A Study of the Observability of Multidimensional Hybrid Linear Systems

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Abstract: A class of multidimensional hybrid linear systems is presented, with the time vector composed by q continuous-time real components and by r discrete-time integer ones, $q, r \geq 1$. The state equation is of multidimensional partial differential-difference type. A generalized variation-of-parameters formula is provided and it is used to obtain the state and the general response of the system.
The fundamental concept of observability is studied for these systems. An observability Gramian is introduced, which is a generalization of the Gramians corresponding to the classical 1D continuous-time and 1D discrete-time systems.
In the case of completely observable systems this Gramian is used to obtain a formula which provides the initial state of the system for any input-output pair. A list of observability criteria is given for time-invariant systems and the duality between the concepts of observability and reachability is emphasized.

Key–Words: multidimensional hybrid systems, observability, observability Gramian, time-varying systems, time-invariant systems.

1 Introduction

In the past three decades a lot of published paper and books have been designed to the theory of two-dimensional (2D) or, more generally, multidimensional (nD) systems, which become a distinct and important branch of the systems theory. The reasons for the increasing interest in this domain are on one side the richness in potential application fields and on the other side the richness and significance of the theoretical approaches. The list of application fields include circuits, control and signal processing, image processing, computer tomography, gravity and magnetic field mapping, seismology, control of multipass processes, etc. From the theoretical point of view, the domain of nD systems needs a specific approach, since many aspects of the 1D systems do not generalize and there are many nD systems phenomena which have no 1D systems counterparts. Various state space 2D discrete-time models have been proposed in literature by Roesser [18], Fornasini-Marchesini [6] Attasi [4], Eising [5] and others. Apparently different, in fact some of them are equivalent.

A quite new subdomain of the 2D systems theory is represented by the 2D hybrid models, whose state equation is of differential-difference type, having a continuous-time variable and a discrete-time one [9], [14], [16], [17]. These hybrid models have applications in various areas such as linear repetitive processes [8], [19], pollution modelling [7], long-wall coal cutting and metal rolling [20] or in iterative learning control synthesis [13].

The concept of observability which is fundamental in control theory was introduced for 1D systems by Kalman in [10], being imposed by engineering problems (see [11]). This concept and its dual (controllability) characterize the minimal systems. In the case of 2D systems some notions such as local and global controllability and observability were introduced, but they are not satisfactory from the point of view of minimality. In [12] the concepts of modal controllability/observability were defined and it was shown that a system is minimal iff it is modally controllable and observable, but these notions do not allow the richness of the characterizations of the 1D notions.

In the present paper the notion of observability is defined and it is analysed for a model of time-variable separable (q, r)-D hybrid (i.e. continuous-discrete) systems, which is a generalization of the class of 2D hybrid system studied in [15], this class being the continuous-discrete time-varying counterpart of Attasi's 2D discrete-time time-invariant model [4].

In Section 2 a variation-of-parameters formula is provided and it is used to obtain the formula of the state of the system, determined by an initial state and a given control, as well as the formula of the general response of the system.

Section 3 analyses the notions of unobservable states and completely observable systems. A (q, r)-D (multidimensional) observability Gramian is defined, which is a natural generalization of the Gramians corresponding to the 1-D continuous-time and discrete-time systems. Using this Gramian necessary and sufficient conditions of observability for time-varying systems are given, as well as the characterization of the subspace of unobservable states. In the case of completely observable systems a formula is provided which determines the initial state by knowing the control and the corresponding output.

Section 5 is devoted to time-invariant (q, r)-D hybrid systems. A list of necessary and sufficient conditions of observability is provided. The geometric characterization of the subspace of unobservable states is given in terms of invariant subspaces included in the kernel of the output matrix.

This study can be continued in many directions such as stability, positivity, 2D generalized systems, linear quadratic optimal control etc., by extending to the present approach some results presented for instance in [1], [2] and [3]. Another direction is represented by the application of this framework to the geometric approach of multitime systems studied in [21]-[25].

We shall use the following notations: $q \in \mathbf{N}$ and $r \in \mathbf{N}$ being the number of continuous and discrete variables respectively, a function $x(t_1, \ldots, t_q; k_1, \ldots, k_r)$, $t_i \in \mathbf{R}$, $k_i \in \mathbf{Z}$ will be sometimes denoted by $x(t; k)$, where $t = (t_1, \ldots, t_q)$, $k = (k_1, \ldots, k_r)$. By $\overline{m}$ with $m \in \mathbf{N}^*$ we denote the set $\{1, 2, \ldots, m\}$ and by $\mathcal{P}(\overline{m})$ the family of all subsets of $\overline{m}$.

By $s \leq t$, $(s < t)$, $s, t \in \mathbf{R}^q$ we mean $s_i \leq t_i$, $(s_i < t_i)$ $\forall i \in \bar{q}$ and a similar signification has $l \leq k$, $l, k \in \mathbf{Z}^r$; $(s; l) < (t; k)$ means $s \leq t$, $l \leq k$ and $(s; l) \neq (t; k)$. For $t^0, t^1 \in \mathbf{R}^q$ and $k^0, k^1 \in \mathbf{Z}^r$, $t^0 < t^1$, $k^0 < k^1$ we denote by $[t^0, t^1]$ and $[k^0, k^1]$ respectively the sets $[t^0, t^1] = \prod_{i=1}^{q} [t_i^0, t_i^1]$ and $[k^0, k^1] = \prod_{i=1}^{r} \{k_j^0, k_j^0 + 1, \ldots, k_j^1\}$.

If $\tau = \{i_1, \ldots, i_l\}$ is a subset of $\overline{m}$, $|\tau| := l$ and $\tilde{\tau} := \overline{m} \setminus \tau$; for $i \in \overline{m}$, $\tilde{i} := \overline{m} \setminus \{i\}$ and $\tilde{\tilde{i}} := \{i + 1, \ldots, m\}$. The notation $(\tau, \delta) \subset (\overline{q}, \overline{r})$ means that τ and δ are subsets of $\overline{q}$ and $\overline{r}$ respectively and $(\tau, \delta) \neq (\overline{q}, \overline{r})$. For $\tau = \{i_1, \ldots, i_l\}$ and $\delta = \{j_1, \ldots, j_h\}$ the operators $\dfrac{\partial}{\partial \tau}$ and σ_δ are defined by

$$\frac{\partial}{\partial \tau} x(t; k) = \frac{\partial^l}{\partial t_{i_1} \ldots \partial t_{i_l}} x(t; k), \ \sigma_\delta x(t; k) = x(t; k + e_\delta)$$

where $e_\delta = e_{j_1} + \ldots + e_{j_h}$, $e_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, 0, \ldots, 0) \in \mathbf{R}^r$; when $\tau = \overline{q}$ and $\delta = \overline{r}$ we denote $\partial / \partial \tau = \partial / \partial t$ and $\sigma_\delta = \sigma$.

If A_i, $i \in \overline{m}$ is a family of matrices, $\sum_{i \in \emptyset} A_i = 0$ and $\prod_{i \in \emptyset} A_i = I$.

If P is a positive definite matrix, one writes $P > 0$.

2 State space representation of (q, r)-D CDSs

The time set of the Attasi-type multidimensional system is $T = \mathbf{R}^q \times \mathbf{Z}^r$, $q, r \in \mathbf{N}^*$.

