$. The nominal model of the system is given by

$$G_p(s) = \frac{-0.5s + 1}{(s+1)(2s+1)} e^{-\overline{\tau}s}, \tag{23}$$

where $\overline{\tau}$ has been selected to be the mean value of the uncertain delay, 0.6 seconds. The frequency responses of the multiplicative errors for different delays and the multiplicative weight are shown in Fig. 2. The multiplicative weight

$$W_T(s) = \frac{s}{0.43s + 1.35}, \tag{24}$$

is chosen to bound the multiplicative errors. Note, by including the time-delay in the nominal model, we are able to reduce the conservativeness in our plant uncertainty. This will increase the size of the set of PID controllers that robustly meet our performance goals.

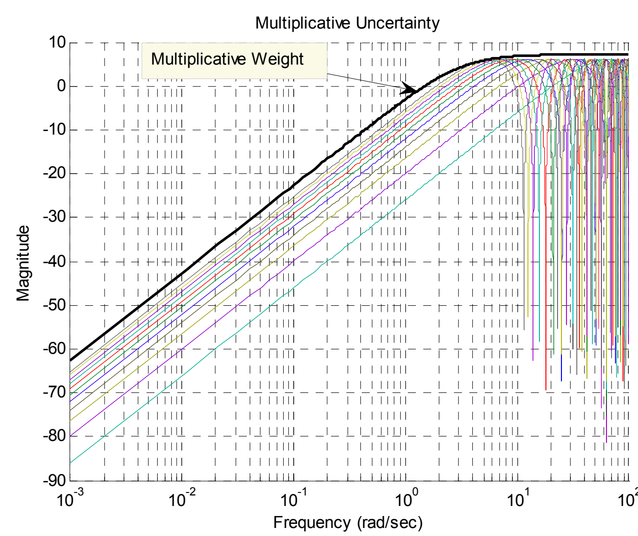

Fig. 2 Multiplicative errors for different communication delays and the multiplicative weight

The closed-loop step response is required to have an overshoot less than 5% and a settling time less than 20 seconds. The sensitivity weight that is chosen to satisfy the performance requirement for the closed-loop system is

$$W_S(s) = \frac{0.39(s+0.26)}{s+0.2}. \qquad (25)$$

Equations (15) and (16) are used in the (K_p, K_i) plane for a fixed value of $\tilde{K}_d = 0.2$. As discussed previously, the PID stability boundary of the nominal system can be found by setting $\gamma = \infty$ in (15) and (16). All PID controllers that satisfy the robust performance constraint in (7) are found by setting $\gamma = 1$ in (15) and (16) for $\theta_S \in [0, 2\pi)$ and $\theta_T \in [0, 2\pi)$, and then finding the intersection of all regions.

The region that satisfies the robust performance constraint and the nominal stability boundary is shown in Fig. 3. The intersection of all regions inside the nominal stability boundary of the (K_p, K_i) plane is the robust performance region. To verify the results, an arbitrary controller from this region is chosen,

$$G_c(s) = 0.35 + \frac{0.18}{s} + 0.2\,s. \qquad (26)$$

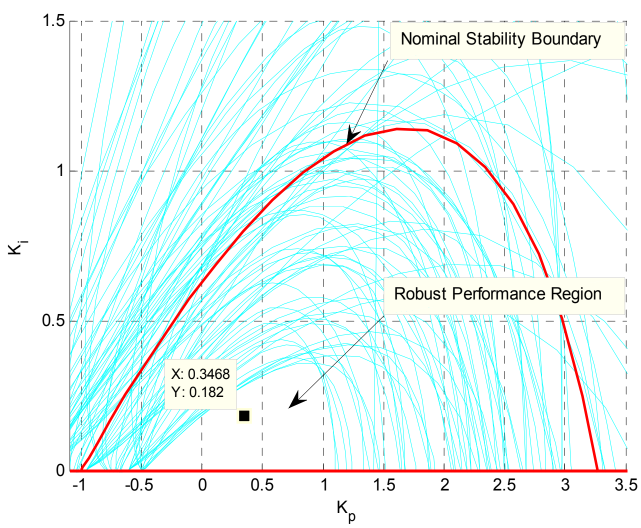

Fig. 3 Nominal stability boundary and robust performance region in the (K_p, K_i) plane for a fixed $\tilde{K}_d = 0.2$

Substituting (23), (24), (25), and (26), into (7), we find that $\left|W_S(j\omega)S(j\omega)\right| + \left|W_T(j\omega)T(j\omega)\right| \le 0.67$. The Bode plot of $\left|W_S(j\omega)S(j\omega)\right| + \left|W_T(j\omega)T(j\omega)\right|$ is shown in Fig. 4. As can be seen, the magnitude of robust performance system is less than one and the design goal is met.

