An alternative collection of structural invariants for matrix pencils under strict equivalence

A. DÍAZ, M. I. GARCÍA-PLANAS
Departament de Matemática Aplicada I
Universitat Politècnica de Catalunya,
C. Minería 1, Esc C, 1º-3ª
08038 Barcelona, Spain
E-mail: maria.isabel.garcia@upc.edu

Abstract:- We consider pairs of matrices (E, A), representing singular linear time invariant systems in the form $E\dot{x}(t) = Ax$ with $E, A \in M_{p\times n}(C)$ under restricted equivalence.

In this paper we obtain an alternative collection of invariants that they permit us to deduce the Kronecker canonical reduced form. As application we obtain a reduced form form systems under proportional and derivative feedback as well proportional and derivative output injection.

Key-Words:- Singular linear systems, canonical reduced form, structural invariants.

AMS Classification: 15A04, 15A21, 93B52.

1 Introduction

We denote by $M_{p\times n}(C)$ the space of complex matrices having p rows and n columns, and in the case which $p = n$ we write $M_n(C)$.

We consider the set $\mathcal{M}$ of pairs of matrices (E, A) representing families of singular linear time invariant systems in the form $E\dot{x} = Ax$ with $E, A \in M_{p\times n}(C)$.

Singular linear systems are found in engineering systems such as electrical circuit network, power systems for example. They can be treated under different point of view.

A manner to understand the properties of the system is treating it by purely algebraic techniques. The main aspect of this approach is defining an equivalence relation preserving these properties.

Having defined an equivalence relation, the standard procedure then is to look for a canonical form, that is to say to look for a pair of matrices which is equivalent to a given pair and which has a simple form from which we can directly read off the properties of the corresponding singular system. In order to obtain the canonical reduced form we try to find a complete system of invariants for pairs of matrices. The proposed approach can be applied to obtain a canonical reduced form of systems where feedbacks and output injections are considered.

In this paper we deal with equivalence relation between systems accepting one or both of the following transformations: basis change in the state space and premultiplication by an invertible manifold. A canonical reduced form can be derived associating a matrix pencil to the system characterized by two sets of minimal indices (row and column minimal indices), and sets of finite and infinite elementary divisors (the classical invariants). In this paper we present an alternative complete system of structural invariants based in computation of the ranks of certain matrices, which permits an easy characterization of the equivalence classes. An alternative system of structural invariants also based in computation of the ranks of certain matrices for standard systems was obtained by García-Planas and Magret in [10], and for generalized regularizable systems a collection of invariants was also obtained by García-Planas in [9]. The Kronecker canonical form provides a variety of applications in control systems theory as we can see in [4], [13] among others.

The paper is organized as follows

In section §2, a brief resume of notations is presented.

In section §3, an equivalence relation is defined.

In section §4, a matrix pencil is associated to the pair of matrices. After to observe that the strict equivalence of the pencils, is related to the equivalence relation of the pairs, it is possible to make use of the Kronecker reduced form for pencils in order to obtain a canonical reduced form for pairs of matrices.

In section §5, A complete system of invariants for pairs of matrices under equivalence relation considered, is obtained.

In section §6, as application we obtain a reduced form of a pencil associated to a singular system under proportional and derivative feedback as well proportional and derivative output injection.

And finally in §7 conclusions are presented.

2 Notations

In the sequel we will use the following notations.

- I_n denotes the n-order identity matrix,
- N denotes a nilpotent matrix in its reduced form $N = \operatorname{diag}(N_1, \dots, N_\ell)$, $N_i = \begin{pmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{pmatrix} \in M_{n_i}(C)$,
- J denotes the Jordan matrix $J = \operatorname{diag}(J_1, \dots, J_t)$, $J_i = \operatorname{diag}(J_{i_1}, \dots, J_{i_s})$, $J_{i_j} = \lambda_i I + N$,
- $L = \operatorname{diag} = (L_1, \dots, L_q)$, $L_j = \begin{pmatrix} I_{n_j} & 0 \end{pmatrix} \in M_{n_j \times (n_j+1)}(C)$,
- $R = \operatorname{diag}(R_1, \dots, R_p)$, $R_{n_j} = \begin{pmatrix} 0 & I_{n_j} \end{pmatrix} \in M_{n_j \times (n_j+1)}(C)$.

3 Equivalence relation

The description equation for mathematical model may be obtained by selecting appropriate state variables. The selection of these variables is not unique having no uniqueness of the model. So we are interested in to study the relationships between the state variable models, defining an appropriate equivalence relation.

The standard transformations in state space: $x(t) = Px_1(t)$, and premultiplication by an invertible matrix: $QE\dot{x}(t) = QAx(t)$, realized over generalized systems relate them in the following manner, two systems are related when one can be obtained from the other by means of one, or both, of the transformations considered. In fact, this transformations define an equivalence relation in the corresponding space of pairs of matrices in the following manner.

Definition 1 *two pairs of matrices $(E_i, A_i) \in \mathcal{M}$, $i = 1, 2$, are equivalent if and only if there exist matrices $P \in Gl(n; \mathbb{C})$ and $Q \in Gl(p; \mathbb{C})$ such that*

$$(E_2, A_2) = \quad (QE_1P, QA_1P),$$

That we can write in a matrix form in the following manner

$$\begin{pmatrix} E_2 & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$$

It is straightforward the following proposition.

