Structural Properties of Linear Generalized Systems

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Abstract: Generalized linear systems are considered, which contain in their state-space representation matrices with elements functions of bounded variation and controls in the space of regulated functions. The Perron-Stieltjes integral is used to obtain a variation-of-parameters formula. On this basis the formula of the state of the system as well as the input-output map are derived. The fundamental concepts of controllability and reachability are analysed in this approach by means of two controllability and reachability Gramians. An optimal control is provided which solves the problem of the minimum energy transfer. The observability of these generalized systems is studied. In the case of completely observable systems a formula is obtained which recovers the initial state from the exterior data. The duality between the concepts of controllability and observability is emphasized as well as Kalman's canonical form.

Key–Words: functions of bounded variation; regulated functions; linear systems; controllability; observability; weighting patterns.

1 Introduction

The Perron-Stieltjes integral with respect to functions of regulated functions which include the functions of bounded variation was defined in [20]. This integral is equivalent to the Kurzweil integral (see [5] and [8]). In this paper, using the results of M.Tvrdý ([10], [11]) concerning the properties of the Perron-Stieltjes integral with respect to regulated functions and the differential equation in this space, a class of generalized linear systems is considered, having the controls in the space of regulated functions and the coefficient matrices of bounded variation. This allows us to extend in this framework the concepts of controllability, observability (see for instance [4]).

The minimal energy transfer problem is analysed and the optimal control is provided.

The observability of these generalized systems is studied. In the case of completely observable systems a formula is obtained which recovers the initial state from the exterior data. The duality between the concepts of controllability and observability is emphasized as well as Kalman's canonical form.

Linear boundary value systems were studied in the same framework in [6].

This study can be continued in many directions such as stability, positivity, 2D generalized systems, linear quadratic optimal control etc., by extending to the present approach some results presented for instance in [1], [2] and [3]. Another direction can be the application of this framework to the geometric approach of multitime systems studied in [12]-[19].

2 Preliminaries

A function $f : [a, b] \to \mathbf{R}$ which posseses finite one side limits $f(t-)$ and $f(t+)$ for any $t \in [a, b]$ (where by definition $f(a-) = f(a)$ and $f(b+) = f(b)$) is said to be *regulated* on $[a, b]$. The set of all regulated functions denoted by $G(a, b)$, endowed with the supremal norm, is a Banach space; the set $BV(a, b)$ of functions of bounded variation on $[a, b]$ with the norm $||f|| = |f(a)| + var_a^b f$ is also a Banach space; the Banach space of n-vector valued functions belonging to $G(a, b)$ and $BV(a, b)$ respectively are denoted by $G^n(a, b)$ and $BV^n(a, b)$ (or simply G^n and BV^n); $BV^{n\times m}$ denotes the space of $n \times m$ matrices with entries in $BV(a, b)$. For a matrix function $U : [a, b] \times [a, b] \to \mathbf{R}^{n\times m}$, let us denote by $\nu(U)$ its two-dimensional Vitali variation (see [9], Definition I.6.1).

A pair $D = (d, s)$ where $d = \{t_0, t_1, \ldots, t_m\}$ is a division of $[a, b]$ (i.e. $a = t_0 < t_1 < \ldots < t_m = b$) and $s = \{s_1, \ldots, s_m\}$ verifies $t_{j-1} \le s_j \le t_j$, $j = 1, \ldots, m$ is called a *partition* of $[a, b]$.

A function $\delta : [a, b] \to (0, +\infty)$ is called a *gauge* on $[a, b]$.

Given a gauge δ, the partition (d, s) is said to be *δ-fine* if $[t_{j-1}, t_j] \subset (s_j - \delta(s_j), s_j + \delta(s_j)), \quad j = 1, \dots, m.$

Given the function $f, g : [a, b] \to \mathbf{R}$ and a partition $D = (d, s)$ of $[a, b]$ let us associate the integral sum

$$S_D(f\Delta g) = \sum_{j=1}^{m} f(s_j)(g(t_j) - g(t_{j-1})).$$

Definition 1. The number $I \in \mathbf{R}$ is said to be the *Perron-Stieltjes* (*Kurzweil*) *integral of f with respect to g from a to b* and it is denoted as $\displaystyle\int_a^b f dg$ or $\displaystyle\int_a^b f(t)dg(t)$ if for any $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$|I - S_D(f\Delta g)| < \varepsilon$$

for all δ-fine partitions D of $[a, b]$.

Given $f \in G(a, b)$ and $g \in G([a, b] \times [a, b])$ we define the differences $\Delta^+, \Delta^-, \Delta$ and $\Delta_s^+, \Delta_s^-, \Delta_s$ by

$$\Delta^+ f(t) = f(t+) - f(t),$$

$$\Delta^- f(t) = f(t) - f(t-),$$

$$\Delta f(t) = f(t+) - f(t-),$$

$$\Delta_s^+ g(t, s) = g(t, s+) - g(t, s),$$

$$\Delta_s^- g(t, s) = g(t, s) - g(t, s-),$$

$$\Delta_s g(t, s) = g(t, s+) - g(t, s-);$$

$\mathbf{D}^-(f)$, $\mathbf{D}^+(f)$ denote respectively the set of the left and right discontinuities of f in $[a, b]$ and similarly with respect to the argument t we can define $\mathbf{D}_t^-(g)$, $\mathbf{D}_t^+(g)$.