Definition 1. A (q, r)-D *continuous-discrete system* (CDS) is a set $\Sigma = (\{A_{ci} | i \in \bar{q}\}, \{A_{dj} | j \in \overline{r}\}, B, C, D)$ with $A_{ci}(t; k)$, $i \in \overline{q}$ and $A_{dj}(t; k)$, $j \in \overline{r}$ commuting $n \times n$ matrices $\forall t \in \mathbf{R}^q$, $\forall k \in \mathbf{Z}^r$ and $B(t; k)$, $C(t; k)$, $D(t; k)$ respectively $n \times m$, $p \times n$ and $p \times m$ real matrices, all these matrices being continuous with respect to $t \in \mathbf{R}^q$ for any $k \in \mathbf{Z}^r$; the state equation is

$$\begin{gathered} \frac{\partial}{\partial t} \sigma x(t; k) = \sum_{(\tau, \delta) \subset (\overline{q}, \overline{r})} (-1)^{q + r - |\tau| - |\delta| - 1} \times \\ \times \left(\prod_{i \in \tilde{\tau}} A_{ci}(t; k) \right) \left(\prod_{j \in \tilde{\delta}} A_{dj}(t; k) \right) \frac{\partial}{\partial \tau} \sigma_\delta x(t; k) + \\ + B(t; k) u(t; k) \end{gathered} \tag{1}$$

and the output equation is

$$y(t; k) = C(t; k) x(t; k) + D(t; k) u(t; k) \tag{2}$$

where

$$x(t; k) = x(t_1, \ldots, t_q; k_1, \ldots, k_r) \in \mathbf{R}^n$$

is the *state*, $u(t; k) \in \mathbf{R}^m$ is the *input* and $y(t; k) \in \mathbf{R}^p$ is the *output*. $X = \mathbf{R}^n$, $U = \mathbf{R}^n$ and $Y = \mathbf{R}^p$ are respectively the *state space*, the *input space* and the *output space* .

For $\tau = \{i_1, \ldots, i_l\} \subset \overline{q}$, $\delta = \{j_1, \ldots, j_h\} \subset \overline{r}$ and $t_i \in \mathbf{R}$, $i \in \tau$, $t_i^0 \in \mathbf{R}$, $i \in \tilde{\tau}$, $k_j \in \mathbf{Z}$, $j \in \delta$, $k_j^0 \in \mathbf{Z}$, $j \in \tilde{\delta}$ we use the notation

$$\begin{gathered} x(t_\tau, t_{\tilde{\tau}}^0; k_\delta, k_{\tilde{\delta}}^0) := x(t_1^0, \ldots, t_{i_1 - 1}^0, t_{i_1}, \\ t_{i_1 + 1}^0, \ldots, t_{i_l - 1}^0, t_{i_l}, t_{i_l + 1}^0, \ldots, t_q^0; \\ k_1^0, \ldots, k_{j_1 - 1}^0, k_{j_1}, k_{j_1 + 1}^0, \ldots, k_{j_h - 1}^0, k_{j_h}, \\ k_{j_h + 1}^0, \ldots, k_{j_r}^0). \end{gathered}$$

Let $\Phi_i(t_i, t_i^0; t_{\tilde{i}}; k)$ be the (continuous) fundamental matrix of $A_{ci}(t;k)$ with respect to the variables t_i, t_i^0, $i \in \overline{q}$, i.e. the unique matrix solution of the system $\frac{\partial Y}{\partial t_i}(t;k) = A_{ci}(t;k)Y(t;k), Y(t_{\tilde{i}}, t_i^0; k) = I$ for any $t_l \in \mathbf{R}$, $l \in \tilde{i}$ and $k \in \mathbf{R}^r$. If A_{ci} is a constant matrix then $\Phi_i(t_i, t_i^0; t_{\tilde{i}}, k) = e^{A_{ci}(t_i - t_i^0)}$.

The *discrete fundamental matrix* $F_j(t; k_j, k_j^0; k_{\tilde{j}})$ of the matrix $A_{dj}(t;k)$ is defined by

$$F_j(t; k_j, k_j^0; k_{\tilde{j}}) = \\ = \begin{cases} A_{dj}(t; k_j - 1, k_{\tilde{j}}) A_{dj}(t; k_j - 2, k_{\tilde{j}}) \dots A_{dj}(t; k_j^0, k_{\tilde{j}}) \\ \qquad \text{for} \quad k_j > k_j^0 \\ I_n \quad \text{for} \quad k = k_j^0, \end{cases}$$

for any $k_h \in \mathbf{Z}$, $h \in \tilde{j}$ and $t \in \mathbf{R}^q$.

$F_j(t; k_j, k_j^0; k_{\tilde{j}})$ is the unique matrix solution of the difference system

$$Y(t; k_j + 1, k_{\tilde{j}}) = A_{dj}(t;k) Y(t; k_j, k_{\tilde{j}}),$$
$$Y(t; k_j^0; k_{\tilde{j}}) = I.$$

If A_{dj} is a constant matrix then

$$F_j(t; k_j, k_j^0; k_{\tilde{j}}) = A_{dj}^{k_j - k_j^0}.$$

Remark 1. Under the hypothesis: (H) "The matrices $A_{dj}(t;k)$ are nonsingular for any $t \in \mathbf{R}^q$, $k \in \mathbf{Z}^r$", the discrete fundamental matrix F_j can be defined for $k_j < k_j^0$ by

$$F_j(t; k_j, k_j^0; k_{\tilde{j}}) = F_j(t; k_j^0, k_j; k_{\tilde{j}})^{-1}.$$

In this case the semigroup property

$$F_j(t; k_j, k_j^1; k_{\tilde{j}}) F_j(t; k_j^1, k_j^0; k_{\tilde{j}}) = F_j(t; k_j, k_j^0; k_{\tilde{j}})$$

holds for any k_j^0, k_j^1, k_j.

Definition 2. The vector $x^0 \in \mathbf{R}^n$ is called an *initial state* of the system Σ if

$$x(t_\tau, t_{\tilde{\tau}}^0; k_\delta, k_{\tilde{\delta}}^0) = \left(\prod_{i \in \tau} \Phi_i(t_i, t_i^0; t_{\tilde{i}}; k) \right) \cdot \\ \cdot \left(\prod_{j \in \delta} F_j(t; k_j, k_j^0; k_{\tilde{j}}) \right) x^0 \tag{3}$$

for any $(\tau, \delta) \subset (\overline{q}, \overline{r})$; equalities (3) are called the *initial conditions of* Σ.

Proposition 1. *The solution of the initial value problem*

$$\frac{\partial}{\partial t} \sigma x(t;k) = \sum_{(\tau,\delta) \subset (\overline{q}, \overline{r})} (-1)^{q+r-|\tau|-|\delta|-1} \cdot \\ \cdot \left(\prod_{i \in \tilde{\tau}} (\sigma_\delta A_{ci}(t;k)) \right) \left(\prod_{j \in \tilde{\delta}} A_{dj}(t,k) \right) \cdot \\ \cdot \frac{\partial}{\partial \tau} \sigma_\delta x(t;k) + f(t;k) \tag{4}$$

with the initial conditions (3) *is given by the generalized variation-of-parameters formula*

$$x(t;k) = \left(\prod_{i=1}^{q} \Phi_i(t_i, t_i^0; t_{\overline{i-1}}^0, t_{\tilde{\tilde{i}}}; k) \right) \cdot \\ \cdot \left(\prod_{j=1}^{r} F_j(t^0; k_j, k_j^0; k_{\overline{j-1}}^0, k_{\tilde{\tilde{j}}}) \right) x^0 + \\ + \int_{t_1^0}^{t_1} \dots \int_{t_q^0}^{t_q} \sum_{l_1 = k_1^0}^{k_1 - 1} \dots \sum_{l_r = k_r^0}^{k_r - 1} \left(\prod_{i=1}^{q} \Phi_i(t_i, s_i; s_{\overline{i-1}}, t_{\tilde{\tilde{i}}}; k) \right) \cdot \\ \cdot \left(\prod_{j=1}^{r} F_j(s; k_j, l_j + 1; l_{\overline{j-1}}, k_{\tilde{\tilde{j}}}) \right) f(s;l) ds_1 \dots ds_q; \tag{5}$$

here $s = (s_1, \dots, s_q)$, $l = (l_1, \dots, l_r)$ *and if for instance* $i = 1$, *then the corresponding variable* $t_{\overline{i-1}} = t_\emptyset^0$ *lacks;* $f : \mathbf{R}^q \times \mathbf{Z}^r \to \mathbf{R}^n$ *is a continuous function with respect to* $t \in \mathbf{R}^q$ *for any* $k \in \mathbf{Z}^r$.