Proposition 1 *The equivalence relation defined over the space of pairs of matrices, is an equivalence relation.*

Clearly, the relation defined possesses reflexivity, transitivity, and symmetry.

Generally, the matrices P and Q which transfer a pair of matrices (E_1, A_1) to the equivalent pair (E_2, A_2) are not unique.

Example 1 The pairs

$$(E_1, A_1) = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$(E_2, A_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

are equivalent because of, for $Q = \begin{pmatrix} 0 & 1/2 \\ 1 & -1/2 \end{pmatrix}$ and $P = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}$ we have that $(E_2, A_2) = (QE_1P, QA_1P)$.

4 Associated pencil

Given a pair of matrices $(E, A) \in \mathcal{M}$ we can associate in a natural way the following matrix pencil $H(\lambda) = \lambda E + A$. It is easy to proof the following proposition

Proposition 2 *Two pairs of matrices in $\mathcal{M}$ are equivalent if and only if the associated pencils are strictly equivalent.*

Proof. Straightforward.

□

As a consequence we can apply Kronecker's theory of pencils of matrices as presented in [7].

Corollary 1 *Let $(E, A) \in \mathcal{M}$ a pair of matrices. Then, the associated pencil is equivalent to $\lambda F + G$ with*

$$F = \begin{pmatrix} L & & & \\ & L^T & & \\ & & I_1 & \\ & & & N \end{pmatrix}, \text{ and } G = \begin{pmatrix} R & & & \\ & R^T & & \\ & & J & \\ & & & I_2 \end{pmatrix}.$$

Remember the definition of eigenvalue of a pair of matrices, necessary for construction of matrix J in the canonical reduced form.

Definition 2 *Let $(E, A) \in \mathcal{M}$ be a pair of matrices and $H(\lambda)$ its associated pencil. The value $\lambda_0 \in \mathbb{C}$ is an eigenvalue of $H(\lambda)$ if and only if*

$$\operatorname{rank} H(\lambda_0) < \operatorname{rank} H(\lambda).$$

We denote by $\sigma(E, A)$, the spectrum of the pencil, that is to say the set of eigenvalues of the pencil:

$$\sigma(E, A) = \{\lambda_i \in \mathbb{C} \mid \operatorname{rank} H(\lambda_i) < \operatorname{rank} H(\lambda)\}.$$

It is easy to observe that $\sigma(E, A)$ is an empty set or it is a finite set.

Example 2 a) Let

$$H(\lambda) = \lambda E + A = \lambda \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

for all $\lambda_0 \neq 0$ we have $\operatorname{rank} H(\lambda_0) = \operatorname{rank} H(\lambda) = 3$ and for $\lambda_0 = 0$ $\operatorname{rank} H(0) = 2 < \operatorname{rank} H(\lambda)$. Then

$$\sigma(E, A) = \{0\}.$$

b) Let

$$H(\lambda) = \lambda E + A = \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

for all λ_0 we have $\operatorname{rank} H(\lambda_0) = \operatorname{rank} H(\lambda) = 3$. Then

$$\sigma(E, A) = \emptyset.$$

Proposition 3 *The eigenvalues are invariant under equivalence relation considered.*

Proof.

$$\begin{aligned} &\operatorname{rank} H_2(\lambda_i) = \operatorname{rank} (\lambda_i E_2 + A_2 = \\ &= \operatorname{rank} Q(\lambda_i E_1 + A_1)P = \\ &= \operatorname{rank} QH_1(\lambda_i)P = \operatorname{rank} H_1(\lambda_i). \end{aligned}$$

We observe that the associate pencil to the pair (E_2, A_2) in example 1, is in its Kronecker reduced form.

We can obtain the Kronecker canonical reduced form computing a collection of discrete and continuous invariants.

Theorem 1 *Let $\big(\lambda E + A\big)$ be a matrix pencil under strict equivalence. Each equivalence class is characterized by the following set of structural invariants.*

i) $\omega_1 \geq \cdots \geq \omega_s \geq 1$*: Segre characteristic of infinite zeroes.*

ii) $k_1(\lambda) \geq \cdots \geq k_{j(\lambda)}(\lambda) \geq 1$*: Segre characteristic of eigenvalue* λ*.*

iii) $\epsilon_1 \geq \cdots \geq \epsilon_{r_\epsilon} > \epsilon_{r_\epsilon+1} = \cdots = \epsilon_r = 0$: *column minimal indices.*

iv) $(\eta_1 \geq \cdots \geq \eta_{l_\eta} > \eta_{l_\eta+1} = \cdots = \eta_l = 0$: *row minimal indices.*

Corollary 2 *Let $\lambda E + A \in M_{p\times n}$ be a matrix pencil. Then*

$$n=\sum_{i=1}^s \omega_i + \sum_{i=1}^u \sum_{j=1}^{j(\lambda_i)} k_j(\lambda_i) + \sum_{i=1}^{l_\eta} \eta_i + \sum_{i=1}^{r_\epsilon} \epsilon_i + r_\epsilon + r_0.$$

$$p=\sum_{i=1}^s \omega_i + \sum_{i=1}^u \sum_{j=1}^{j(\lambda_i)} k_j(\lambda_i) + \sum_{i=1}^{r_\epsilon} \epsilon_i + \sum_{i=1}^{l_\eta} \eta_i + l_\eta + l_0.$$

where $r_0 = r - r_\epsilon$ and $l_0 = l - l_\eta$ denote the number of zero columns and zero rows respectively.