Let us recall some basic properties of the Perron-Stieltjes integral, by following [9] and [10]. The existence theorem of the Perron-Stieltjes integral $\displaystyle\int_a^b f \mathrm{d}g$ for $f \in BV(a, b)$ and $g \in G(a, b)$, due to Tvrdý [11] is essential for our treatment.

Theorem 1 ([9], Theorem I.4.19 and [11], Theorem 2.8). *If $f \in G(a, b)$ and $g \in BV(a, b)$ then the Perron-Stieltjes integrals $\displaystyle\int_a^b f \mathrm{d}g$ and $\displaystyle\int_a^b g \mathrm{d}f$ exist.*

In the sequel we shall denote by $\displaystyle\sum_t$ the sum $\displaystyle\sum_{t \in \mathbf{D}}$ where $\mathbf{D} = \mathbf{D}^-(f) \cup \mathbf{D}^+(f) \cup \mathbf{D}^-(g) \cup \mathbf{D}^+(g)$.

Theorem 2 (integration-by-parts, [10], Theorem 2.15). *If $f \in G(a, b)$ and $g \in BV(a, b)$ then*

$$\begin{aligned} &\textstyle\int_a^b f \mathrm{d}g + \int_a^b g \mathrm{d}f = f(b)g(b) - f(a)g(a) + \\ &\textstyle\sum_t [\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)]. \end{aligned} \quad (1)$$

Theorem 3 ([10], Proposition 2.16). *If $\displaystyle\int_a^b f dg$ exists, then the function $h(t) = \displaystyle\int_a^t f dg$ is defined on $[a, b]$ and*

i) if $g \in G(a, b)$ then $h \in G(a, b)$ and, for any $t \in [a, b]$

$$\begin{aligned} \Delta^+ h(t) &= f(t)\Delta^+ g(t), \\ \Delta^- h(t) &= f(t)\Delta^- g(t); \end{aligned} \quad (2)$$

ii) if $g \in BV(a, b)$ and f is bounded on $[a, b]$, then $h \in BV(a, b)$.

Theorem 4 (substitution, [10], Theorem 2.19). *Let f, g, h be such that h is bounded on $[a, b]$ and the integral $\displaystyle\int_a^b f dg$ exists. Then the integral $\displaystyle\int_a^b h(t)f(t)dg(t)$ exists if and only if the integral $\displaystyle\int_a^b h(t)d\left[\int_a^t f(s)dg(s)\right]$ exists, and in this case*

$$\int_a^b h(t)f(t)\mathrm{d}g(t) = \int_a^b h(t)\left[\int_a^t f(s)\mathrm{d}g(s)\right]. \quad (3)$$

Theorem 5 (Dirichlet formula, [9], Theorem I.4.32). *If $h : [a, b] \times [a, b] \to \mathbf{R}$ is a bounded function and $var_a^b h(s, \cdot) + var_a^b h(\cdot, t) < \infty$, $\forall t, s \in [a, b]$, then for any $f, g \in BV(a, b)$*

$$\begin{aligned} &\int_a^b \mathrm{d}g(t)\left(\int_a^t h(s, t)\mathrm{d}f(s)\right) = \\ &\int_a^b \left(\int_s^b \mathrm{d}g(t)h(s, t)\right)\mathrm{d}f(s) + \\ &+ \sum_t [\Delta^- g(t)h(t, t)\Delta^- f(t) - \\ &- \Delta^+ g(t)h(t, t)\Delta^+ f(t)]. \end{aligned} \quad (4)$$

3 Generalized differential equations

The symbol

$$dx = \mathrm{d}[A]x + \mathrm{d}g \quad (5)$$

where $A \in BV^{n \times n}$ and $g \in G^n(a, b)$ is said to be a *generalized linear differential equation* (GLE) *in the space of regulated functions*.

Definition 2. A function $x : [a, b] \to \mathbf{R}^n$ is said to be a *solution* of GLE (5) if for any $t, t_0 \in [a, b]$ it verifies the equality

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{d}[A(s)]x(s) + g(t) - g(t_0). \quad (6)$$

If x satisfies the initial condition

$$x(t_0) = x_0 \quad (7)$$

for given $t_0 \in [a, b]$ and $x_0 \in \mathbf{R}^n$ then x is called the *solution of the initial value problem* (5), (7).