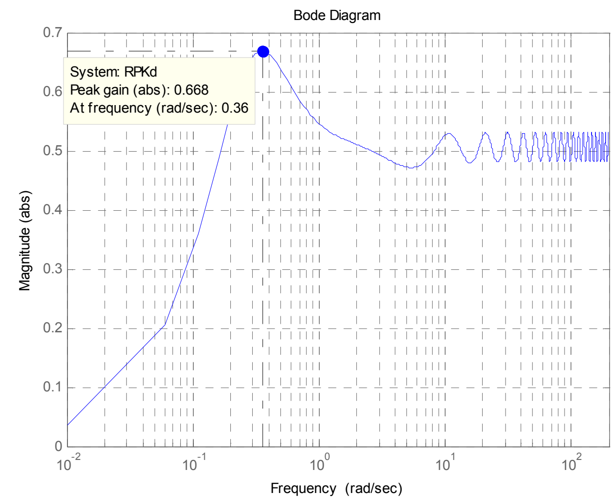

Fig. 4 Magnitude of $\left|W_S(j\omega)S(j\omega)\right| + \left|W_T(j\omega)T(j\omega)\right|$ for $G_c(s) = 0.35 + \frac{0.18}{s} + 0.2\,s$

The second method uses (15) and (19) in the (K_p, K_d) plane for a fixed value of $\tilde{K}_i = 0.1$. As discussed previously, the PID stability boundary of the nominal system can be found by setting $\gamma = \infty$ in (15) and (19). The PID controller is designed to satisfy the robust performance constraint (7) by setting $\gamma = 1$ in (15) and (19) for $\theta_S \in [0, 2\pi)$ and $\theta_T \in [0, 2\pi)$, and finding the intersection of all regions.

The region that satisfied the robust performance constraint and the nominal stability boundary is shown in Fig. 5. The intersection of all regions inside the nominal stability boundary of the (K_p, K_d) plane is the robust performance region. To verify the results, an arbitrary controller from this region is chosen, giving us the PID controller

$$G_c(s) = 0.26 + \frac{0.1}{s} + 0.35\,s. \qquad (27)$$

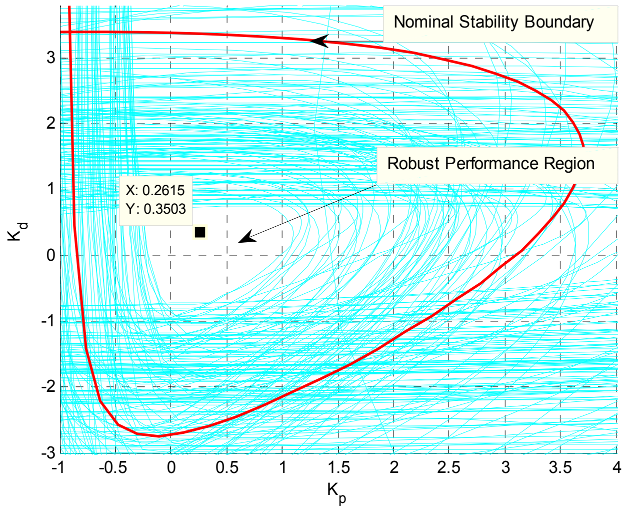

Fig. 5 Nominal stability boundary and robust performance region in the (K_p, K_d) plane for a fixed $\tilde{K}_i = 0.1$

Substituting (23), (24), (25), and (27) into (7) gives $\left|W_S(j\omega)S(j\omega)\right| + \left|W_T(j\omega)T(j\omega)\right| \le 0.65$. The Bode plot of $\left|W_S(j\omega)S(j\omega)\right| + \left|W_T(j\omega)T(j\omega)\right|$ is shown in Fig. 6. As can be seen, the magnitude of robust performance system is less than one, that is, the design goal is met.