Definition 3 *Given a pair of matrices (E, A), we call rank of the pair and we will denote by r_n to the rank of the associated pencil $H(\lambda)$:*

$$r_n = \operatorname{rank}(E, A) = \operatorname{rank}(\lambda E + A).$$

3 Sequences of matrices associated to (E, A)

Let $(E, A) \in \mathcal{M}$ be a pair of matrices, for all $\ell = 1, 2, \dots$ we define the following matrices.

$\mathcal{H}_1 = E$,
$\mathcal{H}_2 = \begin{pmatrix} E & \\ A & E \end{pmatrix}$,
$\vdots$
$$\mathcal{H}_\ell = \begin{pmatrix} E & & & & \\ A & E & & & \\ & A & E & & \\ & & \ddots & \ddots & \\ & & & A & E \end{pmatrix},$$

$\mathcal{C}_1 = \begin{pmatrix} E \\ A \end{pmatrix}$,
$\vdots$
$$\mathcal{C}_\ell = \begin{pmatrix} E & & & \\ A & E & & \\ & A & E & \\ & & \ddots & \ddots \\ & & & E \\ & & & A \end{pmatrix},$$

$\mathcal{O}_1 = \begin{pmatrix} E & A \end{pmatrix}$,
$\vdots$
$$\mathcal{O}_\ell = \begin{pmatrix} E & A & & & \\ & E & A & & \\ & & \ddots & \ddots & \\ & & & E & A \end{pmatrix},$$

$\mathcal{J}_1 = \lambda E + A$,
$\mathcal{J}_2(\lambda) = \begin{pmatrix} \lambda E+A & \\ E & \lambda E+A \end{pmatrix}$,
$\vdots$
$$\mathcal{J}_\ell(\lambda) = \begin{pmatrix} \lambda E+A & & & \\ E & \lambda E+A & & \\ & E & \lambda E+A & \\ & & \ddots & \ddots \\ & & & E & \lambda E+A \end{pmatrix},$$ for all $\lambda \in \mathbb{C}$.

Remark 2 We are interested in $\mathcal{J}_\ell(\lambda)$ for $\lambda \in \sigma(E, A)$. For other values of λ the matrices $\mathcal{J}_\ell(\lambda)$ are full rank for all ℓ.

Proposition 4 *The ranks of the matrices $\mathcal{H}_\ell$, $\mathcal{J}_\ell(\lambda)$, $\mathcal{C}_\ell$, $\mathcal{O}_\ell$, for all $\ell = 1, 2, \dots$ are invariant under equivalence relation considered.*

Proof.

Let (E_1, A_1) and (E_2, A_2) two equivalent pairs of matrices, then there exist invertible matrices Q and P such that $(E_2, A_2) = (QE_1P, QA_1P)$. So,

$$\begin{pmatrix} E_2 & & & & \\ A_2 & E_2 & & & \\ & A_2 & E_2 & & \\ & & \ddots & \ddots & \\ & & & A_2 & E_2 \end{pmatrix} =$$
$$\begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix} \begin{pmatrix} E_1 & & & & \\ A_1 & E_1 & & & \\ & A_1 & E_1 & & \\ & & \ddots & \ddots & \\ & & & A_1 & E_1 \end{pmatrix} \begin{pmatrix} P & & \\ & \ddots & \\ & & P \end{pmatrix}.$$

Consequently, matrices $\mathcal{H}_\ell(E_1, A_1)$ and $\mathcal{H}_\ell(E_2, A_2)$ have the same rank.

Analogously, we can prove the invariance of ranks of the other matrices. □

We denote by

- $r^{\mathcal{H}}_\ell = \operatorname{rank} \mathcal{H}_\ell$,
- $r_\ell(\lambda) = \operatorname{rank} \mathcal{J}_\ell(\lambda)$,
- $r^{\mathcal{C}}_\ell = \operatorname{rank} \mathcal{C}_\ell$,
- $r^{\mathcal{O}}_\ell = \operatorname{rank} \mathcal{O}_\ell$.

Example 3

Let (E, A) with $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

1. $r_2 = 1$
2. $r^{\mathcal{H}}_\ell = \ell$
3. $r_\ell(\lambda) = \ell$
4. $r^{\mathcal{C}}_\ell = \ell + 1$
5. $r^{\mathcal{O}}_\ell = \ell$

Theorem 2 *Let $(E, A) \in \mathcal{M}$ be a pair of matrices of rank r_n. Then, for all $\ell = 1, 2, \dots$*

$$\begin{aligned} r^{\mathcal{H}}_\ell &= r_n\ell - \textstyle\sum_{i=1}^{s} \min\{\ell, \omega_i\} \\ r_\ell(\lambda) &= r_n\ell - \textstyle\sum_{i=1}^{j(\lambda)} \min\{\ell, k_i(\lambda)\} \\ r^{\mathcal{C}}_\ell &= r_n\ell + \textstyle\sum_{i=1}^{r_\epsilon} \min\{\ell, \epsilon_i\} \\ r^{\mathcal{O}}_\ell &= r_n\ell + \textstyle\sum_{i=1}^{l_\eta} \min\{\ell, \eta_i\} \end{aligned}$$

Proof. Because of proposition 3, it suffices to compute these collection of numbers, for an equivalent pair such that the associated pencil is in its Kronecker reduced form. □

(A similar result is obtained in [12]).