Theorem 6 ([9], Theorem III.2.10). *Assume that for any $t \in [a, b]$ the matrix $A \in BV^{n\times n}$ verifies the conditions*

$$\det[I + \Delta^{+}A(t)] \neq 0 \quad (8)$$

and

$$\det[I - \Delta^{-}A(t)] \neq 0. \quad (9)$$

Then there exists a unique matrix valued function $U : [a, b] \times [a, b] \to \mathbf{R}^{n\times n}$ such that, for any $(t, s) \in [a, b] \times [a, b]$

$$U(t, s) = I + \int_{s}^{t} d[A(\tau)]U(\tau, s). \quad (10)$$

$U(t, s)$ is called the fundamental matrix solution of the homogeneous equation

$$dx = d[A]x \quad (11)$$

and has the following properties, for any $\tau, t, s \in [a, b]$:

$$U(t, s) = U(t, \tau)U(\tau, s); \quad (12)$$

$$U(t, t) = I; \quad (13)$$

$$\begin{array}{l} U(t+, s) = [I + \Delta^{+}A(t)]U(t, s); \\ U(t-, s) = [I - \Delta^{-}A(t)]U(t, s); \\ U(t, s+) = U(t, s)[I + \Delta^{+}A(s)]^{-1}; \\ U(t, s-) = U(t, s)[I - \Delta^{-}A(s)]^{-1}; \end{array} \quad (14)$$

$$U(t, s)^{-1} = U(s, t); \quad (15)$$

there exists a constant $M > 0$ such that

$$\begin{array}{l} |U(t, s)| + var_a^b U(t, \cdot) + var_a^b U(\cdot, s)+ \\ +\nu(U) < M. \end{array} \quad (16)$$

The twodimensional variation of U is finite on $[a, b] \times [a, b]$, i.e. $v_{[a,b]\times[a,b]}(U) < \infty$.

Some methods for the calculus of the fundamental matrix $U(t, s)$ were provided in [7].

From [9], Theorem III.3.1 and [11], Proposition 2.5, one obtains

Theorem 7 (Variation-of-constants formula). *If $A \in BV^{n\times n}$ satisfies the conditions* (8) *and* (9), *then the initial value problem* (5), (7) *has a unique solution given by*

$$\begin{array}{l} x(t) = U(t, t_0)x_0 + g(t) - g(t_0)- \\ - \int_{t_0}^{t} d_s[U(t, s)](g(s) - g(t_0)). \end{array} \quad (17)$$

If $g \in G^n$ $(g \in BV^n)$ then $x \in G^n$ $(x \in BV^n)$.

4 Generalized linear systems

In this section we shall study linear systems which are controlled by inputs over $G(a, b)$.

Definition 3. A *generalized linear system* (GLS) Σ is an ensemble $\Sigma = (A(\cdot), B(\cdot), C(\cdot), D(\cdot)) \in BV^{n\times n} \times BV^{n\times m} \times BV^{p\times n} \times BV^{p\times m}$ with the state space representation

$$\mathrm{d}x(t) = \mathrm{d}[A(t)]x(t) + B(t)\mathrm{d}u(t), \quad (18)$$

$$y(t) = C(t)x(t) + D(t)u(t), \quad t \in [a, b]; \quad (19)$$

$x \in G^n$, $u \in G^m$, $y \in G^p$ are the *state*, the *input* and the *output* of the system, respectively.

Proposition 1. *If*

$$\det[(I - \Delta^{-}A(t))(I + \Delta^{+}(t))] \neq 0 \quad (20)$$

for any $t \in [a, b]$, then the input-output map of Σ is given by

$$\begin{array}{l} y(t) = C(t)U(t, a)x_0+ \\ + \int_{a}^{t} C(t)U(t, s)B(s)du(s)+ \\ \sum_{a\le s<t} C(t)U(t, a)\Delta_s^{+}(U(a, s))B(s)\Delta^{+}u(s)- \\ \sum_{a<s\le t} C(t)U(t, a)\Delta_s^{-}(U(a, s))B(s)\Delta^{-}u(s)+ \\ +D(t)u(t) \end{array} \quad (21)$$

where $x_0 = x(a)$ is the initial state of the system.

Proof. The equation (18) is of the form (6) where $g(t) = \int_{a}^{t} B(s)\mathrm{d}u(s)$. It follows from [9, Corollary I 4.12] and from Theorem 6 respectively that

$$\begin{array}{l} \Delta^{+}g(t) = \lim_{s\downarrow t} \int_a^s B(r)\mathrm{d}u(r) - \int_a^t B(r)\mathrm{d}u(r) = \\ = B(t)\Delta^{+}u(t), \\ \Delta^{-}g(t) = B(t)\Delta^{-}u(t) \end{array}$$

and

$$\begin{aligned}\Delta_s^+ U(t,s) = U(t,s+) - U(t,s) = \\ = U(t,s)((I+\Delta^+ A(s))^{-1} - I) = \\ = U(t,a)U(a,s)((I+\Delta^+ A(s))^{-1} - I) = \\ = U(t,a)(U(a,s+) - U(a,s)) = \\ = U(t,a)\Delta_s^+ U(a,s);\end{aligned}$$

similarly

$$\Delta_s^- U(t,s) = U(t,a)\Delta_s^- U(a,s).$$

If A satisfies the assumption (20), the solution of (18) is

$$\begin{aligned}x(t) = x(a) + \textstyle\int_a^t B(s)\mathrm{d}u(s) - \\ - \textstyle\int_a^t \mathrm{d}_s[U(t,s)] \int_a^s B(r)\mathrm{d}u(r).\end{aligned}$$

