A Unified Approach for Sensitivity Design of PID Controllers in the Frequency Domain

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Abstract: - In this paper a graphical technique is introduced for finding all continuous-time and discrete-time proportional integral derivative (PID) controllers that satisfy an H_∞ sensitivity constraint of an arbitrary order transfer function with time delay. These problems can be solved by finding all achievable PID controllers that simultaneously stabilize the closed-loop characteristic polynomial and satisfy constraints defined by a set of related complex polynomials. A key advantage of this procedure is that it only depends on the frequency response of the system. The delta operator is used to describe the controllers, because it not only possesses numerical properties superior to the discrete-time shift operator, but also converges to the continuous-time controller as the sampling period approaches zero. A unified approach allows us to use the same procedure for discrete-time and continuous-time H_∞ sensitivity design of PID controllers.

Key-Words: - Sensitivity design, frequency response, delta domain, PID controllers, and time delay

1 Introduction

Because of the extensive use of proportional integral derivative (PID) controllers in industry, there has been a significant effort to determine the set of all PID controllers that meet certain design goals. As the intent of this research is to develop design methods that can be applied in industry, these methods should possess several key attributes. First, they should be applicable to a broad set of plants. In order for the methods to be applicable in the process control industry, it is particularly important that they handle time-delays. Ideally, the design methods would be simple to understand and easy to implement. Methods that depend only on the frequency response of the system eliminate the need for a plant model, which may not be available in some applications. Finally, designs done directly in the digital domain allow for easy computer implementation.

Not surprisingly, most of the early work in this area sought to find all continuous-time PID controllers that stabilized the nominal plant model. Much of the early work in this area was done by Bhattacharyya and colleagues and assumed knowledge of the system transfer function model [1], [2]. Many of these results depend on generalizations of the Hermite-Biehler theorem [3]. They developed results based on theorems by Pontryagin and a generalized Nyquist criterion [4]. The method introduced by Tan in [5] broke the numerator and denominator of the plant transfer function into even and odd parts. In [6], [7], and [8] a new method, which did not involve complex mathematical derivations, was used to solve the problem of stabilizing an arbitrary order transfer function, when only the frequency response of the plant transfer function was known.

Beyond stability, investigators have also looked at performance and robustness. The authors in [5] and [7] found regions where the controllers were guaranteed to meet certain gain and phase margin requirements. PID controllers that also satisfy gain crossover, phase crossover, and bandwidth requirements for double integrator systems with delay were found in [9]. In [10] and [11], the parameters of PID controller were determined using a metaheuristic algorithm method. In [10], the metaheuristic algorithm method was used to adjust the PID parameters to meet the performance requirement for a pouring task. In [12], the authors used a fractional PID controller to meet the performance requirement for an active magnetic bearing system. In this paper, an adaptive genetic algorithm was used to determine the PID controller parameters that optimized a multi-objective cost function. In [13], constrained pole assignment was used for design of PD controllers for a double integrator plant model with time delays or time constant.

As these controllers must be implemented on real systems, design methods that deal with robustness are of particular importance. In [14] and [15], Saeki and colleagues looked at different methods for H_∞ controller design of PID controllers. Ho used a generalization of the Hermite-Biehler theorem for H_∞ PID design [16]. Tantaris, Keel, and Bhattacharyya looked at design of first-order controllers in [17]. Unfortunately, these methods that dealt with robustness did not work directly with systems with time-delay, which are prevalent in the process control industry. In [18], Keel and Bhattacharyya did allow for time-delays in the nominal model when they investigated the stability problem for plants with no poles or zeros on the $j\omega$ axis and a known time delay. All of the methods in [1]-[18] are based in continuous-time systems.

As more and more controllers are implemented as digital compensators, design method that work directly in the digital domain become more important. Unfortunately, most of the work in this area has concentrated on design of continuous-time PID controllers. In [19], the delta operator was used to obtain a unified approach for finding stability boundaries of PID controllers for arbitrary order transfer functions with time delay in the frequency domain. The delta operator was used to describe controllers in the discrete-time, because it not only provides numerical properties superior to the discrete-time shift operator, but also converges to the continuous-time as the sampling period approaches zero [20], [21]. In [22], Suchomski used the delta operator to design robustly stable PID controllers for low order known system transfer functions.

In [23], [24], [25], [26] and [27] the authors of this paper developed techniques for finding all achievable continuous-time PID controllers that simultaneously stabilize the closed-loop system and satisfy an H_∞ sensitivity, complementary sensitivity, weighted sensitivity, robust stability constraint, or robust performance constraint. In this paper the goal is to define a unified approach for continuous-time and discrete-time sensitivity design of PID controllers. This method is applicable for single-input-single-output (SISO) proper transfer functions of any order with time delay. A unified approach using the delta operator allows us to use the same procedure for discrete-time and continuous-time H_∞ sensitivity design. As this work builds upon the straight-forward development in [19], it does not require the plant transfer function model, but only the frequency response of the system. If the plant transfer function is known, we can apply the same procedures by first computing the frequency response.