It is easy to deduce the following results.

Corollary 3 *For all pair $(E, A) \in \mathcal{M}$ we have*

$$\begin{aligned} s &= r_n - r^{\mathcal{H}}_1, \\ j(\lambda) &= r_n - r_1(\lambda), \\ r_\epsilon &= r^{\mathcal{C}}_1 - r_n, \\ l_\eta &= r^{\mathcal{O}}_1 - r_n, \text{ and} \\ r_n &= r^{\mathcal{H}}_{n+1} - r^{\mathcal{H}}_n = r^{\mathcal{C}}_{n+1} - r^{\mathcal{C}}_n = \\ &= r^{\mathcal{O}}_{n+1} - r^{\mathcal{O}}_n = r_{n+1}(\lambda) - r_n(\lambda). \end{aligned}$$

Proof. For $\ell = 1$, we have

$$\begin{aligned} r^{\mathcal{H}}_1 &= r_n \cdot 1 - \textstyle\sum_{i=1}^{s} \min(1, \omega_1) \\ &= r_n - (1 + \overset{s}{\ldots} + 1) = r_n - s. \end{aligned}$$

Analogously, we can deduce the other expressions.

□

Example 4 Taking the same pair in the example 3, we have

1. $s = 0$
2. $j(\lambda) = 0$
3. $r_\epsilon = 1$
4. $l_\eta = 0$

Corollary 4 *For all pair $(E, A) \in \mathcal{M}$ we have*

$$\begin{aligned} r_n &= \textstyle\sum_{i=1}^{s} \omega_i + \sum_{i=1}^{u} \sum_{j=1}^{j(\lambda_i)} k_j(\lambda_i) + \\ &+ \textstyle\sum_{i=1}^{r_\epsilon} \epsilon_i + \sum_{i=1}^{l_\eta} \eta_i. \end{aligned}$$

Theorem 3 *Let $(E, A) \in \mathcal{M}$ a pair of matrices. Then*

1. *The numbers*
$$\begin{aligned} s_1 &= r^{\mathcal{H}}_1 - 2r^{\mathcal{H}}_0 \\ s_{\ell+1} &= r^{\mathcal{H}}_{\ell+1} - 2r^{\mathcal{H}}_\ell + r^{\mathcal{H}}_{\ell-1},\ \ell = 1, 2, \dots \end{aligned}$$
determine the quantity of blocks of size ℓ corresponding to the infinite zeroes in the Kronecker reduced form of the associated pencil.

2. *The numbers*
$$\begin{aligned} j_1(\lambda) &= r_1(\lambda) - 2r_0(\lambda) \\ j_{\ell+1}(\lambda) &= r_{\ell+1}(\lambda) - 2r_\ell(\lambda) + r_{\ell-1}(\lambda), \\ &\quad \ell = 1, 2, \dots \end{aligned}$$
determine the quantity of Jordan blocks of size ℓ associated to the eigenvalue λ of the pencil $H(\lambda)$.

Proof. It suffices to compute the ranks of the matrices $\mathcal{H}_i$ of a Kronecker reduced form pencil. □

Remark 3 let $(E, A) \in \mathcal{M}$ a pair of matrices and $H(\lambda)$ its associated pencil. If λ is not in the spectrum of $H(\lambda)$, we have that $j_\ell(\lambda) = 0,\ \ell = 1, 2, \dots$.

Theorem 4 ([17]) *Let $(E, A) \in \mathcal{M}$ be a pair of matrices. Then,*

1. *the numbers*

$$r_1 = -r_2^{\mathcal{C}} + 2r_1^{\mathcal{C}}$$
$$r_\ell = -r_{\ell+1}^{\mathcal{C}} + 2r_\ell^{\mathcal{C}} - r_{\ell-1}^{\mathcal{C}},\ \ell = 2,3,\dots$$

determine the quantity of column minimal indices of size ℓ appearing in $\lambda E + A$,

2. *the numbers*

$$l_1 = -r_2^{\mathcal{O}} + 2r_1^{\mathcal{O}}$$
$$l_\ell = -r_{\ell+1}^{\mathcal{O}} + 2r_\ell^{\mathcal{O}} - r_{\ell-1}^{\mathcal{O}},\ \ell = 2,3,\dots$$

determine the quantity of row minimal indices of size ℓ appearing in $\lambda E + A$.

5 An alternative complete systems of invariants

Now we construct an alternative method to obtain the canonical reduced form of a pencil $\lambda E + A$. These method is deduced from the following numbers $r_\ell^{\mathcal{H}}$, $r_\ell(\lambda)$, $r_\ell^{\mathcal{C}}$ y $r_\ell^{\mathcal{O}}$, $\ell = 1,2,\dots$.