By use of the integration-by-parts formula (1) we obtain

$$\begin{aligned} x(t) &= U(t,a)x_0 + \\ &+ \int_a^t U(t,s)\mathrm{d}\left[\int_a^s B(r)\mathrm{d}u(r)\right] + \\ &+ \sum_{a\le s<t} \Delta_s^+ U(t,s)\Delta^+\left(\int_a^s B(r)\mathrm{d}u(r)\right) - \\ &- \sum_{a<s\le t} \Delta_s^- U(t,s)\Delta^-\left(\int_a^s B(r)\mathrm{d}u(r)\right)\end{aligned}$$

and, by Theorem 4, the state of the system Σ is given by

$$\begin{aligned} x(t) &= U(t,a)x_0 + \int_a^t U(t,s)B(s)\mathrm{d}u(s) + \\ &+ \sum_{a\le s<t} U(t,a)\Delta_s^+(U(a,s))B(s)\Delta^+ u(s) - \\ &- \sum_{a<s\le t} U(t,a)\Delta_s^-(U(a,s))B(s)\Delta^- u(s).\end{aligned} \tag{22}$$

Now we replace $x(t)$ given by (22) in the output equation (19) and it results that the input-output map of the GLS (18), (19) has the form (21). □

Remark 1. By Theorem 3, x and y are regulated vector functions. If $u \in BV^n$, then x is of bounded variation and so is y. Given the matrix $A \in BV^{n\times n}$ we consider the set of *admissible controls*

$$\begin{aligned}\mathcal{U}(a,b) = \{u \in G^m(a,b) | \mathbf{D}^+(A) \cap \mathbf{D}^+(u) = \emptyset, \\ \mathbf{D}^-(A) \cap \mathbf{D}^-(u) = \emptyset\}.\end{aligned}$$

Then the formulae (21) and (22) have a simplified form and we obtain

Corollary 1. *If $u \in \mathcal{U}(a,b)$ then the state equation and the input-output map of the system Σ respectively have the form*

$$x(t) = U(t,a)x_0 + \int_a^t U(t,s)B(s)\mathrm{d}u(s) \tag{23}$$

and

$$\begin{aligned} y(t) = C(t)U(t,a)x_0 + \textstyle\int_a^t C(t)U(t,s)B(s)\mathrm{d}u(s) + \\ + D(t)u(t).\end{aligned} \tag{24}$$

In the sequel we shall denote $\mathcal{U}(a,b)$ by $\mathcal{U}$ if there is no confusion and sometimes we shall consider for simplicity only admissible control $u \in \mathcal{U}$.

Remark 2. Classical dynamical systems with the state representation $\dot{x}(t) = \tilde{A}(t)x(t) + B(t)u(t)$ are particular cases of GLS with the absolutely continuous drift matrix $A(t) = \int_a^t \tilde{A}(s)\mathrm{d}s$ and the controls $u(t) = \int_a^t \tilde{u}(s)\mathrm{d}s$. In this, case (18) becomes the usual state equation $\dot{x}(t) = \tilde{A}(t)x(t) + B(t)\tilde{u}(t)$ and $U(t,s) = \Phi_{\tilde{A}}(t,s)$ where $\Phi_{\tilde{A}}(t,s)$ is the fundamental matrix of the system $\dot{x}(t) = \tilde{A}(t)x(t)$, and the input-output map (24) becomes the usual one

$$\begin{aligned} y(t) = C(t)\Phi_{\tilde{A}}(t,a)x_0 + \\ + \textstyle\int_a^t C(t)\Phi_{\tilde{A}}(t,s)B(s)u(s)\mathrm{d}s + D(t)u(t).\end{aligned}$$

5 Controllability of Generalized Linear Systems

Let us consider the system

$$\mathrm{d}x(t) = \mathrm{d}[A(t)]x(t) + B(t)\mathrm{d}u(t) \tag{25}$$

where $A \in BV^{n\times n}$, $B \in BV^{n\times m}$ and
$\det(I - \Delta^- A(t))(I + \Delta^+(A(t)) \neq 0, t \in \mathbf{R}$.

The couple $(x_0, t_0) \in \mathbf{R}^n$ is called a *phase* of the system if x_0 is the state of the system at time t_0.

Definition 4. A phase (x,t) is *G-controllable* (*controllable*) if there exists some s, $s > t$ and some control $u \in G^m(t,s)$ $(u \in \mathcal{U}(t,x))$ which transfers (x,t) to the phase $(0,s)$. A phase (x,t) is *reachable* if there exists some s, $s < t$ and some control $u \in G^m(s,t)$, $(u \in \mathcal{U}(s,t))$ which transfers the phase $(0,s)$ to (x,t).