The remainder of this paper is organized as follows. The design methodology is introduced in Section 2. A numerical example that demonstrates the application of this method is presented in Section 3. Finally, the results of this paper are summarized in Section 4.

2 Design Methodology

A SISO continuous-time plant transfer function with time delay τ is defined as

$$G_p(s) = G(s)e^{-\tau s}. \tag{1}$$

The equivalent model in the delta domain, when the output of plant is sampled and a zero-order hold is placed at the input, can be found from [20] as

$$G_p(\gamma) = \frac{\gamma}{1+\gamma T_0} T\left[L^{-1}\left\{\frac{1}{s}G_p(s)\right\}\right], \tag{2}$$

where T_0 is the sampling period, T is the generalized transform, and γ, as defined in [21], is given by

$$\gamma = \begin{cases} s, & T_0 = 0 \\ \dfrac{e^{sT_0}-1}{T_0}, & T_0 \neq 0 \end{cases}. \tag{3}$$

Consider the SISO system shown in Fig. 1, where $G_p(\gamma)$ is the plant and $G_c(\gamma)$ is the PID controller. The reference input signal and the error signal are r and z, respectively. The output of the controlled plant is y. The PID controller is defined as

$$G_c(\gamma) = K_p + \frac{K_i}{\gamma} + K_d \frac{\gamma}{1+T_0\gamma}, \quad (4)$$

where K_p, K_i, and K_d are the proportional, integral, and derivative gains, respectively. The error signal that we want to minimize is defined as

$$z = r - y. \quad (5)$$

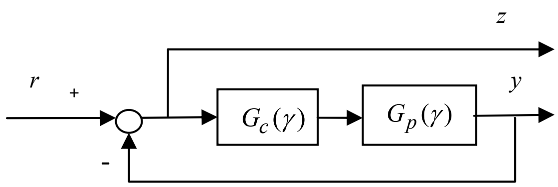

Fig. 1 Block diagram of sensitivity function

The transfer functions in Fig. 1 can be expressed in the frequency domain. The plant transfer function can be written in terms of its real and imaginary parts as

$$G_p(\beta) = R_e(\beta) + jI_m(\beta), \quad (6)$$

where $\beta = \begin{cases} j\omega & T_0 = 0 \\ \dfrac{e^{j\omega T_0} - 1}{T_0} & T_0 \neq 0 \end{cases}$. The PID controller is defined in the frequency domain as

$$G_c(\beta) = K_p + \frac{K_i}{\beta} + K_d \frac{\beta}{1+T_0\beta}. \quad (7)$$

The deterministic values of K_p, K_i, and K_d for which the closed-loop characteristic polynomial is Hurwitz stable have been found in [19] (with a small difference in the parameterization of the PID controllers). In this paper, the problem is to find all PID controllers that stabilize the system and satisfy the sensitivity constraint

$$\left\| S(\beta) \right\|_\infty \leq \gamma_0, \quad (8)$$

where $S(\beta) = \dfrac{1}{1+G_p(\beta)G_c(\beta)}$ is the sensitivity function and γ_0 is a positive real scalar. The complex function in (8) can be written in terms of its magnitude and phase angle as

$$\left\| \left| S(\beta) \right| e^{j\angle S(\beta)} \right\| \leq \gamma_0 \qquad \forall \omega. \quad (9)$$

If (9) holds, then for each value of β

$$S(\beta)e^{j\theta_S} \leq \gamma_0, \quad (10)$$

must be true for some $\theta_S \in [0, 2\pi)$, where $\theta_S = -\angle S(\beta)$. Consequently, all PID controllers that satisfy (8) must lie at the intersection of all controllers that satisfy (10) for all $\theta_S \in [0, 2\pi)$ [24].

To accomplish this, for each value of $\theta_S \in [0, 2\pi)$, we will find all PID controllers on the boundary of (10). It is easy to show from (10), that all the PID controllers on the boundary must satisfy

$$P(\omega, \theta_S, \gamma_0, T_0) = 0, \quad (11)$$

where $P(\omega, \theta_S, \gamma_0, T_0) = 1 + G_p(\beta)G_c(\beta) - \dfrac{1}{\gamma_0} e^{j\theta_S}$. Note that (11) reduces to the frequency response of the standard closed-loop characteristic polynomial as $\gamma_0 \to \infty$. Substituting (6), (7), and $e^{j\theta_S} = \cos\theta_S + j\sin\theta_S$ into (11), and solving for the real and imaginary parts yields

$$X_{Rp}K_p + X_{Ri}K_i + X_{Rd}K_d = Y_R, \quad (12)$$

and

$$X_{Ip}K_p + X_{Ii}K_i + X_{Id}K_d = Y_I, \quad (13)$$

where

$$X_{Rp} = \omega R_e(\beta),$$

$$X_{Ri} = -\frac{T_0}{2}\omega R_e(\beta) + I_m(\beta)\frac{\left(\cos\left(\omega T_0\right) + 1\right)}{2\mathrm{sinc}\left(\omega T_0\right)},$$