Definition 4 *For all pair of matrices (E, A), we will call infinite zeros numbers and we will write $r_i^{\mathcal{CO}}$, to*

$$\begin{aligned} r_1^{\mathcal{CO}} &= r_n - r_1^{\mathcal{H}} \\ r_2^{\mathcal{CO}} &= r_1^{\mathcal{H}} - r_2^{\mathcal{H}} + r_n \\ &\vdots \\ r_\ell^{\mathcal{CO}} &= r_{\ell-1}^{\mathcal{H}} - r_\ell^{\mathcal{H}} + r_n, \quad \ell = 2,3,\dots \end{aligned}$$

Proposition 5 *Let $(E, A) \in \mathcal{M}$ be a pair and $H(\lambda)$ its associated pencil. Each $r_\ell^{\mathcal{CO}}, \ell = 1,2,\dots$ determine the quantity of blocks corresponding to the infinite zeroes of size greater than $\ell - 1$, that they appear in $H(\lambda)$. The $r^{\mathcal{CO}}$-numbers verify:*

$$r_1^{\mathcal{CO}} \geq r_2^{\mathcal{CO}} \geq \cdots \geq r_{\ell_1}^{\mathcal{CO}} \geq r_{\ell_1+1}^{\mathcal{CO}} = \cdots = 0.$$

Proof.

$$\begin{aligned} s =& \ r_n - r_1^{\mathcal{H}} \\ s - s_1 =& \ r_1^{\mathcal{H}} - r_2^{\mathcal{H}} + r_n \\ &\vdots \\ s - \textstyle\sum_{i=1}^{\ell-1} s_i =& \ r_{\ell-1}^{\mathcal{H}} - r_\ell^{\mathcal{H}} + r_n, \\ \ell = 2,3,\dots& \end{aligned}$$

□

Corollary 5 *The infinite zeroes indices $(\omega_1, \omega_2, \dots, \omega_s)$ are the conjugate partition of the set of non zero numbers $r_\ell^{\mathcal{CO}}$.*

Definition 5 *For all pair of matrices (E, A), we will call Jordan numbers corresponding to the eigenvalue λ and we will write $r_i^{\overline{\mathcal{CO}}}(\lambda)$, to*

$$\begin{aligned} r_1^{\overline{\mathcal{CO}}}(\lambda) &= r_n - r_1(\lambda) \\ r_2^{\overline{\mathcal{CO}}}(\lambda) &= r_1(\lambda) - r_2(\lambda) + r_n \\ &\vdots \\ r_\ell^{\overline{\mathcal{CO}}}(\lambda) &= r_{\ell-1}(\lambda) - r_\ell(\lambda) + r_n, \\ \ell =& \ 2,3,\dots \end{aligned}$$

Proposition 6 *Let $(E, A) \in \mathcal{M}$ be a pair and $H(\lambda)$ its associated pencil. Each $r_\ell^{\overline{\mathcal{CO}}}(\lambda), \ell = 1,2,\dots$ determine the quantity of Jordan blocks of size greater than $\ell - 1$ associated to the eigenvalue λ appearing in $\lambda E + A$. The $r^{\overline{\mathcal{CO}}}$-numbers verify:*

$$\begin{aligned} r_1^{\overline{\mathcal{CO}}}(\lambda) \geq \cdots \geq r_{\ell(\lambda)}^{\overline{\mathcal{CO}}}(\lambda) \geq \\ \geq r_{\ell(\lambda)+1}^{\overline{\mathcal{CO}}}(\lambda) = \cdots = 0. \end{aligned}$$

Proof.

$$\begin{aligned} j(\lambda) =& \ r_n - r_1(\lambda) \\ j(\lambda) - j_1(\lambda) =& \ r_1(\lambda) - r_2(\lambda) + r_n \\ &\vdots \\ j(\lambda) - \textstyle\sum_{i=1}^{\ell-1} j_i(\lambda) =& \ r_{\ell-1}(\lambda) - r_\ell(\lambda) + r_n, \\ \ell = 2,3,\dots.& \end{aligned}$$

□

Corollary 6 *The Segre characteristic $(k_1(\lambda), k_2(\lambda), \dots, k_{j(\lambda)}(\lambda))$ is the conjugate partition of non zero $r_\ell^{\overline{\mathcal{CO}}}(\lambda)$ numbers.*

Definition 6 *For all pair of matrices (E, A), we will call column minimal numbers and we will write $r_i^{\mathcal{C}\overline{\mathcal{O}}}$ to*

$$\begin{aligned} r_0^{\mathcal{C}\overline{\mathcal{O}}} &= n - r_n \\ r_1^{\mathcal{C}\overline{\mathcal{O}}} &= r_1^{\mathcal{C}} - r_n \\ r_2^{\mathcal{C}\overline{\mathcal{O}}} &= r_2^{\mathcal{C}} - r_1^{\mathcal{C}} - r_n \\ &\vdots \\ r_\ell^{\mathcal{C}\overline{\mathcal{O}}} &= r_\ell^{\mathcal{C}} - r_{\ell-1}^{\mathcal{C}} - r_n, \quad \ell = 2, 3, \dots \end{aligned}$$