From (22) and (23) we obtain:

Proposition 2. *A phase* (x,t) *is G-controllable (controllable) iff* (26) ((27)) *holds for some* $s>t$ *and* $u\in G^m(t,s)$ $(u\in\mathcal{U}(t,s))$:

$$\begin{aligned} x=&-\int_t^s U(t,r)B(r)\mathrm{d}u(r)-\\ &-\sum_{t\le r<s}\Delta_r^+(U(t,r))B(r)\Delta^+u(r)+\\ &+\sum_{t<r\le s}\Delta_r^-(U(t,r))B(r)\Delta^-u(r); \end{aligned} \tag{26}$$

respectively

$$x=-\int_t^s U(t,r)B(r)\mathrm{d}u(r). \tag{27}$$

Proof. We replace in (22) ((23)) $x(t)$ by 0, x_0 by x, t by s, a by t, s by r, we premultiply the obtained relation by $U(t,s)$ and (26) ((27)) results by using (12) and (15). □

In a similar way we can prove

Proposition 3. *A phase* (x,t) *is G-reachable (reachable) iff* (28) ((29)) *holds for some* $s<t$ *and some control* $u\in G^m(s,t)$ $(u\in\mathcal{U}(s,t))$:

$$\begin{aligned} x=&\int_s^t U(t,r)B(r)\mathrm{d}u(r)+\\ &+\sum_{s\le r<t}U(t,s)\Delta_r^+(U(s,r))B(r)\Delta^+u(r)-\\ &-\sum_{s<r\le t}U(t,s)\Delta_r^-(U(s,r))B(r)\Delta^-u(r); \end{aligned} \tag{28}$$

$$x=\int_s^t U(t,r)B(r)\mathrm{d}u(r). \tag{29}$$

Now let us consider the symmetric, non-negative matrices

$$\mathcal{C}(t,s)=\int_t^s U(t,r)B(r)B(r)^TU(t,r)^T\mathrm{d}r,\quad t<s \tag{30}$$

$$\mathcal{A}(s,t)=\int_s^t U(t,r)B(r)B(r)^TU(t,r)^T\mathrm{d}r,\quad s<t \tag{31}$$

called the *controllability Gramian* and *reachability Gramian* of Σ respectively.

We denote by $\mathcal{R}(M)$ and $\mathcal{N}(M)$ the range and the kernel of a linear operator M.

Theorem 8. *It is possible to transfer the phase* (x_0,t_0) *to* (x_1,t_1) *iff the vector* $U(t_0,t_1)x_1-x_0$ *belongs to* $\mathcal{R}(\mathcal{C}(t_0,t_1))$.

Proof. *Sufficiency*. By hypothesis, $\exists v\in X=\mathbf{R}^n$ such that $U(t_0,t_1)x_1-x_0=\mathcal{C}(t_0,t_1)v$ hence by using (15) $U(t_0,t_1)^{-1}=U(t_1,t_0)$ we get

$$\begin{aligned} x_1=&U(t_1,t_0)x_0+U(t_1,t_0)\mathcal{C}(t_0,t_1)v=U(t_1,t_0)x_0\\ &+\int_{t_0}^{t_1}U(t_1,s)B(s)B(s)^TU(t_0,s)^Tvds; \end{aligned}$$

therefore, if we consider the control $u(t)=\int_{t_0}^t B(s)^TU(t_0,s)^Tvds$ we have $u\in\mathcal{U}(t_0,t_1)$ and by using (23) one obtains

$$x(t_1)=U(t_1,t_0)x_0+\int_{t_0}^{t_1}U(t_1,s)B(s)\mathrm{d}u(s)=x_1.$$

Necessity. Let us suppose that it is possible to transfer the phase (x_0,t_0) to (x_1,t_1); again by (23) this is equivalent to

$$\begin{aligned} 0=&U(t_1,t_0)(U(t_0,t_1)x_1-x_0)+\\ &+\textstyle\int_{t_0}^{t_1}U(t_1,s)B(s)\mathrm{d}u(s), \end{aligned}$$

for some $u\in\mathcal{U}(t_0,t_1)$ hence it is possible to transfer (x_2,t_0) (where $x_2\overset{def}{=}U(t_0,t_1)x_1-x_0$) to $(0,t_1)$ using the control $u_1=-u$.

Since $\mathcal{C}(t_0,t_1)$ is symmetrical, we have the following orthogonal direct-sum decomposition of the state space: $X=\mathcal{R}\otimes\mathcal{N}$ where $\mathcal{R}=\mathcal{R}(\mathcal{C}(t_0,t_1))$ and $\mathcal{N}=\mathcal{N}(\mathcal{C}(t_0,t_1))$; then $x_2=x_3+x_4$, where $x_3\in\mathcal{R}$ and $x_4\in\mathcal{N}$.

From sufficiency it results (with $0,x_3$ instead of x_1,x_0) that (x_3,t_0) can be transferred to $(0,t_1)$ by a control u_3. It results by linearity that it is possible to transfer (x_4,t_0) to $(0,t_1)$ using the control $u_4=u_1-u_3$; indeed, by (23) we have

$$\begin{aligned} x(t_1)=&U(t_1,t_0)x_4+\\ &+\int_{t_0}^{t_1}U(t_1,s)B(s)\mathrm{d}(u_1(s)-u_2(s))=\\ =&U(t_1,t_0)x_2+\int_{t_0}^{t_1}U(t_1,s)B(s)\mathrm{d}u_1(s)-\\ &-\left(U(t_1,t_0)x_3+\int_{t_0}^{t_1}U(t_1,s)B(s)\mathrm{d}u_3(s)\right)=0 \end{aligned}$$