Proof. We shall prove (5) firstly for the case $q = r = 1$, that is for the equation

$$\frac{\partial}{\partial t} x(t; k+1) = A_c(t; k+1) x(t; k+1) + \\ + A_d(t;k) \frac{\partial}{\partial t} x(t;k) - A_c(t;k) A_d(t;k) x(t;k) + f(t;k) \tag{6}$$

with the initial conditions

$$x(t; k_0) = \Phi(t, t_0; k_0) x_0, \quad x(t_0; k) = F(t_0; k, k_0) x_0 \tag{7}$$

where $\Phi(t, t_0; k_0)$ is the fundamental matrix of $A_c(t;k)$ and $F(t;k,k_0)$ is the discrete fundamental matrix of $A_d(t;k)$.

Let us consider the vector

$$z(t;k) = \frac{\partial}{\partial t} x(t;k) - A_c(t;k) x(t;k). \tag{8}$$

From (7) and the first property of the fundamental matrix we have

$$z(t;k_0) = \frac{\partial}{\partial t} x(t;k_0) - A_c(t;k_0) x(t;k_0) = \\ = \frac{\partial}{\partial t} \Phi(t; t_0, k_0) x_0 - A_c(t;k_0) \Phi(t; t_0, k_0) x_0 = 0,$$

hence $z(t;k_0) = 0, \forall t \geq t_0$. Equation (6) can be written as the 1D difference equation $z(t;k+1) = A_j(t;k)z(t;k) + f(t;k)$ whose solution, given by the discrete-time variation of parameters formula, is

$$z(t;k) = F(t;k,k_0)z(t,k_0) + \sum_{l=k_0}^{k-1} F(t;k,l+1)f(t,l)$$

hence

$$z(t,k) = \sum_{l=k_0}^{k-1} F(t;k,l+1)f(t,l) \tag{9}$$

since $z(t,k_0) = 0$. But the equation (8) can be written as the differential equation $\frac{\partial}{\partial t}x(t;k) = A_c(t;k)x(t;k) + z(t;k)$ and by the variation of parameters formula its solution is

$$x(t;k) = \Phi(t,t_0;k)x(t_0;k) + \int_{t_0}^{t} \Phi(t,s;k)z(s;k)ds.$$

By replacing $x(t_0;k)$ and $z(s;k)$ by their expressions (7) and (9) we get

$$\begin{array}{c} x(t;k) = \Phi(t,t_0;k)F(t_0;k,k_0)x_0 + \\ \int_{t_0}^{t} + \sum_{l=k_0}^{k-1} \Phi(t,s;k)F(s;k,l+1)f(s,l) \end{array} \tag{10}$$

hence formula (5) is true for $q = 1, r = 1$.

Assume that formula (5) is true for some $q, r \in \mathbf{N}$ and consider the equation (4) with $q + 1$ instead of q. We introduce the function

$$\begin{array}{c} z(t_1;t_2,\ldots,t_{q+1};k_1,\ldots,k_r) = \dfrac{\partial x}{\partial t_1}(t_1,t_2,\ldots,t_{q+1}; \\ k_1,\ldots,k_r) - A_{c1}(t_1,t_2,\ldots,t_{q+1};k_1,\ldots,k_r)\cdot \\ \cdot x(t_1,t_2,\ldots,t_{q+1};k_1,\ldots,k_r). \end{array} \tag{11}$$

Then the equation (4) of order $(q+1,r)$ can be written as an equation (q,r) of the same type with z instead of x and the initial conditions (3) give null initial conditions for z.

By the induction assumption the corresponding solution is

$$\begin{array}{l} z(t_1;t_2,\ldots,t_{q+1};k_1,\ldots,k_r) = \int_{t_2^0}^{t_2} \cdots \int_{t_{q+1}^0}^{t_{q+1}} \\ \sum_{l_1=k_1^0}^{k_1-1} \cdots \sum_{l_r=k_r^0}^{k_r-1} \left(\prod_{i=2}^{q+1} \Phi_i(t_i,s_i;t_1;s_{\overline{i-1}\setminus\{1\}},t_{\tilde{\tilde{i}}};k \right) \cdot \\ \cdot \left(\prod_{j=1}^{r} F_j(t_1;s_2,\ldots,s_{q+1};k_j,l_j+1;l_{\overline{j-1}};k_{\tilde{\tilde{j}}} \right) \cdot \\ \cdot f(s;l)ds_2 \ldots ds_{q+1}. \end{array} \tag{12}$$

But the variation of parameters formula gives the solution (11)

$$\begin{array}{l} x(t_1,t_2,\ldots,t_{q+1};k_1,\ldots,k_r) = \\ = \Phi_1(t_1,t_1^0;t_2,\ldots,t_{q+1};k_1,\ldots,k_r)\cdot \\ \cdot x(t_1^0;t_2,\ldots,t_{q+1};k_1,\ldots,k_r) + \\ + \int_{t_1^0}^{t_1} \Phi_1(t_1,s_1;t_2,\ldots,t_{q+1};k_1,\ldots,k_r)\cdot \\ \cdot z(s_1;t_2,\ldots,t_{q+1};k_1,\ldots,k_r)ds_1 \end{array}$$

and, by replacing z (12) one obtains formula (5) for the case $(q+1,r)$.

Similarly one can derive the case $(q,r+1)$ from (q,r), hence we proved by induction (5) for any $q, r \in \mathbf{N}$. □

Theorem 1. *The state of the system* Σ (1) *determined by the initial state* $x_0 \in \mathbf{R}^n$ *and the control* u *is*

$$\begin{array}{l} x(t;k) = \left(\prod_{i=1}^{q} \Phi_i(t_i,t_i^0;t_{\overline{i-1}}^0,t_{\tilde{\tilde{i}}};k) \right) \cdot \\ \cdot \left(\prod_{j=1}^{r} F_j(t^0;k_j,k_j^0;k_{\overline{j-1}}^0,k_{\tilde{\tilde{j}}}) \right) x^0 + \\ + \int_{t_1^0}^{t_1} \cdots \int_{t_q^0}^{t_q} \sum_{l_1=k_1^0}^{k_1-1} \cdots \sum_{l_r=k_r^0}^{k_r-1} \left(\prod_{i=1}^{q} \Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\tilde{\tilde{i}}};k) \right) \cdot \\ \cdot \left(\prod_{j=1}^{r} F_j(s;k_j,l_j+1;l_{\overline{j-1}},k_{\tilde{\tilde{j}}}) \right) \cdot \\ \cdot B(s;l)u(s;l)ds_1 \ldots ds_q. \end{array} \tag{13}$$

Proof. Equation (1) has the form (4) with $f(t;k) = B(u;k)u(t;k)$ and (13) results from (5) by replacing $f(t;k)$. □

By replacing the state $x(t;k)$ (13) in the output equation (2) we obtain formula (14) below of the general response of the system.