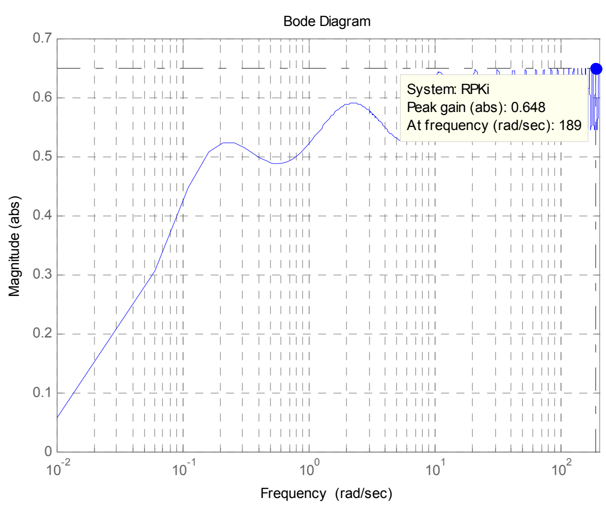

Fig. 6 Magnitude of $\left|W_S(j\omega)S(j\omega)\right| + \left|W_T(j\omega)T(j\omega)\right|$ for

$$G_c(s) = 0.26 + \frac{0.1}{s} + 0.35\,s$$

The third method is applied in the (K_i, K_d) plane for a fixed value of $\tilde{K}_p = 0.5$. Plots of $K_p(\omega,\theta_S,\theta_T,\gamma)$ at $\gamma = \infty$ and $K_p(\omega,\theta_S,\theta_T,\gamma)$ (from (15)) for values of $\theta_S \in [0,2\pi)$ and $\theta_T \in [0,2\pi)$ are shown in Fig. 7. For each curve, the ω_is are the frequencies at which the chosen value for $K_p(\omega_i,\theta_S,\theta_T,\gamma) = \tilde{K}_p = 0.5$. Each ω_i for this chosen constant coefficient of $\tilde{K}_p$ is substituted into (22) to find the required boundaries. In addition, we have the boundary at $K_i(0,\theta_S,\theta_T,\gamma) = 0$.

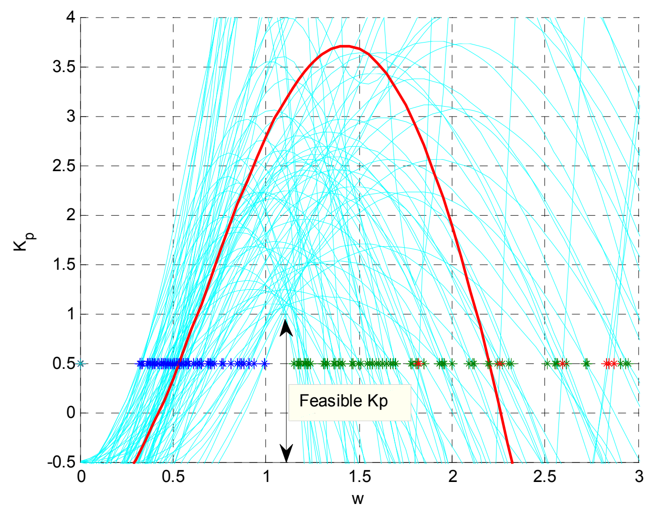

Fig. 7 Plots of $K_p(\omega,\theta_S,\theta_T,\gamma)$ versus ω used to find values of ω_i, which $\tilde{K}_p = 0.5$

The region that satisfied the robust performance constraint and the nominal stability boundary is shown in Fig 8. The intersection of all regions inside the nominal stability boundary of the (K_i, K_d) plane is the robust performance region. To verify the results, an arbitrary controller from this region is chosen, giving us the PID controller