Proposition 7 *Let $(E, A) \in \mathcal{M}$ be a pair of matrices and $H(\lambda)$ its associated pencil each $r_i^{\mathcal{C}\overline{\mathcal{O}}}$ number determine the quantity of column minimal indices of size greater than $\ell - 1$ that appear in $\lambda E + A$. These $r_i^{\mathcal{C}\overline{\mathcal{O}}}$-numbers verify*

$$r_0^{\mathcal{C}\overline{\mathcal{O}}} \geq r_1^{\mathcal{C}\overline{\mathcal{O}}} \geq \dots \geq r_{\ell_2-1}^{\mathcal{C}\overline{\mathcal{O}}} \geq r_{\ell_2}^{\mathcal{C}\overline{\mathcal{O}}} = \dots = 0$$

Proof.

$$\begin{aligned} r = r_\epsilon + r_0 = &\quad n - r_n \\ r_\epsilon = &\quad r_1^{\mathcal{C}} - r_n \\ r_\epsilon - r_1 = &\quad r_2^{\mathcal{C}} - r_1^{\mathcal{C}} - r_n \\ &\vdots \\ r_\epsilon - \textstyle\sum_{i=1}^{\ell-1} r_i = &\quad r_\ell^{\mathcal{C}} - r_{\ell-1}^{\mathcal{C}} - r_n, \ \ell = 2, 3, \dots \end{aligned}$$

□

Corollary 7 *let $(k_1^\epsilon, \dots, k_{r_\epsilon}^\epsilon, k_{r_\epsilon+1}^\epsilon, \dots, k_r^\epsilon)$ be the conjugate partition of the non zero $r_i^{\mathcal{C}\overline{\mathcal{O}}}$-numbers. Then the non-negative numbers*

$$(\epsilon_1, \dots, \epsilon_{r_\epsilon}, 0, \dots, 0) = \\ (k_1^\epsilon - 1, \dots, k_{r_\epsilon}^\epsilon - 1, k_{r_\epsilon+1}^\epsilon - 1, \dots, k_r^\epsilon - 1)$$

coincide with the column minimal indices that appear in $\lambda E + A$.

As a consequence we have.

Corollary 8 *$\lambda E + A$ has column full rank if and only if $r_n = n$.*

Corollary 9 *Let $\lambda E + A$ be a pencil having column full rank. Then, for each ℓ, matrices $\mathcal{C}_\ell(E, A)$ have column full rank.*

Proof.

$$r_1^{\mathcal{C}\overline{\mathcal{O}}} + \dots + r_\ell^{\mathcal{C}\overline{\mathcal{O}}} = r_\ell^{\mathcal{C}} - \ell r_n = r_\ell^{\mathcal{C}} - \ell n = 0.$$

□

Definition 7 *For all pair of matrices (E, A), we will call row minimal numbers and we will write $r_i^{\overline{\mathcal{C}}\mathcal{O}}$ to*

$$\begin{aligned} r_0^{\overline{\mathcal{C}}\mathcal{O}} &= p - r_n \\ r_1^{\overline{\mathcal{C}}\mathcal{O}} &= r_1^{\mathcal{O}} - r_n \\ r_2^{\overline{\mathcal{C}}\mathcal{O}} &= r_2^{\mathcal{O}} - r_1^{\mathcal{O}} - r_n \\ &\vdots \\ r_\ell^{\overline{\mathcal{C}}\mathcal{O}} &= r_\ell^{\mathcal{O}} - r_{\ell-1}^{\mathcal{O}} - r_n, \quad \ell = 2, 3, \dots \end{aligned}$$

Proposition 8 *Let $(E, A) \in \mathcal{M}$ be a pair of matrices and $H(\lambda)$ its associated pencil each $r_i^{\overline{\mathcal{C}}\mathcal{O}}$ number determine the quantity of row minimal indices of size greater than $\ell - 1$ that appear in $\lambda E + A$. These $r_i^{\overline{\mathcal{C}}\mathcal{O}}$-numbers verify*

$$r_0^{\overline{\mathcal{C}}\mathcal{O}} \geq \dots \geq r_{\ell_3-1}^{\overline{\mathcal{C}}\mathcal{O}} \geq r_{\ell_3}^{\overline{\mathcal{C}}\mathcal{O}} = \dots = 0$$

Proof.

$$\begin{aligned} l = l_\eta + l_0 = &\quad p - r_n \\ l_\eta = &\quad r_1^{\mathcal{O}} - r_n \\ l_\eta - l_1 = &\quad r_2^{\mathcal{O}} - r_1^{\mathcal{O}} - r_n \\ &\vdots \\ l_\eta - \textstyle\sum_{i=1}^{\ell-1} = &\quad r_\ell^{\mathcal{O}} - r_{\ell-1}^{\mathcal{O}} - r_n, \ell = 2, 3, \dots \end{aligned}$$

□

Corollary 10 *Let $(k_1^\eta, \dots, k_{l_\eta}^\eta, k_{l_\eta+1}^\eta, \dots, k_l^\eta)$ be the conjugate partition of the non-zero $r_i^{\overline{\mathcal{C}}\mathcal{O}}$ numbers. Then*

$$(\eta_1, \dots, \eta_{l_\eta}, 0, \dots, 0) = \\ (k_1^\eta - 1, \dots, k_{l_\eta}^\eta - 1, k_{l_\eta+1}^\eta - 1, \dots, k_l^\eta - 1)$$

coincide with the row minimal indices that its appear in $\lambda E + A$.