Theorem 2. *The input-output map of the* (q,r)-D

CDS Σ (1), (2) *is*

$$y(t;k) = C(t;k)\left(\prod_{i=1}^{q}\Phi_i(t_i,t_i^0;t^0_{\overline{i-1}},t_{\tilde{\bar{i}}},k)\right)\cdot$$
$$\cdot\left(\prod_{j=1}^{r}F_j(t^0;k_j;k^0_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)x^0+$$
$$+\int_{t_1^0}^{t_1}\dots\int_{t_q^0}^{t_q}\sum_{l_1=k_1^0}^{k_1-1}\dots\sum_{l_r=k_r^0}^{k_r-1}C(t;k)\cdot \tag{14}$$
$$\cdot\left(\prod_{i=1}^{q}\Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\tilde{\bar{i}}};k)\right)\cdot$$
$$\cdot\left(\prod_{j=1}^{r}F_j(s;k_j,l_j;l_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)\cdot$$
$$\cdot B(s;l)u(s;l)ds_1\dots ds_q + D(t;k)u(t;k).$$

3 Observability of time-varying (q,r)-D CDSs

For some $(t^0;k^0),(t^1;k^1),(t;k)\in T$ with $(t^0;k^0)<(t^1;k^1)$ we denote by P the multiple interval $P=[t^0;t^1]\times[k^0;k^1]$, by $\int_{t^0}^{t}$ the multiple integral $\int_{t_1^0}^{t_1}\cdots\int_{t_q^0}^{t_q}$, by $\sum_{l=k^0}^{k-1}$ the sum $\sum_{l_1=k_1^0}^{k_1-1}\cdots\sum_{l_r=k_r^0}^{k_r-1}$ and $ds=ds_1\cdots ds_q$.

A triplet $(t,k,\tilde{x})\in\mathbf{R}^q\times\mathbf{Z}^r\times\mathbf{R}^n$ is said to be a *phase* of Σ if $\exists u:T\to\mathbf{R}^m$ and $x^0\in\mathbf{R}^n$ such that $\tilde{x}=x(t;k)$ where $x(t;k)$ is given by (13). In this case one says that the control u transfers the phase (t^0,k^0,x^0) to the phase $(t,k,\tilde{x})$.

Definition 3. A phase (t^0,k^0,x) is said to be *unobservable/unobservable on P* if for any control $u(t;k)$ it provides the same output $y(t;k)$ as the phase $(t^0,k^0,0)$ for any $(t;k)\geq(t^0;k^0)$/for any $(t;k)\in P$.

A state x is said to be *unobservable at* $(t^0;k^0)$*/unobservable on P* if the phase (t^0,k^0,x) is unobservable/unobservable on P.

Proposition 2. *The phase* (t^0,k^0,x) *is unobservable/unobservable on* P *if and only if*

$$C(t;k)\left(\prod_{i=1}^{q}\Phi_i(t_i,t_i^0;t^0_{\overline{i-1}},t_{\tilde{\bar{i}}},k)\right)\cdot$$
$$\cdot\left(\prod_{j=1}^{r}F_j(t^0;k_j;k^0_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)x^0=0. \tag{15}$$

for any $(t;k)\in T$, $(t;k)\geq(t^0;k^0)$ *for any* $(t;k)\in P$.

Proof. We replace x^0 by 0 in (14) and we obtain the output provided by the null initial state and by an arbitrary control u:

$$y_0(t;k)=\int_{t_1^0}^{t_1}\dots\int_{t_q^0}^{t_q}\sum_{l_1=k_1^0}^{k_1-1}\dots\sum_{l_r=k_r^0}^{k_r-1}C(t;k)\cdot$$
$$\cdot\left(\prod_{i=1}^{q}\Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\tilde{\bar{i}}};k)\right)\cdot$$
$$\cdot\left(\prod_{j=1}^{r}F_j(s;k_j,l_j;l_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)B(s;l)u(s;l)ds_1\dots ds_q+$$
$$+D(t;k)u(t;k).$$

It results that the state x^0 is unobservable at $(t^0;k^0)$/unobservable on P if and only if $y_0(t;k)=y(t;k)$ for any $(t;k)\geq(t^0;k^0)$/for any $(t;k)\in P$ (where the output $y(t;k)$ is given by (14)). Obviously, this condition is equivalent to (15). □

Formula (15) as well as all the following formulas concerning observability involve only the drift matrices A_{ci},A_{dj} and the matrix C, therefore in the sequel we shall consider (q,r)-D CDSs reduced to the form $\Sigma=(\{A_{ci}|i\in\bar{q}\},\{A_{dj}|j\in\bar{r}\},C)$.

Definition 4. The system $\Sigma=(\{A_{ci}|i\in\bar{q}\},\{A_{dj}|j\in\bar{r}\},C)$ is said to be *completely observable* at $(t^0;k^0)$/ *completely observable* on P if there is no state $x\in\mathbf{R}^n$, $x\neq0$ unobservable at $(t^0;k^0)$/ unobservable on P.

Definition 5. The matrix

$$\mathcal{O}_\Sigma(t^0,t^1;k^0,k^1)=\int_{t^0}^{t^1}\sum_{l=k^0}^{k^1-1}\cdot$$
$$\cdot\left(C(t;k)\left(\prod_{i=1}^{q}\Phi_i(t_i,t_i^0;t^0_{\overline{i-1}},t_{\tilde{\bar{i}}},k)\right)\cdot\right.$$
$$\left.\cdot\left(\prod_{j=1}^{r}F_j(t^0;k_j;k^0_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)\right)^T\cdot \tag{16}$$
$$\cdot\left(C(t;k)\left(\prod_{i=1}^{q}\Phi_i(t_i,t_i^0;t^0_{\overline{i-1}},t_{\tilde{\bar{i}}},k)\right)\cdot\right.$$
$$\left.\cdot\left(\prod_{j=1}^{r}F_j(t^0;k_j;k^0_{\overline{j-1}},k_{\tilde{\bar{j}}})\right)\right)dt.$$

is called the *observability Gramian of* Σ *on* P.

Obviously $\mathcal{O}_\Sigma=\mathcal{O}_\Sigma(t^0,t^1;k^0,k^1)$ is a symmetrical positive semidefinite $n\times n$ matrix.

Proposition 3. *The state* x *is unobservable on* P *if and only if*

$$\mathcal{O}_\Sigma(t^0,t^1;k^0,k^1)x=0 \tag{17}$$

Proof. *Necessity*. If the state x is unobservable on P one obtains by (16) and by Proposition 2, for any $(t;k) \in P$:

$$\mathcal{O}_\Sigma(t^0,t^1;k^0,k^1)x = = \int_{t^0}^{t^1} \sum_{l=k^0}^{k^1-1} \left(C(t;k) \left(\prod_{i=1}^{q} \Phi_i(t_i,t_i^0;t^0_{\overline{i-1}},t_{\tilde{\tilde{i}}},k) \right) \cdot \left(\prod_{j=1}^{r} F_j(t^0;k_j;k^0_{\overline{j-1}},k_{\tilde{\tilde{j}}}) \right) \right)^T \cdot \left(C(t;k) \left(\prod_{i=1}^{q} \Phi_i(t_i,t_i^0;t^0_{\overline{i-1}},t_{\tilde{\tilde{i}}},k) \right) \cdot \left(\prod_{j=1}^{r} F_j(t^0;k_j;k^0_{\overline{j-1}},k_{\tilde{\tilde{j}}}) \right) \right) x dt = 0.$$

Sufficiency. From (17) one obtains

$$x^T \mathcal{O}_\Sigma(t^0,t^1;k^0,k^1)x = 0,$$

hence

$$\int_{t^0}^{t^1} \sum_{l=k^0}^{k^1-1} \| C(t;k) \left(\prod_{i=1}^{q} \Phi_i(t_i,t_i^0;t^0_{\overline{i-1}},t_{\tilde{\tilde{i}}},k) \right) \cdot \left(\prod_{j=1}^{r} F_j(t^0;k_j;k^0_{\overline{j-1}},k_{\tilde{\tilde{j}}}) \right) x \|^2 dt = 0.$$

Since the integrand is a sum of non-negative functions, it results that (17) holds a.e. on P, hence again by Proposition 2 the state x is unobservable on P. $\square$

Similarly, we get

Proposition 4. *The state x is unobservable at $(t^0;k^0)$ if and only if* (17) holds for any $(t^1;k^1) \in T$ with $(t^0;k^0) < (t^1;k^1)$.