$$G_c(s) = 0.5 + \frac{0.2}{s} + 0.46\,s. \tag{28}$$

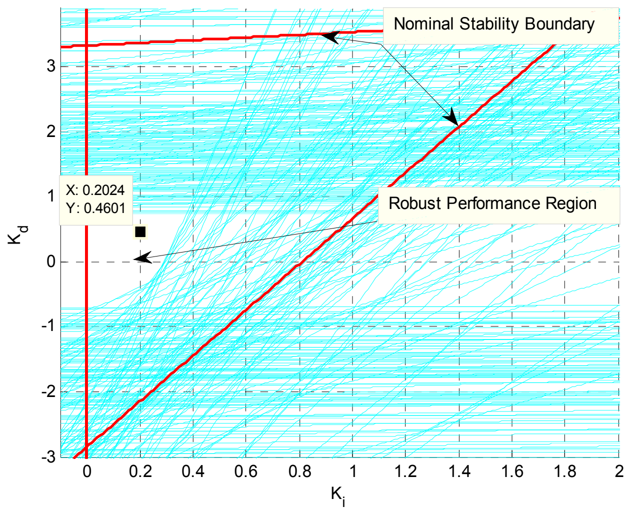

Fig. 8 Nominal stability boundary and robust performance region in the (K_i, K_d) plane for a fixed $\tilde{K}_p = 0.5$

Substituting (23) , (24), (25), and (27) into (7) gives $\left|W_S(j\omega)S(j\omega)\right| + \left|W_T(j\omega)T(j\omega)\right| \le 0.74$ The Bode plot of $\left|W_S(j\omega)S(j\omega)\right| + \left|W_T(j\omega)T(j\omega)\right|$ is shown in Fig. 9. As can be seen, the magnitude of robust performance system is less than one, that is, the design goal is met.

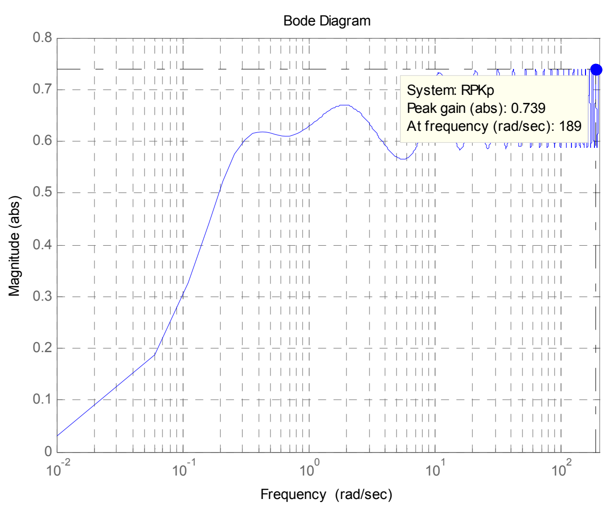

Fig. 9 Magnitude of $\left|W_S(j\omega)S(j\omega)\right| + \left|W_T(j\omega)T(j\omega)\right|$ for

$$G_c(s) = 0.5 + \frac{0.2}{s} + 0.46\,s$$

Step responses of the closed-loop system with the PID controller in (28) and various time-delays between 0.1 and 1.1 seconds are shown in Fig. 10. As can be seen, the closed-loop step responses all have an overshoot less than 5% and a setting time less than 20 seconds. The maximum setting time is 14.5 seconds and the maximum percent overshoot is 1.02%.

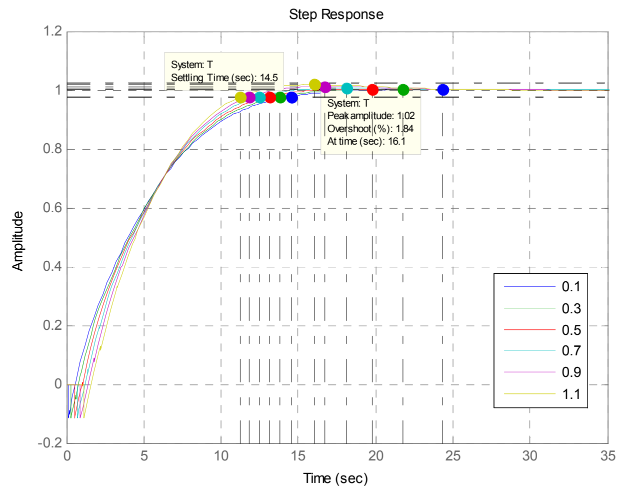

Fig. 10 Step response of the closed loop system for various time delays

4 Conclusion

A graphical technique was introduced for finding all achievable PID controllers that satisfy the robust performance constraint of an arbitrary-order transfer function with an unknown time-delay. This method is simple to understand and requires only the frequency response of the system. A numerical example with an unknown time-delay in the feedback path was presented to demonstrate the application of this method. By including the time-delay in the nominal model, we were able to reduce the conservativeness in our plant uncertainty and increase the size of the set of PID controllers that satisfied the robust performance requirements.

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