Corollary 11 *$\lambda E + A$ has full row rank if and only if $r_n = p$.*

Corollary 12 *Let $\lambda E + A$ a pencil having full row rank. Then for all ℓ, the matrices $\mathcal{O}_\ell(E, A)$ have full row rank.*

Proof.

$$r_1^{\overline{\mathcal{C}}\mathcal{O}} + \dots + r_\ell^{\overline{\mathcal{C}}\mathcal{O}} = r_\ell^{\mathcal{O}} - \ell r_n = r_\ell^{\mathcal{O}} - \ell p = 0$$

□

As a consequence we have the following results.

Theorem 7 *The pencil $H(\lambda) = \lambda E + A$ has full column and row rank if and only if $p = n$ and $r_n = n$.*

Theorem 8 *For all pair of matrices $(E, A) \in \mathcal{M}$, the collection of numbers*

i) $r_1^{\mathcal{C}\mathcal{O}} \geq \dots \geq r_{\ell_1}^{\mathcal{C}\mathcal{O}} \geq r_{\ell_1+1}^{\mathcal{C}\mathcal{O}} = \dots = 0,$

ii) $r_0^{\mathcal{C}\overline{\mathcal{O}}} \geq \dots \geq r_{\ell_2-1}^{\mathcal{C}\overline{\mathcal{O}}} \geq r_{\ell_2}^{\mathcal{C}\overline{\mathcal{O}}} = \dots = 0,$

iii) $r_0^{\overline{\mathcal{C}}\mathcal{O}} \geq \dots \geq r_{\ell_3-1}^{\overline{\mathcal{C}}\mathcal{O}} \geq r_{\ell_3}^{\overline{\mathcal{C}}\mathcal{O}} = \dots = 0,$

iv) $r_1^{\overline{\mathcal{C}\mathcal{O}}}(\lambda) \geq \dots \geq r_{\ell(\lambda)}^{\overline{\mathcal{C}\mathcal{O}}}(\lambda) \geq r_{\ell(\lambda)+1}^{\overline{\mathcal{C}\mathcal{O}}}(\lambda) = \dots = 0, \quad \lambda \in \mathbb{C}$

constitute a complete system of invariants.

Proof. The non-zero r-numbers permit us to deduce the collection of numbers

i) $\omega_1 \geq \dots \geq \omega_s \geq 1$

ii) $k_1(\lambda) \geq \dots \geq k_{j(\lambda)}(\lambda) \geq 1, \quad \lambda \in \sigma(E, A)$

iii) $\epsilon_1 \geq \dots \geq \epsilon_{r_\epsilon} > \epsilon_{r_\epsilon+1} = \dots = \epsilon_r = 0$

iv) $\eta_1 \geq \dots \eta_{l_\eta} > \eta_{l_\eta+1} = \dots = \eta_l = 0$

that correspond with the structural invariants of the associated pencil to the pair of matrices.

□

6 Application to the systems under proportional and derivative feedback as well proportional and derivative output injection

We consider the set of quadruples of matrices (E, A, B, C) representing families of singular linear time invariant systems in the form

$$\left.\begin{aligned} E\dot{x}(t) &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \qquad (1)$$

with $E, A \in M_{p\times n}(C)$, $B \in M_{p\times m}(C)$ and $C \in M_{q\times n}(C)$.

These equations arise in theoretical areas as differential equations on manifolds as well as in applied areas as systems theory and control, [14], [16].

Many interesting and useful equivalence relations between singular systems have been defined. We deal with the equivalence relation accepting one or more, of the following transformations: basis change in the state space, input space, output space, operations of state and derivative feedback, state and derivative output injection and to pre-multiply the first equation in (1) by an invertible matrix. That is to say.

Definition 8 *Two quadruples (E_i, A_i, B_i, C_i), $i = 1, 2$, are equivalent if and only if there exist matrices $P \in Gl(n; \mathbb{C})$,$Q \in Gl(p; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $S \in Gl(q; \mathbb{C})$, $F_E^B, F_A^B \in M_{m\times n}(\mathbb{C})$, $F_E^C, F_A^C \in M_{p\times q}(\mathbb{C})$ such that*

$$\begin{aligned} E_2 &= QE_1P + QB_1F_E^B + F_EC_1P, \\ A_2 &= QA_1P + QB_1F_A^B + F_AC_1P, \\ B_2 &= QB_1R, \\ C_2 &= SC_1P, \end{aligned}$$

To consider proportional and derivative feedback as well proportional and derivative output injection is implicit in definition of controllability and observability character of the system.

Definition 9 *A system (E, A, B, C) is controllable if and only if*

$$\begin{cases} \operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = n, \\ \operatorname{rank}\begin{pmatrix} sE + A & B \end{pmatrix} = n, \ \forall s \in \mathbb{C}, \end{cases}$$

Proposition 10 *Let (E, A, B, C) be a controllable system. Then, there exists:*

a) a derivative feedback F_E^B such that $\operatorname{rank}\left(E + BF_E^B\right) = n$,

b) a proportional feedback F_A^B such that $\operatorname{rank}\left(A + BF_A^B\right) = n$.