$$X_{Rd} = \omega^2 \begin{pmatrix} R_e(\beta) \dfrac{\sin(\omega T_0)\operatorname{sinc}(\omega T_0)}{\cos(\omega T_0)+1} - \\ I_m(\beta)\operatorname{sinc}(\omega T_0) \end{pmatrix},$$

$$Y_R = \omega\left(\frac{1}{\gamma_0}\cos\theta_S - 1\right),$$

$$X_{Ip} = \omega I_m(\beta),$$

$$X_{Ii} = -R_e(\beta)\frac{\left(\cos(\omega T_0)+1\right)}{2\operatorname{sinc}(\omega T_0)} - \frac{T_0}{2}\omega I_m(\beta),$$

$$X_{Id} = \omega^2 \begin{pmatrix} R_e(\beta)\operatorname{sinc}(\omega T_0) + \\ I_m(\beta)\dfrac{\sin(\omega T_0)\operatorname{sinc}(\omega T_0)}{\cos(\omega T_0)+1} \end{pmatrix},$$

$$Y_I = \frac{\omega}{\gamma_0}\sin\theta_S \ .$$

This is a three-dimensional system in terms of the controller parameters K_p, K_i, and K_d. The boundary of (11) can be found in the ($K_{p,}$ K_i) plane for a fixed value of K_d. After setting K_d to the fixed value $\tilde{K}_d$, (12) and (13) can be rewritten as

$$\begin{bmatrix} X_{Rp} & X_{Ri} \\ X_{Ip} & X_{Ii} \end{bmatrix}\begin{bmatrix} K_p \\ K_i \end{bmatrix} = \begin{bmatrix} Y_R - X_{Rd}\tilde{K}_d \\ Y_I - X_{Id}\tilde{K}_d \end{bmatrix}. \qquad (14)$$

Solving (14), for all $\omega \neq 0$ and $\theta_S \in [0, 2\pi)$, gives the following equations:

$$K_p(\omega,\theta_S,\gamma_0,T_0) = \frac{\dfrac{1}{\gamma_0}\begin{pmatrix} R_e(\beta)\begin{pmatrix} \left(\cos(\omega T_0)+1\right)\left(\cos\theta_S - \gamma_0\right) - \\ \sin(\omega T_0)\sin\theta_S \end{pmatrix} + \\ I_m(\beta)\begin{pmatrix} \left(\cos(\omega T_0)+1\right)\sin\theta_S + \\ \sin(\omega T_0)\left(\cos\theta_S + \gamma_0\right) \end{pmatrix} \end{pmatrix}}{\left|G_p(\beta)\right|^2\left(\cos(\omega T_0)+1\right)} \qquad (15)$$

and

$$K_i(\omega,\theta_S,\gamma_0,T_0) = \tilde{K}_d\omega^2\frac{2\operatorname{sinc}^2(\omega T_0)}{\cos(\omega T_0)+1} + \frac{\dfrac{\omega}{\gamma_0}\begin{pmatrix} R_e(\beta)\left(-2\operatorname{sinc}(\omega T_0)\sin\theta_S\right) + \\ I_m(\beta)\left(2\operatorname{sinc}(\omega T_0)\left(\cos\theta_S - \gamma_0\right)\right) \end{pmatrix}}{\left|G_p(\beta)\right|^2\left(\cos(\omega T_0)+1\right)}, \qquad (16)$$

where $\left|G_p(\beta)\right|^2 = R_e^{\,2}(\beta) + I_m^{\,2}(\beta)$. Setting $\omega = 0$ in (14) , we obtain

$$\begin{bmatrix} 0 & X_{Ri}(0) \\ 0 & X_{Ii}(0) \end{bmatrix}\begin{bmatrix} K_p \\ K_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad (17)$$

and conclude that $K_p(0,\theta_S,\gamma_0,T_0)$ is arbitrary and $K_i(0,\theta_S,\gamma_0,T_0) = 0$, unless $I_m(0) = R_e(0) = 0$, which holds only when $G_p(s)$ has a zero at the origin. By letting $T_0 \to 0$ in (15) and (16), the continuous-time sensitivity boundaries are found as

$$K_p(\omega,\theta_S,\gamma_0,0) = \frac{\dfrac{1}{\gamma_0}\begin{pmatrix} R_e(\omega)\left(\cos\theta_S - \gamma_0\right) + \\ I_m(\omega)\left(\sin\theta_S\right) \end{pmatrix}}{\left|G_p(j\omega)\right|^2}, \qquad (18)$$

and

$$K_i(\omega,\theta_S,\gamma_0,0) = \omega^2\tilde{K}_d + \frac{\dfrac{\omega}{\gamma_0}\left(-R_e(\omega)\left(\sin\theta_S\right) + I_m(\omega)\left(\cos\theta_S - \gamma_0\right)\right)}{\left|G_p(j\omega)\right|^2}, \qquad (19)$$

where $\left|G_p(j\omega)\right|^2 = R_e^{\,2}(\omega) + I_m^{\,2}(\omega)$, $R_e(\omega)$, and $I_m(\omega)$ are the real and imaginary parts of the continuous-time plant transfer function, respectively.