hence

$$x_4=-\int_{t_0}^{t_1}U(t_0,s)B(s)\mathrm{d}u_4(s).$$

Since $x_4\in\mathcal{N}$ we have that $0=x_4^T\mathcal{C}(t_0,t_1)x_4=$

$$\begin{aligned} &=\int_{t_0}^{t_1}x_4^TU(t_0,s)B(s)B(s)^TU(t_0,s)^Tx_4ds=\\ &=\int_{t_0}^{t_1}||B(s)^TU(t_0,s)^Tx_4||^2ds; \end{aligned}$$

the integrand is non-negative, hence $B(s)^T U(t_0,s)^T x_4 = 0$ almost everywhere on $[t_0,t_1]$. It implies that $||x_4||^2 = x_4^T x_4 =$

$$= -\int_{t_0}^{t_1} u_4(s)^T B(s)^T U(t_0,s)^T x_4 ds = 0$$

(no state $0 \neq x_4 \in \mathcal{N}$ is controllable), hence $x_4 = 0$ and $x_2 = x_3 \in \mathcal{R}$. □

If we replace in Theorem 8 (x_0, x_1) by $(0, x)$ and $(x, 0)$ respectively, we obtain:

Corollary 2. *A phase (x,t) is reachable iff there exists some moment $t_0 \leq t$ such that*

$$x \in U(t,t_0)\mathcal{R}(\mathcal{C}(t_0,t)).$$

A phase (x,t_0) is *controllable iff there exists $t_f \geq t_0$ such that $x \in \mathcal{R}(\mathcal{C}(t_0,t_f))$.*

In a similar manner we can prove

Theorem 9. *A phase (x,t) is reachable iff $x \in \mathcal{R}(\mathcal{A}(t_0,t))$ for some $t_0 < t$.*

Corollary 3. *If $t_1(t_0)$ is any value of t_1 for which $\mathcal{C}(t_0,t_1)$ has maximal rank, then the set $X_c(t_0)$ of states which are controllable at time t_0 is $X_c(t_0) = \mathcal{R}(\mathcal{C}(t_0,t_1(t_0)))$.*

Proof. Clearly,

$$X_c(t_0) = \bigcup_{t \geq t_0} \mathcal{R}(\mathcal{C}(t_0,t)).$$

If $\mathcal{C}(t_0,t_1(t_0))$ has the maximal rank then $\mathcal{R}(\mathcal{C}(t_0,t)) = \mathcal{R}(\mathcal{C}(t_0,t_1(t_0)))$ for any $t \geq t_0$, hence $X_c(t_0) = \mathcal{R}(\mathcal{C}(t_0,t_1(t_0)))$. □ Let $t_0,t,t_f \in \mathbf{R}$

be some fixed moments $t_0 < t < t_f$.

Definition 5. The system Σ is said to be *completely controllable* on $[t,t_f]$ (*completely reachable* on $[t_0,t]$) if every phase (x,t) is controllable (reachable) during the period $[t,t_f]$ $([t_0,t])$.

Theorem 10. *Σ is completely controllable (reachable) on $[a,b]$ iff rank $\mathcal{C}(a,b) = n$ (i.e. the matrix $\mathcal{C}(a,b)$ is positive definite).*

Proof. The above condition is equivalent to $X = \mathcal{R}(\mathcal{C}(a,b))$, hence any phase $(x,t_0) \in \mathbf{R} \times X$ where $X = \mathbf{R}^n$ is controllable (on (a,b)), i.e. Σ is completely controllable on $[a,b]$. Since $U(b,a)$ is nonsingular, it results that $U(b,a)\mathcal{R}(\mathcal{C}(a,b)) = U(b,a)X = X$; therefore any state x gives a reachable phase (x,b) and Σ is completely reachable on $[a,b]$. □

Now let us consider the general case, when the maximal rank is rank $\mathcal{C}(a,t_1(a)) = r < n$. We consider the orthogonal direct-sum decomposition of X defined by $X = X_1(t) \oplus X_2(t)$, where

$$X_1(t) = U(t,a)\mathcal{R}(\mathcal{C}(a,t_1(a)))$$

and

$$X_2(t) = U^T(a,t)\mathcal{N}(\mathcal{C}(a,t_1(a)))$$

(using the notation $(M\mathcal{V} = \{Mv | v \in \mathcal{V}\}$ for any matrix M and for any linear space $\mathcal{V}$). Let us denote $\mathcal{C} = \mathcal{C}(a,t_1(a))$. Since $U(t,a)$ is nonsingular for any t,t_0, it results that $\dim X_1(t) = \dim\mathcal{R}(\mathcal{C}(a,t_1(a))) = \operatorname{rank}\mathcal{C}(a,t_1(a)) = r$, $\dim X_2(t) = \dim\mathcal{N}(\mathcal{C}(a,t_1(a))) = \operatorname{rank}\mathcal{C}(a,t_1(a)) = n-r$,