Proposition 3 gives by paraphrase:

Corollary 1. *The set of the states of the* (q,r)-D CDS Σ *which are unobservable on P is the subspace of X*

$$X_{uo} = \ker \mathcal{O}_\Sigma(t^0,t^1;k^0,k^1). \tag{18}$$

By definition, the system Σ is completely observable on P if and only if $\ker \mathcal{O}_\Sigma(t^0,t^1;k^0,k^1) = \{0\}$, condition which is equivalent to the fact that $\mathcal{O}_\Sigma(t^0,t^1;k^0,k^1)$ has the full rank n. We proved:

Theorem 3. *The system Σ is completely observable on P if and only if*

$$\operatorname{rank} \mathcal{O}_\Sigma(t^0,t^1;k^0,k^1) = n \tag{19}$$

The concept of observability is connected to the property of Σ to allow the determining of the state of the system from the information about the exterior signals (the control u and the output y). The solution of this problem is given by the following result.

Theorem 4. *Assume that the system $\Sigma = (\{A_{ci}|i \in \bar{q}\},\{A_{dj}|j \in \bar{r}\},C)$ is complete observable on P. If the control $u(t;k)$ produces the output $y(t;k)$, $\forall(t;k) \in P$, then the initial state x^0 is given by the formula*

$$x^0 = \mathcal{O}_\Sigma(t^0,t^1;k^0,k^1)^{-1} \int_{t^0}^{t^1} \cdot \sum_{l=k^0}^{k^1-1} \left(C(t;k) \left(\prod_{i=1}^{q} \Phi_i(t_i,t_i^0;t^0_{\overline{i-1}},t_{\tilde{\tilde{i}}},k) \right) \cdot \left(\prod_{j=1}^{r} F_j(t^0;k_j;k^0_{\overline{j-1}},k_{\tilde{\tilde{j}}}) \right) \right)^T \tilde{y}(t;k)dt. \tag{20}$$

where

$$\tilde{y}(t;k) = y(t;k) - + \int_{t_1^0}^{t_1} \ldots \int_{t_q^0}^{t_q} \sum_{l_1=k_1^0}^{k_1-1} \ldots \sum_{l_r=k_r^0}^{k_r-1} C(t;k) \cdot \left(\prod_{i=1}^{q} \Phi_i(t_i,s_i;s_{\overline{i-1}},t_{\tilde{\tilde{i}}};k) \right) \cdot \left(\prod_{j=1}^{r} F_j(s;k_j,l_j;l_{\overline{j-1}},k_{\tilde{\tilde{j}}}) \right) B(s;l)u(s;l)ds_1 \ldots ds_q - -D(t;k)u(t;k). \tag{21}$$

Remark 2. If we replace the initial state x^0 given by (20) and (21) in (13) we obtain the state $x(t;k)$ for any $(t;k) \in P$; therefore if a system Σ is completely observable one can recover the whole trajectory $x(t;k)$, $(t;k) \in P$ of the system from the exterior data.

4 Observability of time-invariant (q,r)-D CDSs

The system $\Sigma = (\{A_{ci}|i \in \bar{q}\},\{A_{dj}|j \in \bar{r}\};\mathbf{C})$ is said to be *time-invariant* (or stationary) if all its matrices are constant. Since in this case the initial moment (t^0,k^0) is not relevant, we can consider it $(t^0,k^0) = (0,0) \in T_+ := \mathbf{R}_+^q \times \mathbf{Z}_+^r$. The fundamental matrices become $\Phi_i(t_i,0;t_{\tilde{i}};h) = e^{A_{ci}t_i}$ and $(F_j(t;h_j,0;h_{\tilde{j}}) = A_{dj}^{k_j}$ and the input-output map (14) can be written in the form

$$y(t,k) = C\left(\exp\left(\sum_{i=1}^{q} A_{ci}t_i\right)\right)\left(\prod_{j=1}^{r} A_{dj}^{k_j}\right)x^0 + \\ + \int_0^t \sum_{l=j}^{k-1} C\left(\exp\left(\sum_{i=1}^{q} A_{ci}(t_i - s_i)\right)\right)\cdot \\ \cdot\left(\prod_{j=1}^{r} A_{dj}^{k_j-l_j-1}\right) Bu(s,l)ds_1\ldots ds_q. \quad (22)$$

In this case, the observability Gramian (which will be denoted $\mathcal{O}_\Sigma(t,k)$ instead of $\mathcal{O}_\Sigma(t^0,t;k^0,k)$ becomes

$$\mathcal{O}_\Sigma(t,k) = \int_0^t \sum_{l=0}^{k}\left(\prod_{j=1}^{r}(A_{dj}^T)^{l_j}\right)\cdot \\ \cdot\left(\exp\left(\sum_{i=1}^{q} A_{ci}^T s_i\right)\right) C^T C\cdot \\ \cdot\left(\exp\left(\sum_{i=1}^{q} A_{ci}s_i\right)\right)\left(\prod_{j=1}^{r} A_{dj}^{l_j}\right) ds_1\ldots ds_q \quad (23)$$

If the state x is unobservable at $(t^0,k^0)=(0,0)$ we will say that x is *unobservable*. From Proposition 2 and Theorem 3 we obtain

Proposition 6. *The state $x\in X$ is unobservable if and only if*

$$C\left(\exp\left(\sum_{i=1}^{q} A_{ci}t_i\right)\right)\left(\prod_{j=1}^{r} A_{dj}^{k_j}\right)x = 0, \\ \forall(t,k)\in T_+. \quad (24)$$

Theorem 5. *The system $\Sigma = (\{A_{ci}|i\in \bar q\},\{A_{dj}|j\in\bar r\},C)$ is completely observable if and only if* rank $\mathcal{O}_\Sigma(t,k)=n$ *for any* $(t,k)\in T_+$.

In this case of time-invariant systems we can use a simpler controllability matrix instead of the controllability Gramian.

Definition 6. The *observability matrix* of the system Σ is

$$O_\Sigma = [C^T A_{c1}^T C^T \ldots (A_{c1}^T)^{n-1}C^T\ldots \\ \ldots\left(\prod_{\alpha\in\gamma}(A_{c\alpha}^T)^{i_\alpha}\right)\left(\prod_{\beta\in\delta}(A_{d\beta}^T)^{j_\beta}\right)C^T\ldots \\ \ldots\left(\prod_{\alpha=1}^{q}(A_{c\alpha}^T)^{n-1}\right)\left(\prod_{\beta=1}^{r}(A_{d\beta}^T)^{n-1}\right)C^T]^T \quad (25)$$

where we consider subsets $\gamma\subset\bar q$ and $\delta\subset\bar r$ and numbers i_α and j_β verifying $0\le i_\alpha, j_\beta\le n-1$, $\forall\alpha\in\gamma$ and $\forall\beta\in\delta$; $\prod_{\alpha\in\emptyset}A_\alpha = I$.