The procedure can be repeated in the (K_p, K_d) plane. After setting K_i to a fixed value $\tilde{K}_i$, (12) and (13) can be rewritten as

$$\begin{bmatrix} X_{Rp} & X_{Rd} \\ X_{Ip} & X_{Id} \end{bmatrix}\begin{bmatrix} K_p \\ K_d \end{bmatrix} = \begin{bmatrix} Y_R - X_{Ri}\tilde{K}_i \\ Y_I - X_{Ii}\tilde{K}_i \end{bmatrix}. \quad (20)$$

Solving (20) for all $\omega \neq 0$, $\theta_S \in [0, 2\pi)$ gives the same expression as (15) for $K_p(\omega, \theta_S, \gamma_0, T_0)$ and the following equation for $K_d(\omega, \theta_S, \gamma_0, T_0)$,

$$K_d(\omega, \theta_S, \gamma_0, T_0) = \frac{\tilde{K}_i\left(\cos\left(\omega T_0\right)+1\right)}{\omega^2 2\mathrm{sinc}^2\left(\omega T_0\right)} + \frac{\frac{1}{\gamma_0}\left(R_e(\beta)\left(\sin\theta_S\right) + I_m(\beta)\left(\gamma_0 - \cos\theta_S\right)\right)}{\omega\left|G_p(\beta)\right|^2\left(\mathrm{sinc}\left(\omega T_0\right)\right)}. \quad (21)$$

At $\omega = 0$, $\tilde{K}_i$ must be equal to zero for a solution to exist. Furthermore, as $I_m(0) = 0$ for all real plants, $K_d(0, \theta_S, \gamma_0, T_0)$ is arbitrary and

$$K_p(0, \theta_S, \gamma_0, T_0) = \frac{\frac{1}{\gamma_0}\cos\theta_S - 1}{R_e(0)}. \quad (22)$$

Letting $T_0 \to 0$ in (15) and (21) gives the same expression as (18) for $K_p(\omega, \theta_S, \gamma_0, 0)$ and for the continuous-time sensitivity boundary of $K_d(\omega, \theta_S, \gamma_0, 0)$ the following expression:

$$K_d(\omega, \theta_S, \gamma_0, 0) = \frac{\tilde{K}_i}{\omega^2} + \frac{\frac{1}{\gamma_0}\left(R_e(\omega)\left(\sin\theta_S\right) + I_m(\omega)\left(\gamma_0 - \cos\theta_S\right)\right)}{\omega\left|G_p(j\omega)\right|^2}, \quad (23)$$

Lastly, the solution is found in the (K_i, K_d) plane. After setting K_p to a fixed value of $\tilde{K}_p$, (12) and (13) are rewritten as

$$\begin{bmatrix} X_{Ri} & X_{Rd} \\ X_{Ii} & X_{Id} \end{bmatrix}\begin{bmatrix} K_i \\ K_d \end{bmatrix} = \begin{bmatrix} Y_R - X_{Rp}\tilde{K}_p \\ Y_I - X_{Ip}\tilde{K}_p \end{bmatrix}. \quad (24)$$

Although the coefficient matrix is singular, a solution will exist in two cases. First, at $\omega = 0$ $K_d(0, \theta_S, \gamma_0, T_0)$ is arbitrary and $K_i(0, \theta_S, \gamma_0, T_0) = 0$, unless $I_m(0) = R_e(0) = 0$, which holds only when the plant has a zero at the origin. In such a case, a PID compensator should be avoided as the PID pole cancels the zero at the origin and the system becomes internally unstable. A second set of solutions occurs at any frequency ω_i, where $K_p(\omega_i, \theta_S, \gamma_0, T_0)$ (from(15)) is equal to $\tilde{K}_p$. At these frequencies, $K_d(\omega_i, \theta_S, \gamma_0, T_0)$ and $K_i(\omega_i, \theta_S, \gamma_0, T_0)$ must satisfy the following straight line equation

$$K_d(\omega_i, \theta_S, \gamma_0, T_0) = \frac{\tilde{K}_i\left(\cos\left(\omega_i T_0\right)+1\right)}{\omega_i^2 2\mathrm{sinc}^2\left(\omega_i T_0\right)} + \left(\frac{\frac{1}{\gamma_0}\left(R_e(\beta_i)\left(\sin\theta_S\right) + I_m(\beta_i)\left(\gamma_0 - \cos\theta_S\right)\right)}{\omega_i\left|G_p(\beta_i)\right|\left(\mathrm{sinc}\left(\omega_i T_0\right)\right)}\right). \quad (25)$$