hence

$$\dim X_1(t) + \dim X_2(t) = n = \dim X.$$

Let x_1, x_2 be such that $x_1 \in X_1(t)$, $x_2 \in X_2(t)$, that is $x_1 \in U(t,a)\mathcal{C}\bar{x}$, $x_2 \in U^T(a,t)\hat{x}$ for some $\bar{x}$ and $\hat{x}$ with $\bar{x} \in X$ and $\mathcal{C}\hat{x} = 0$. Then $x_1^T x_2 = \bar{x}^T \mathcal{C}^T U^T(t,a) U^T(a,t)\hat{x} = \bar{x}^T\mathcal{C}\hat{x} = 0$, (i.e. $x_1 \perp x_2$), hence $X_1(t) \perp X_2(t)$ and $X = X_1(t) \oplus X_2(t)$. Therefore, any $x(t) \in X$ can be uniquely written as $x(t) = x_1(t) + x_2(t)$ with $x_1(t) \in X_1(t)$, $x_2(t) \in X_2(t)$, or $x = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}$ and $X_1(t) = \left\{ \begin{bmatrix} \tilde{x}_1(t) \\ 0 \end{bmatrix} \middle| \tilde{x}_1 \in \mathbf{R}^r \right\}$, $X_2(t) = \left\{ \begin{bmatrix} 0 \\ \tilde{x}_2(t) \end{bmatrix} \middle| \tilde{x}_2 \in \mathbf{R}^{n-r} \right\}$ in a basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_1, \mathcal{B}_2$ are two bases in $X_1(t), X_2(t)$ respectively.

Theorem 11. *In terms of the above direct-sum decomposition, the fundamental equation of the linear system Σ have the form*

$$\begin{aligned} dx_1(t) = d[\tilde{A}_{11}(t)]x_1(t) \quad &+ d[\tilde{A}_{12}(t)]x_2(t) + \\ &+ \tilde{B}_1(t)du(t), \\ dx_2(t) = \quad & d[A_{22}(t)]x_2(t) \end{aligned} \tag{32}$$

Proof. If $x_1(t) \in X_1(t)$, then $x_1(t) \in U(t,t_0)\mathcal{C}\bar{x}$ for some $\bar{x} \in X$; then, by Theorem 6 and (10) we get

$$\begin{aligned} d[A(t)]x_1(t) = d[A(t)]U(t,a)\mathcal{C}\bar{x} = d[U(t,a)\mathcal{C}\bar{x}] = \\ = d[x_1(t)]. \end{aligned}$$

In a basis corresponding to the above direct-sum decomposition it results (for the block representation of A as $\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$) that for any $x(t) = \begin{bmatrix} \tilde{x}_1(t) \\ 0 \end{bmatrix} \in X_1(t)$,

$$\mathrm{d}[\tilde{A}(t)]x(t) = \mathrm{d}\left[\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}\right]\begin{bmatrix} \tilde{x}_1(t) \\ 0 \end{bmatrix} =$$
$$= \begin{bmatrix} \mathrm{d}[\tilde{A}_{11}]\mathrm{d}\tilde{x}_1(t) \\ \mathrm{d}[\tilde{A}_{21}]\mathrm{d}\tilde{x}_1(t) \end{bmatrix} = \begin{bmatrix} \mathrm{d}\tilde{x}_1(t) \\ 0 \end{bmatrix},$$
$$\forall \tilde{x}_1(t) \in X_1(t), \text{ hence } \quad \tilde{A}_{21} = 0.$$

If $x_2 \in X_2(t)$, there exists $\hat{x} \in \mathcal{N}(\mathcal{C})$ such that $x_2 = U^T(a,t)\hat{x}$. It was proved in Theorem 8 (with x_4 instead of $\hat{x}$) that $B^T(t)U^T(a,t)\hat{x} = 0$ a.e., hence $B^T(t)x_2 = 0$ for any $x_2 \in X_2(t)$. It results that any column of $B(t)$ belongs to $X_1(t)$, hence the matrix $B(t)$ in the above direct-sum decomposition has the form $\tilde{B}(t) = \begin{bmatrix} \tilde{B}_1(t) \\ 0 \end{bmatrix}$. □

By Corollary 2, $X_1(t)$ is the linear space of reachable states of X.

6 Minimum energy transfer

We can prove some results which are similar to those given in Section 5 by replacing the controllability Gramian $\mathcal{C}(a,b)$ by the reachability Gramian $\mathcal{A}(a,b)$, for instance:

Theorem 12. *The system* (25) *is completely reachable on* $[a,b]$ *iff the matrix* $\mathcal{A}(a,b)$ *is positive definite.*

Proof. *Sufficiency*. if $\mathcal{A} = \mathcal{A}(a,b) > 0$ then its inverse $\mathcal{A}^{-1}$ exists and the control

$$\tilde{u}(t) = \int_a^t B(t)^T U(b,t)^T \mathcal{A}^{-1}(x_1 - U(b,a)x_0)\mathrm{d}t \tag{33}$$

transfers the phase (x_0, a) to (x_1, b) for any $x_0, x_1 \in X$; indeed, by (23) and (3) the state provided by the control $\tilde{u}$ is

$$\begin{aligned} &x(b) = U(b,a)x_0 + \\ &+\textstyle\int_a^b U(b,t)B(t)B(t)^T U(b,t)^T \mathrm{d}t\mathcal{A}^{-1}(x_1 - \\ &-U(b,a)x_0) = U(b,a)x_0 + \mathcal{A}\mathcal{A}^{-1}(x_1 - \\ &-U(b,a)x_0) = x_1. \end{aligned}$$

By taking $x_0 = 0$ one obtains that any state $x_1 \in X$ is reachable on $[a,b]$, hence Σ is completely reachable on $[a,b]$.