Theorem 6. *The system $\Sigma = (\{A_{ci}|i\in\bar q\},\{A_{dj}|j\in\bar r\},C)$ is completely observable if and only if*

$$\text{rank } O_\Sigma = n. \quad (26)$$

Proof. *Necessity.* Let us assume that (26) fails, i.e. rank $\mathcal{O}_\Sigma < n$ since $\mathcal{O}_\Sigma$ has n columns. Then there exists $x\in X\setminus\{0\}$ such that $\mathcal{O}_\Sigma x = 0$, which implies $Cx=0$, $CA_{c1}x = 0,\ldots, C\left(\prod_{\beta\in\delta}A_{d\beta}^{j_\beta}\right)\left(\prod_{\alpha\in\gamma}A_{c\alpha}^{i_\alpha}\right)x = 0,\ldots, C\left(\prod_{\beta=1}^{r}A_{d\beta}^{n-1}\right)\left(\prod_{\alpha=1}^{q}A_{c\alpha}^{n-1}\right)x=0.$

By the Hamilton-Cayley Theorem applied to the matrices A_{ci}, $i\in\bar q$ and A_{dj}, $j\in\bar r$ (and taking into account the commutativity of these matrices), we can prove that

$$C\left(\prod_{i=1}^{q}A_{ci}^{a_i}\right)\left(\prod_{j=1}^{r}A_{dj}^{k_j}\right)x = 0,\ \forall a_i,k_j\in\mathbf{N}. \quad (27)$$

Then

$$C\left(\exp\left(\sum_{i=1}^{q}A_{ci}t_i\right)\right)\left(\prod_{j=1}^{r}A_{dj}^{k_j}\right)x = \\ = C\left(\prod_{i=1}^{q}\left(\sum_{a_i=0}^{\infty}\frac{A_{ci}^{a_i}t_i^{a_i}}{a_i!}\right)\right)\left(\prod_{j=1}^{r}A_{dj}^{k_j}\right)x = \\ = \sum_{a_i=0}^{\infty}\left(\prod_{i=1}^{q}\frac{t_i^{a_i}}{a_i!}\right)C\left(\prod_{i=1}^{q}A_{ci}^{a_i}\right)\left(\prod_{j=1}^{r}A_{dj}^{k_j}\right)x = 0.$$

It results by Proposition 6 that the state x is unobservable, hence Σ is not completely observable.

Sufficiency. Assume that Σ is not completely observable, hence by Proposition 6 equality (24) holds for some $x\in X\setminus\{0\}$. By deriving successively this equality with respect to t_i, $i\in\bar q$ and by taking $t=(t_1,\ldots,t_q)=(0,\ldots,0)$ we obtain (27). Then $\mathcal{O}_\Sigma x = 0$, hence rank $\mathcal{O}_\Sigma < n$. □

From the proof of Theorem 6 we also deduce

Corollary 2. *A state $x\in X$ is unobservable if and only if x verifies* (27).

Corollary 3. *The set of all unobservable states of* Σ is $X_{uo} = \text{Ker } O_\Sigma$.

Theorem 7. *The system Σ is completely observable if and only if equality* (27) *implies* $x=0$.

The following theorem emphasizes the duality relation between the concepts of observability and reachability.

Definition 7. The system $\hat\Sigma = (\{\hat A_{ci}|i\in\bar q\},\{\hat A_{dj}|j\in\bar r\},\hat B,\hat C,\hat D)$ is called the *dual* of the system $\Sigma = (\{A_{ci}|i \in \bar{q}\}, \{A_{dj}|j \in \bar{r}\}, B, C, D)$ if $\forall i \in \bar{q}$, $\hat{A}_{ci} = A_{ci}^T$, $\forall j \in \bar{r}$, $\hat{A}_{dj} = A_{dj}^T$, $\hat{B} = C^T$, $\hat{C} = B^T$, $\hat{D} = D^T$.

Theorem 8. *The system Σ is completely observable if and only if its dual $\hat{\Sigma}$ is completely controllable.*

Proof. Obviously, the observability matrix $\mathcal{O}_\Sigma$ (18) of Σ and the controllability matrix of $\hat{\Sigma}$, $C_{\hat{\Sigma}}$ (see Definition 4 in [17]) verify $\mathcal{O}_\Sigma^T = C_{\hat{\Sigma}}$, hence rank $\mathcal{O}_\Sigma =$ rank $C_{\hat{\Sigma}}$. The proof is complete by Theorem 6 (and by Theorem 4.3 in [14], which states that $\hat{\Sigma}$ is completely reachable iff rank $C_{\hat{\Sigma}} = n$). □

Now we can give a geometric characterization of the set of unobservable states of Σ.

Theorem 9. The set X_u of all unobservable states of Σ is the greatest subspace of X which is $(\{A_{ci}|i \in \bar{q}\}, \{A_{dj}|j \in \bar{r}\})$ - invariant and it is contained in Ker C.

Proof. By Corollary 2,

$$X_{uo} = \{x \in X | C \left(\prod_{i=1}^{q} A_{ci}^{a_i}\right) \left(\prod_{j=1}^{r} A_{dj}^{k_j}\right) x = 0, \quad \forall a_i, k_j \in \mathbf{N}\}.$$

Obviously, for $a_i = 0$, $\forall i \in \bar{q}$ and $k_j = 0$, $\forall j \in \bar{r}$ it results than $Cx = 0$ $\forall x \in X_{uo}$, hence $X_{uo} \subset$ Ker C. For any $\iota \in \bar{q}$, $x \in X_{uo}$ implies

$$C \left(\prod_{i=1}^{q} A_{ci}^{a_i}\right) \left(\prod_{j=1}^{r} A_{dj}^{k_j}\right) (A_{c\iota} x) =$$

$$= C \left(\prod_{i=1}^{q} A_{ci}^{b_i}\right) \left(\prod_{j=1}^{r} A_{dj}^{k_j}\right) x = 0$$

where $b_i = \begin{cases} a_i + 1 & \text{for} \quad i = \iota \\ a_i & \text{for} \quad i = \tilde{\iota} \end{cases}$, hence $A_{c\iota} x \in X_{uo}$.

Similarly, we can prove that $A_{dj} x \in X_{uo}$, $\forall j \in \bar{r}$, hence X_{uo} is invariant with respect to A_{ci}, $\forall i \in \bar{q}$ and A_{dj}, $\forall j \in \bar{r}$. Now let us assume that $\mathcal{V}$ is a subspace of X $(\{A_{ci}|i \in \bar{q}\}, \{A_{dj}|j \in \bar{r}\})$-invariant included in Ker C. Let x be any element of $\mathcal{V}$. Since $\mathcal{V} \subset$ Ker C, $Cx = 0$. $\mathcal{V}$ being invariant with respect to the drift matrices we obtain $A_{ci} x \in \mathcal{V}$ and $A_{dj} \in \mathcal{V}$, $\forall i \in \bar{q}$, $\forall j \in \bar{r}$ and by recurrence, $\left(\prod_{i=1}^{q} A_{ci}^{a_i}\right) \left(\prod_{j=1}^{r} A_{dj}^{k_j}\right) x \in \mathcal{V} \subset$ Ker C, $\forall a_i, k_j \in \mathbf{N}$, hence $C \left(\prod_{i=1}^{q} A_{ci}^{a_i}\right) \left(\prod_{j=1}^{r} A_{dj}^{k_j}\right) x = 0$, i.e. $x \in X_{uo}$. Therefore $\mathcal{V} \subset X_{uo}$. □

From Theorem 5 and Theorem 9 we obtain

Theorem 10. *The system Σ is completely observable if $\{0\}$ is the greatest subspace of X which is $(\{A_{ci}|i \in \bar{q}\}, \{A_{dj}|j \in \bar{r}\})$-invariant and included in Ker C.*

We denote by $\sigma(A)$ the spectrum of a matrix A.