Letting $T_0 \to 0$ in (25) we can get the following expression for the continuous-time case

$$K_d\left(\omega_i, \theta_S, \gamma_0, 0\right) = \frac{K_i}{\omega_i^2} + \frac{\frac{1}{\gamma_0}\left(R_e(\omega_i)\left(\sin\theta_S\right) + I_m(\omega_i)\left(\gamma_0 - \cos\theta_S\right)\right)}{\omega_i\left|G_p(j\omega_i)\right|^2}. \quad (26)$$

3 Example

In this section, a numerical example is used to demonstrate the application of this method. Consider the second-order plant transfer function from [7],

$$G_p(s) = \frac{-0.25(s-2)}{(s+1)(s+0.5)} e^{-0.6s}. \quad (27)$$

The goal is to find all discrete-time PID controllers that stabilize the closed-loop system and satisfy the H_∞ sensitivity constraint in (8) for $\gamma_0 = 2$, when the sampling period is $T_0 = 0.1$ seconds. Using (2), the discrete-time delta-domain equivalent of the system in (27) is given by

$$G_p(\gamma) = \frac{-0.2(\gamma-2.3)}{(\gamma+0.96)(\gamma+0.48)} (1+T_0\gamma)^{-0.6\left(\frac{1}{T_0}\right)}. \quad (28)$$

Equations (15) and (16) are used in the (K_p, K_i) plane for a fixed value of $\tilde{K}_d = 0.2$. Equation (6) is used to find the real and imaginary parts of (28) in the frequency domain. As discussed previously, the PID stability boundary of the nominal system can be found by setting $\gamma_0 = \infty$ in (15) and (16). All PID controllers that satisfy the H_∞ sensitivity constraint in (8) are found by setting $\gamma_0 = 2$ in (15) and (16) for $\theta_S \in [0, 2\pi)$ and finding the intersection of all regions.

The stability boundary and the region that satisfies the H_∞ sensitivity constraint are shown in Fig. 2. The intersection of all regions inside the stability boundary of the (K_p, K_i) plane is the H_∞ sensitivity region for the discrete-time system.

To verify the results, an arbitrary controller from this region is chosen, giving us the discrete-time PID controller as

$$G_c(\gamma) = 0.57 + \frac{0.11}{\gamma} + \frac{0.2\gamma}{1+0.1\gamma}. \quad (29)$$

The Bode response of the sensitivity function is shown in Fig. 3. As can be seen, $\|S(\beta)\|_\infty = 1.25$, which is less than $\gamma_0 = 2$. The design goal is met for the discrete-time system.

The closed loop step response of the system with the PID controller in (29) is shown in Fig. 4. As can be seen, the closed-loop step response of the discrete-time system is stable and has no overshoot, a settling time of 38.5 seconds and zero steady state error.

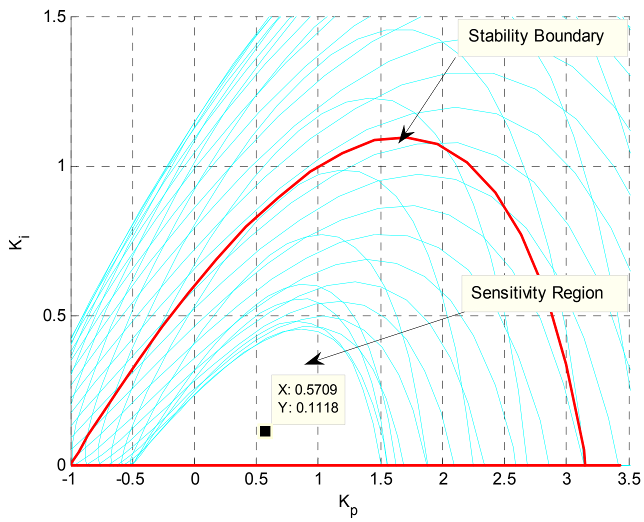

Fig. 2. Stability boundary and sensitivity region of discrete-time system in the (K_p, K_i) plane