Definition 10 *A system (E, A, B, C) is observable if and only if*

$$\begin{cases} \operatorname{rank}\begin{pmatrix} E \\ C \end{pmatrix} = n, \\ \operatorname{rank}\begin{pmatrix} sE + A \\ C \end{pmatrix} = n, \ \forall s \in \mathbb{C}, \end{cases}$$

Proposition 11 *Let (E, A, B, C) be a observable system. Then, there exists:*

a) a derivative output injection F_E^C such that rank $\left(E + F_E^C C\right) = n$,

b) a proportional output injection F_A^C such that rank $\left(A + F_A^C C\right) = n$.

Given a quadruple of matrices (E, A, B, C), we can associate the following matrix pencil.

$$P(\lambda) = \begin{pmatrix} \lambda E + A & \lambda B & B \\ \lambda C & 0 & 0 \\ C & 0 & 0 \end{pmatrix},$$

and we have the following proposition.

Proposition 9 *Two quadruples are equivalent under equivalent relation considered if and only if the associates matrix pencils are strictly equivalent.*

So, we can compute the complete systems of invariants obtained in §5, to obtain qualitative properties of the system.

Corollary 13 *Let $P(\lambda)$ be a matrix pencil associated to the quadruple (E, A, B, C). Then $P(\lambda)$ its is equivalent to the pencil given in Corollary 1, $\lambda F + G$ with*

$$F = \begin{pmatrix} L & & & \\ & L^T & & \\ & & I_1 & \\ & & & N \end{pmatrix}, \quad G = \begin{pmatrix} R & & & \\ & R^T & & \\ & & J & \\ & & & I_2 \end{pmatrix}$$

Remark 4 Given a quadruple of matrices (E, A, B, C), will call eigenvalues of the quadruple to the eigenvalues of the associate pencil $P(\lambda)$, and we denote by $\sigma(E, A, B, C)$ the spectrum of the pencil.

Obviously, the collection of eigenvalues of a quadruple are invariant under equivalence relation considered.

7 Conclusion

We consider pairs of matrices (E, A), representing singular linear time invariant systems in the form $E\dot{x}(t) = Ax(t)$ with $E, A \in M_{p\times n}(C)$ under equivalence that accept basis change in the state space and premultiplication by an invertible matrix. After to observe that this equivalence corresponds with strict equivalence defined over associated pencil $\lambda E + A$ and the Kronecker reduced form can be used, in this paper we obtain an alternative collection of invariants that they permit us to deduce the canonical reduced form.

References

[1] I. Brás. *Diagonally Stable Tridiagonal Switched Linear Systems*, Wseas Transaction on Mathematics. **8**, (9), pp. 520-529, (2009).

[2] S. L. Campbell. "Singular Systems of Differential Equations". *Pitman*, San Francisco, (1980).

[3] L. Dai "Singular Control Systems". *Springer Verlag*. New York, (1989).

[4] J. Demmel, B. Kågström, *Accurate solutions of ill-posed problems in control theory*, SIAM J. Matrix Anal. Appl. **9**, pp. 126-145, (1988).

[5] Díaz A., García-Planas M.I., Tarragona S., *Local Perturbations of generalized systems under feedback and derivative feedback.* Computers and Mathematics with Applications, **56**, (4) pp 988-1000, (2008).

[6] Mª I. García-Planas, A. Díaz, *Canonical forms for multi-input reparable singular systems.* Wseas Transactions on Mathematics. **6** (4), pp. 601-608, (2007).

[7] F. R. Gantmacher, The Theory of Matrices, Vol. 1, 2, *Chelsea*, New York, (1959).

[8] Mª I. García-Planas, *A geometric approach of generalized linear systems.* Wseas trans. Circuits and Systems. vol 5 (3), pp. 1347-1352, (2004).

[9] Mª I. García-Planas, *Bifurcation diagrams of generic families of singular systems under proportional and derivative feedback*, Wseas Transactions on Mathematics. **7**, (2), pp. 48-58, (2008).

[10] Mª I. García-Planas, M.D. Magret, *An alternative System of Structural Invariants for Quadruples of Matrices*, *Linear Algebra and its Applications.* **291**, (1-3), pp. 83-102, (1999).

[11] Mª I. García-Planas, M.D. Magret, S. Tarragona, *Relationship between versal deformations under different equivalence relations in the space of singular systems.* Wseas trans. Circuits and Systems. vol 3 (10) pp. 2217-2222, (2004).

[12] S. Iwata, R. Shimizu, *Combinatorial analysis of generic matrix pencils.* SIAM J.Matrix Anal. Appl. 29, pp. 245-259, (2007).

[13] A.S. Morse, Structural invariants of linear multivariable systems, SIAM J. Contr. **11**, pp. 446-465, (1973).

[14] H.H. Rosenbrock, *Structural Properties of Linear Dynamical Systems*, Int. J. Control, **20**, pp. 191-202, (1974).

[15] Tannenbaum A., *Invariance and System Theory: Algabraic and geometric Aspects*, Lecture Notes in Math. 845, Springer-Verlag, (1981).

[16] J.S. Thorp, *The Singular Pencil of Linear Dynamical System*, System. Int. J. Control, **18** (3), pp. 557-596, (1973).

[17] Williamson J., *On the equivalence of two singular matrix pencils*, Proc. Edin. Math. Soc., Series 2, **4**, (1934/36).