Theorem 11. *The system Σ is completely observable if and only if there is no vector $x \in \mathbf{R}^n \setminus \{0\}$ such that*

$$C \left(\prod_{i=1}^{q} (s_i I - A_{ci})^{-1}\right) \left(\prod_{j=1}^{r} (z_j I - A_{dj})^{-1}\right) x = 0, \tag{28}$$

$\forall s_i \in \mathbf{C} \setminus \sigma(A_{ci}),\ i \in \bar{q}, \forall z_j \in \mathbf{C} \setminus \sigma(A_{dj}),\ j \in \bar{r}.$

Proof. Since for a square matrix A

$$(zI - A)^{-1} = \sum_{k=0}^{\infty} A^k z^{-k-1}, \forall z \in \mathbf{C} \setminus \sigma(A),$$

equality (28) is equivalent to

$$\sum_{a \geq 0} \sum_{k \geq 0} \left(C \left(\prod_{i=1}^{q} A_{ci}^{a_i}\right) \left(\prod_{j=1}^{r} A_{dj}^{k_j}\right) x \right) \left(\prod_{i=1}^{q} s_i^{-a_i - 1}\right) \left(\prod_{j=1}^{r} z_j^{-k_j - 1}\right) = 0.$$

This multiple Laurent series is null if and only if all its coefficients are equal to zero, condition which is equivalent to (27) hence Theorem 11 is a consequence of Theorem 7. □

Definition 8. Two systems $\Sigma = (\{A_{ci}|i \in \bar{q}\}, \{A_{dj}|j \in \bar{r}\}, B, C, D)$ and $\tilde{\Sigma} = (\{\tilde{A}_{ci}|i \in \bar{q}\}, \{\tilde{A}_{dj}|j \in \bar{r}\}, \tilde{B}, \tilde{C}, \tilde{D})$ are *isomorphic* if there exists a nonsingular matrix U such that $\tilde{A}_{ci} = U^{-1} A_{ci} U$, $\forall i \in \bar{q}$, $\tilde{A}_{dj} = U^{-1} A_{dj} U$, $\forall j \in \bar{r}$, $\tilde{B} = U^{-1} B$, $\tilde{C} = CU$, $\tilde{D} = D$.

The next result gives the canonical form of the unobservable systems.

Theorem 12. *The system Σ is not completely observable if and only if it is isomorphic to a system $\tilde{\Sigma}$ with*

$$\tilde{A}_{ci} = \begin{bmatrix} A_{11,ci} & 0 \\ A_{21,ci} & A_{22,ci} \end{bmatrix}, \forall i \in \bar{q}, \quad \tilde{A}_{dj} = \begin{bmatrix} A_{11,dj} & 0 \\ A_{21,dj} & A_{22,dj} \end{bmatrix}, \forall j \in \bar{r},\ \tilde{C} = [C_1\ 0], \tag{29}$$

where $A_{11,ci}, A_{11,dj} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ *and* $\tilde{n} < n$. *For* $\tilde{n} = \text{rank}\ \mathcal{O}_\Sigma$, *the subsystem* $\Sigma_1 = (\{A_{11,ci}|i \in \bar{q}\}, \{A_{11,dj}|j \in \bar{r}\}, C_1)$ *is completely observable.*

Proof. Let $\tilde{n} = \text{rank}\ \mathcal{O}_\Sigma$. Since Σ is not completely observable rank $\mathcal{O}_\Sigma < n$, hence $\tilde{n} < n$ and by Corollary 3, $\tilde{n} = \dim X_{uo}$. If we consider a basis of X_{uo} and we complete it to a basis $\mathcal{B}$ of $X = \mathbf{R}^n$, we obtain the direct sum decomposition $X = X_1 \oplus X_2$, where $X_2 = X_{uo}$. Let U be the transition matrix. Let us denote by $\tilde{A}_{ci}, \tilde{A}_{dj}, \tilde{C}$ the matrices corresponding to A_{ci}, A_{dj} and C in the basis B. A state x becomes $\tilde{x} = U^{-1}x$ and since $X_{uo} = X_2$ we obtain

$$X_2 = \left\{ \tilde{x} = \left[\begin{array}{c} 0 \\ x_2 \end{array} \right] \middle| x_2 \in \mathbf{R}^{n-\tilde{n}} \right\}.$$

Let us partition the matrices corresponding to the direct sum decomposition, i.e. the matrices of the isomorphic system $\tilde{\Sigma}$ as

$$\tilde{A}_{ci} = \left[\begin{array}{cc} A_{11,ci} & A_{12,ci} \\ A_{21,ci} & A_{22,ci} \end{array} \right],$$

$$\tilde{A}_{dj} = \left[\begin{array}{cc} A_{11,dj} & A_{12,dj} \\ A_{21,dj} & A_{22,dj} \end{array} \right],\ \tilde{C} = [C_1\ C_2],$$

hence $A_{11,ci}, A_{11,dj}$ are $\tilde{n} \times \tilde{n}$ matrices and C_1 is a $p \times \tilde{n}$ matrix. Since by Theorem 9 X_2 is A_{ci}-invariant, we have $\tilde{A}_{ci}\tilde{x} \in X_2$, hence

$$\left[\begin{array}{cc} A_{11,ci} & A_{12,ci} \\ A_{21,ci} & A_{22,ci} \end{array} \right] \left[\begin{array}{c} 0 \\ x_2 \end{array} \right] =$$

$$= \left[\begin{array}{cc} A_{12,ci} & x_2 \\ A_{22,ci} & x_2 \end{array} \right] \in X_2,\ \forall x_2 \in \mathbf{R}^{n-\tilde{n}}.$$

Then $A_{12,ci}x_2 = 0$, $\forall x_2 \in \mathbf{R}^{n-\tilde{n}}$, hence $A_{12,ci} = 0$, $\forall i \in \bar{q}$. Similarly, we can prove that $A_{12,dj} = 0$, $\forall j \in \bar{r}$. Since again by Theorem 9 X_2 is included in Ker C, it results that (in the basis $\mathcal{B}$) $[C_1\ C_2] \left[\begin{array}{c} 0 \\ x_2 \end{array} \right] = 0$, $\forall x_2 \in \mathbf{R}^{n-n_2}$, hence $C_2 = 0$ and the matrices of the system have the form (29).

Now, by (29) we can prove that

$$\tilde{C} \left(\prod_{i=1}^{q} \tilde{A}_{ci} \right) \left(\prod_{j=1}^{r} \tilde{A}_{dj} \right) =$$

$$= \left[C_1 \left(\prod_{i=1}^{q} \tilde{A}_{11,ci} \right) \left(\prod_{j=1}^{r} \tilde{A}_{11,dj} \right) \quad 0 \right]$$

where 0 is a $p \times (n - \tilde{n})$ null matrix. Then the controllability matrix of $\tilde{\Sigma}$ is formed by blocks having this structure.