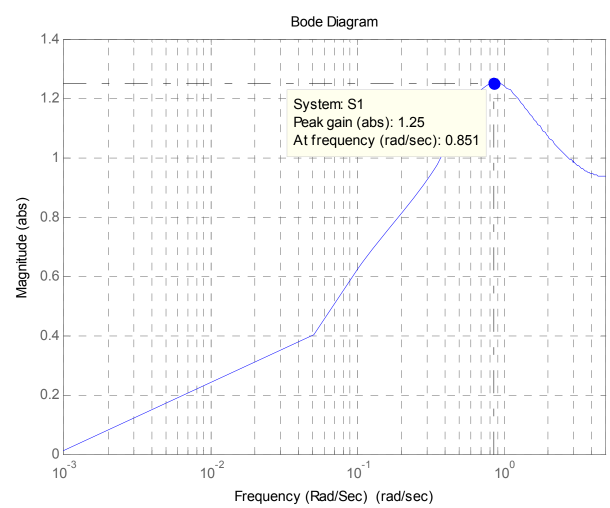

Fig. 3 Magnitude of discrete-time sensitivity function frequency response

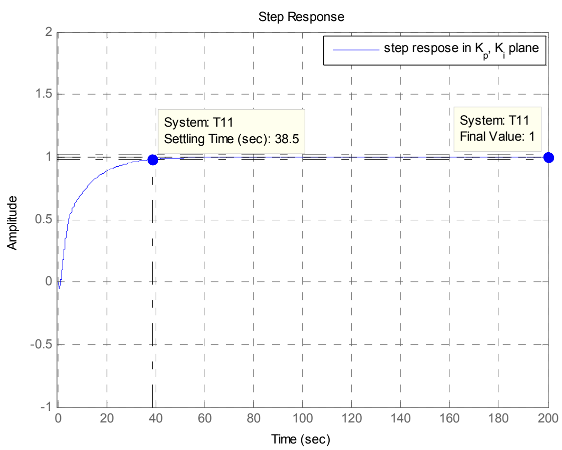

Fig. 4 Closed loop step response of discrete-time system

In order to compare these results with the continuous-time PID controller, equations (18) and (19) are used in the (K_p, K_i) plane for a fixed value of $\tilde{K}_d = 0.2$. The stability boundary and the region that satisfies the H_∞ sensitivity constraint are shown in Fig. 5. The intersection of all regions inside the stability boundary of the (K_p, K_i) plane is the H_∞ sensitivity region for the continuous-time system.

Comparing Fig. 2 and Fig. 5, we can see that while the stability boundary and the H_∞ sensitivity regions are similar for the discrete-time and continuous-time controllers, the continuous-time regions are larger.

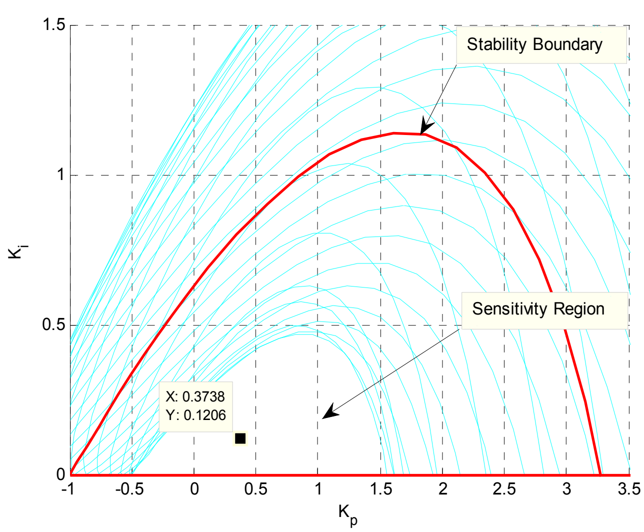

Fig. 5 Stability boundary and sensitivity region of continuous-time system in the (K_p, K_i) plane

The second method uses (15) and (21) in the (K_p, K_d) plane for a fixed value of $\tilde{K}_i = 0.1$. The PID controller is designed to satisfy the weighted sensitivity constraint with $\gamma_0 = 2$. The region that satisfies the sensitivity constraint and the stability boundary is shown in Fig. 6. The intersection of all regions inside the stability boundary of the (K_p, K_d) plane is the sensitivity region for the discrete-time system.

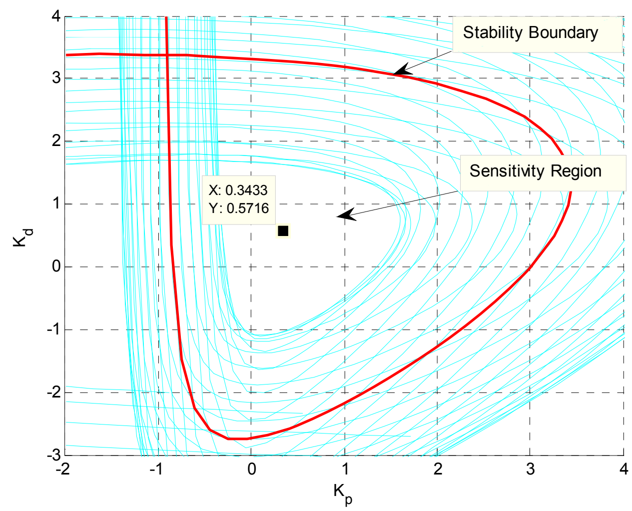

Fig. 6 Stability boundary and sensitivity region of discrete-time system in the (K_p, K_d) plane

To verify the results, an arbitrary controller from this region is chosen, giving us the PID controller

$$G_c(\gamma) = 0.34 + \frac{0.1}{\gamma} + \frac{0.57}{1 + 0.1\gamma}. \quad (30)$$

The substitution of (28) and (30) into (8) gives $\|S(\beta)\|_\infty = 1.23$. As the magnitude of the sensitivity function is less than 2, the design goal is met for the discrete-time system.

In order to compare these results with the continuous-time PID controller, equations (18) and (23) are used in the (K_p, K_d) plane for a fixed value of $\tilde{K}_i = 0.1$. The stability boundary and the region that satisfies the H_∞ sensitivity constraint are shown in Fig. 7. The intersection of all regions inside the stability boundary of the (K_p, K_d) plane is the H_∞ sensitivity region for the continuous-time system.

Comparing Fig. 6 and Fig. 7, we can see that while the stability boundary and the H_∞ sensitivity regions are similar for the discrete-time and continuous-time controllers, the continuous-time regions are larger.

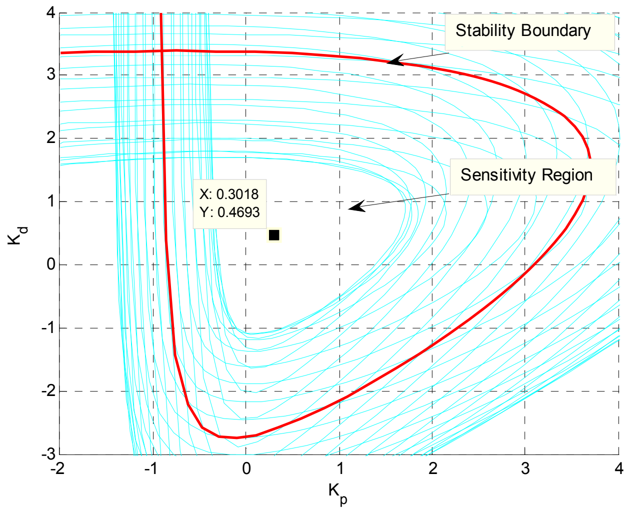

Fig. 7 Stability boundary and sensitivity region of continuous-time system in the (K_p, K_d) plane

The third method is applied in the (K_i, K_d) plane for a fixed value of $\tilde{K}_p = 0.5$. Plots of $K_p(\omega, \theta_S, \infty, T_0)$ and $K_p(\omega, \theta_S, \gamma_0, T_0)$ (from (15)) for various values of $\theta_S \in [0, 2\pi)$ are shown in Fig. 8 for the discrete-time system. For each curve, the ω_i are the frequencies at which $K_p(\omega, \theta_S, \gamma_0, T_0) = \tilde{K}_p = 0.5$. Each ω_i is substituted into (25) to find the required boundaries. In addition, we have the boundary at $K_i(0, \theta_S, \gamma_0, T_0) = 0$ for the discrete-time system.

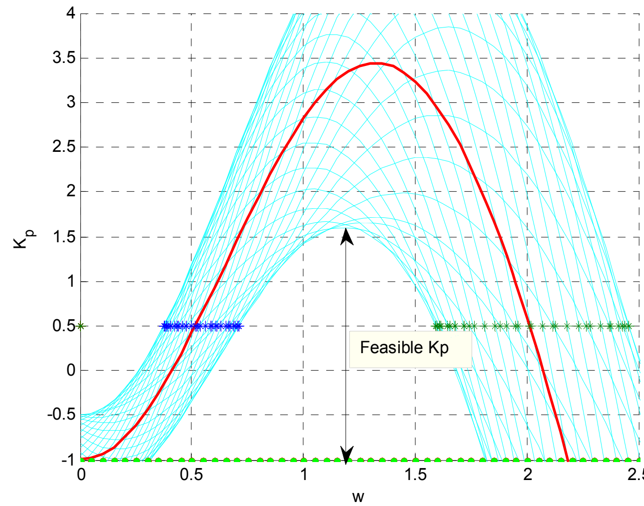

Fig. 8 Plots of $K_p(\omega, \theta_S, \gamma_0, T_0)$ versus ω used to find values of ω_i for discrete-time system

The region that satisfies the sensitivity constraint and the stability boundary is shown in Fig. 9. The intersection of all regions inside the stability boundary of the (K_i, K_d) plane is the H_∞ sensitivity region for the discrete-time system.

To verify the results, an arbitrary controller from this region is chosen, giving us the discrete-time PID controller

$$G_c(\gamma) = 0.5 + \frac{0.24}{\gamma} + \frac{0.69\,\gamma}{1 + 0.1\gamma}. \qquad (31)$$

The substitution of (28) and (31) into (8) gives $\|S(\beta)\|_\infty = 1.29$. As the magnitude of sensitivity function is less than 2, the design goal is met for the discrete-time system.