By Hamilton-Cayley Theorem applied to matrices $A_{11,ci}$ and $A_{11,dj}$ it results that $A_{11,ci}^{\tilde{n}+k}$ are linear combination of matrices $A_{11,ci}^{l}, l = 0, \tilde{n} - 1$, for any $k \geq 0$ and similarly for $A_{11,dj}^{\tilde{n}+k}$. Then, if $\mathcal{O}_{\Sigma_1}$ is the observability matrix of the subsystem Σ_1, since $\mathcal{O}_{\tilde{\Sigma}} = \mathcal{O}_\Sigma U$ and U is nonsingular, we get

$$\dim \Sigma_1 = \tilde{n} = \text{rank}\ \mathcal{O}_\Sigma = \text{rank}\ \mathcal{O}_{\tilde{\Sigma}} = \text{rank}\ \mathcal{O}_{\Sigma_1},$$

hence Σ_1 is completely observable.

Conversely if Σ is isomorphic to a system $\tilde{\Sigma}$ (29), since the blocks of $\mathcal{O}_{\tilde{\Sigma}}$ have the above structure, obviously rank $\mathcal{O}_\Sigma = \text{rank}\ \mathcal{O}_{\tilde{\Sigma}} \leq \tilde{n} < n$, hence Σ is not completely observable. □

We can restate this theorem as

Theorem 13. *The system* Σ *is completely observable if and only if it is not isomorphic to a system of the form* (29).

Theorem 14. *The system* Σ *is completely observable if and only if there is no common eigenvector of the matrices* A_{ci}, $i \in \bar{q}$ *and* A_{dj}, $j \in \bar{r}$, *belonging to* Ker C.

Proof. *Necessity*. Let us assume that $\exists x \in \mathbf{R}^n \setminus \{0\}$ such that $A_{ci}x = \lambda_i x$, $A_{dj}x = \mu_j x$ for some $\lambda_i, \mu_j \in \mathbf{C}$, $\forall i \in \bar{q}, \forall j \in \bar{r}$ and $Cx = 0$. Then we can prove by induction that

$$C \left(\prod_{i=1}^{q} \tilde{A}_{ci}^{a_i} \right) \left(\prod_{j=1}^{r} \tilde{A}_{dj}^{k_j} \right) x =$$

$$= \left(\prod_{i=1}^{q} \lambda_i^{a_i} \right) \left(\prod_{j=1}^{r} \mu_j^{k_j} \right) Cx = 0,$$

$\forall a_i, k_j \geq 0$, $i \in \bar{q}$, $i \in \bar{r}$, therefore $\mathcal{O}_\Sigma\ x = 0$ which implies that rank $\mathcal{O}_\Sigma < n$ and Σ is not completely observable.

Sufficiency. Let us assume that Σ is not completely observable. Then there exists $v \in \mathbf{R}^n \setminus \{0\}$ such that $\mathcal{O}_\Sigma\ v = 0$. Let us denote by S_1 the subspace $X_{uo} = \text{Ker}\ \mathcal{O}_\Sigma$, hence S_1 is a proper subspace of $\mathbf{R}^n$, and by Theorem 9 S_1 is $(\{A_{ci}|i \in \bar{q}\}, \{A_{dj}|j \in \bar{r}\})$-invariant and it is contained in Ker C. Being A_{c1}-invariant, S_1 contains an eigenvector v_1 of A_{c1}, corresponding to some eigenvalue λ_1. Then the set $S_2 = \{x \in \mathbf{C}^n | A_{c1}\ x = \lambda_1 x\}$ is a proper subspace of $\mathbf{C}^n$ as well as $S_3 = S_1 \cap S_2$. Following the ideas in the proof of Theorem 9 we can show that S_2 (and S_3 too) is $(\{A_{ci}|i \in \bar{q}\}, \{A_{dj}|j \in \bar{r}\})$-invariant. Then S_3 contains an eigenvector v_2 of A_{c2} corresponding to an eigenvalue λ_2, and since $S_3 \subset S_2$, v_2 is an eigenvector of A_{c1} too. Now we consider the proper subspaces $S_4 = \{x \in \mathbf{C}^n | A_{c1}\ x = \lambda_1 x, A_{c2}\ x = \lambda_2 x\}$ and $S_5 = S_3 \cap S_4$ and so on. Finally we obtain a sequence of proper subspaces of $\mathbf{C}^n$, $X_{uo} = S_1 \supset$

$S_3 \supset S_5 \supset \ldots \supset S_{2k-1} \supset \ldots \supset S_{2(q+r)-1}$ when $S_{2(q+r)-1} = \{x \in \mathbf{C}^n | A_{ci}\, x = \lambda_i x,\ i \in \bar{q},\ A_{dj}\, x = \mu_j x,\ j \in \bar{r}\} \neq \{0\}$ and $S_{2(q+r)-1} \subset X_{uo} \subset \operatorname{Ker} C$ hence there exists a common eigenvector x of the drift matrices of Σ which belongs to $\operatorname{Ker} C$. □

We can derive from Theorem 14 a Belevitch-Hautus-Popov type criterion of observability

Theorem 15. The system Σ is completely observable if and only if for any $\lambda_i, \mu_j \in \mathbf{C}$, $i \in \bar{q}$, $j \in \bar{r}$

$$\begin{aligned}&\operatorname{rank}\, [C^T\ \lambda_1 I - A_{c1}^T\ \ldots\ \lambda_q I - A_{cq}^T \\ &\mu_1 I - A_{d1}^T\ \ldots\ \mu_r I - A_{dr}^T]^T = n.\end{aligned} \tag{30}$$

Proof. Let us denote the matrix in (30) by M. The existence of $\lambda_i, \mu_j \in \mathbf{C}$, $i \in \bar{q}$, $j \in \bar{r}$ such that rank$M < n$ is equivalent to the existence of $v \in \mathbf{R}^n \setminus \{0\}$ such that $Mv = 0$, i.e. such that $Cv = 0$, $A_{ci}\, v = \lambda_i v$, $i \in \bar{q}$, $A_{dj} v = \mu_j v$, $j \in \bar{r}$ which is equivalent by Theorem 14 to the fact that Σ is not completely observable. □

Corollary 4. *The system Σ is completely observable if and only if* (30) *holds for any $\lambda_i \in \sigma(A_{ci})$, $i \in \bar{q}$ and $\mu_j \in \sigma(A_{dj})$, $j \in \bar{r}$.*

Proof. This statement is an imediate consequence of Theorem 15, since for any square matrix A, $\lambda \in \sigma(A)$ if and only if $\det(\lambda I - A) = 0$. Then rank$(\lambda I - A) = n$ if and only if $\lambda \in \mathbf{C} \setminus \sigma(A)$, hence obviously (30) holds for $\lambda_i \in \mathbf{C} \setminus \sigma(A_{ci})$ and $\mu_1 \in \mathbf{C} \setminus \sigma(A_{dj})$, $i \in \bar{q}$, $j \in \bar{r}$. □

Conclusion. This paper studies a class of multidimensional hybrid linear systems from the point of view of observability. In the case of time-varying systems necessary and sufficient conditions are expressed by introducing a suitable observability Gramian. The connection between the concepts of observability and reachability is emphasized. For time-invariant systems ten criteria of observability are obtained. The geometric characterization of the subspace of unobservable states is given. This study can be continued for other concepts such as stability, stabilizability and detectability of multidimensional hybrid systems and it can be applied to the problems of minimal realizations.

Aknowledgement. This paper is partially supported by the Grant CNCSIS 86/ 2007 and by the 15-th Italian-Romanian Executive Programme of S&T Co-operation for 2006-2008, University Politehnica of Bucharest.

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