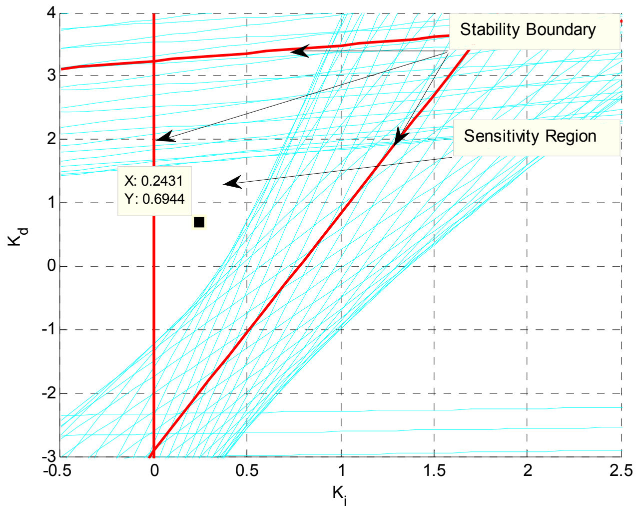

Fig. 9 Stability boundary and sensitivity region of discrete-time system in the (K_i, K_d) plane

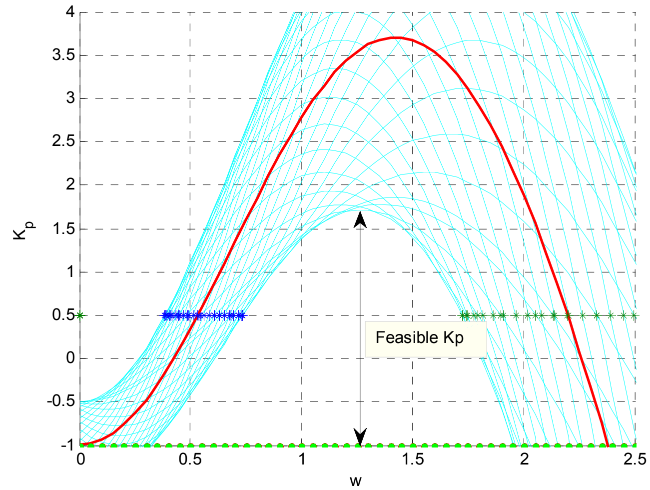

Fig. 10 Plots of $K_p(\omega,\theta_S,\gamma_0,0)$ versus ω used to find values of ω_i for continuous -time system

In order to compare these results with the continuous-time PID controller, plots of $K_p(\omega,\theta_S,\infty,0)$ and $K_p(\omega,\theta_S,\gamma_0,0)$ (from (18)) for various values of $\theta_S \in [0,2\pi)$ are shown in Fig. 10. For each curve, the ω_i are the frequencies at which $K_p(\omega,\theta_S,\gamma_0,0)=\tilde{K}_p=0.5$. Each ω_i is substituted into (26) to find the required boundaries. In addition, we have the boundary at $K_i(0,\theta_S,\gamma_0,0)=0$ for the continuous-time system. The stability boundary and the region that satisfies the H_∞ sensitivity constraint are shown in Fig. 11. The intersection of all regions inside the stability boundary of the (K_i, K_d) plane is the H_∞ sensitivity region for the continuous-time system.

Comparing Fig. 9 and Fig. 11, we can see that while the stability boundary and the H_∞ sensitivity regions are similar for the discrete-time and continuous-time controllers, the continuous-time regions are larger.

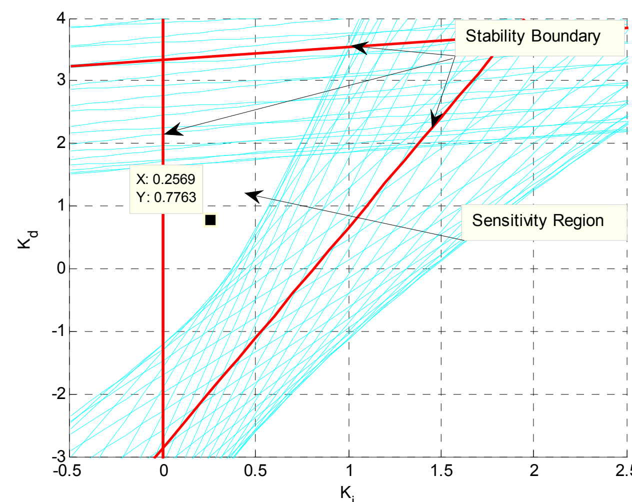

Fig. 11 Stability boundary and sensitivity region of continuous-time system in the (K_i, K_d) plane

4 Conclusions

A graphical technique was introduced for finding all achievable continuous-time or discrete-time PID controllers that satisfy an H_∞ sensitivity constraint for an arbitrary-order transfer function with time delay. This method is simple to understand and requires only the frequency response of the plant. A numerical example with a time delay was presented to demonstrate the application of this method. It was shown that the continuous-time and discrete-time designs can be understood under a common framework through the delta operator.

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