Necessity: Let us suppose that $\mathcal{A}$ is not positive definite, hence there exists a vector $x \in \mathbf{R}^n$, $x \neq 0$, such that $x^T\mathcal{A}x = 0$; since $U(b,s)B(s) \in BV^{n\times m}$, from

$$\int_a^b x^T U(b,s)B(s)B(s)^T U(b,s)^T x\mathrm{d}s = 0$$

it results that

$$x^T U(b,s)B(s) = 0 \quad \text{a.e. in} \quad [a,b]. \tag{34}$$

If the phase $(0,a)$ can be transfered to (x,b), then (23) gives $x = \int_a^b U(b,s)B(s)\mathrm{d}u(s)$. From this equality we obtain by use of (34) that $0 < x^T x = \int_a^b x^T U(b,s)B(s)\mathrm{d}u(s) = 0$, contradiction. □

Definition 6. For any $u \in \mathcal{U}(a,b)$ the number $E_u = \int_a^b u(t)^T \mathrm{d}u(t)$ is said to be the *energy associated to the control* u.

Theorem 13. *If the system* Σ *is completely reachable on* $[a,b]$, *then the control* $\tilde{u}$ *given by* (33) *transfers the phase* (x_0, t_0) *to* (x_1, t_1) *with minimum expenditure of energy, i.e.* $\int_a^b \tilde{u}(t)^T d\tilde{u}(t) \leq \int_a^b u(t)^T du(t)$ *for all* $u \in \mathcal{U}(a,b)$ *which transfers* x_0 *to* x_1 *(*$\tilde{u}$ *is the "minimum energy" control).*

Proof. From the proof of Theorem 12 it results that the control $\tilde{u}$ (33) transfers the state x_0 to the state x_1 in the interval $[a,b]$. If u is another control which transfers x_0 to x_1 in the interval $[a,b]$ and $u = \tilde{u} + u_1$ for some $u_1 \in \mathcal{U}$, then

$$x_1 = U(b,a)x_0 + \int_a^b U(b,t)B(t)\mathrm{d}\tilde{u}(t)$$

and

$$\begin{aligned} x_1 &= U(b,a)x_0 + \int_a^b U(b,t)B(t)\mathrm{d}u(t) = \\ &= U(b,a)x_0 + \int_a^b U(b,t)B(t)\mathrm{d}\tilde{u}(t) + \\ &+ \int_a^b U(b,t)B(t)\mathrm{d}u_1(t), \end{aligned}$$

hence

$$\int_a^b U(b,t)B(t)\mathrm{d}u_1(t) = 0;$$

it follows that $\int_a^b \tilde{u}(t)^T \mathrm{d}u_1(t) = \int_a^b (x_1^T - x_0^T U(b,a)^T)\mathcal{A}^{-1}U(b,t)B(t)\mathrm{d}u_1(t) = 0$. Consequently

$$E_u = \int_a^b u(t)^T \mathrm{d}u(t) = \int_a^b (\tilde{u}(t)^T + u_1(t)^T)\mathrm{d}(\tilde{u}(t)+ \\ +\tilde{u}_1(t)) = \int_a^b \tilde{u}(t)^T \mathrm{d}\tilde{u}(t) + \int_a^b u_1(t)^T \mathrm{d}u_1(t)+ \\ +2\int_a^b \tilde{u}(t)^T \mathrm{d}u_1(t) = \int_a^b \tilde{u}(t)^T \mathrm{d}\tilde{u}(t)+ \\ +\int_a^b u_1(t)^T \mathrm{d}u_1(t) \geq \int_a^b \tilde{u}(t)^T \mathrm{d}\tilde{u}(t) = E_{\tilde{u}}$$

□

7 Observability

Let us consider the GLS Σ with the state space representation (18), (19).

Definition 7. The system Σ is *completely observable* over the time interval $[a,b]$ if for any pair of distinct initial states $x_{01}, x_{02} \in \mathbf{R}^n$ and for any input $u \in \mathcal{U}[a,b]$, the outputs y_1 and y_2 corresponding to x_{01} and x_{02} are distinct, that is $y_1(t) \neq y_2(t)$ for $t \in [a,b]$.

Definition 8. The matrix

$$\mathcal{O} = \mathcal{O}(a,b) = \int_a^b U(t,a)^T C(t)^T C(t) U(t,a)\mathrm{d}t \tag{35}$$

is called the *observability Gramian* of the system Σ.

Obviously $\mathcal{O}(a,b)$ is an $n \times n$ symmetric nonnegative matrix.

Theorem 14. *The system Σ is completely observable over $[a,b]$ iff the observability Gramian $\mathcal{O}(a,b)$ is positive definite.*

Proof. By the input-output map (21) we get the following outputs, for $i = 1, 2$:

$$y_i(t) = C(t)U(t,a)x_{0i}+ \\ +\int_a^t C(t)U(t,s)B(s)\mathrm{d}u(s)+ \\ +\sum_{a\leq s<t} C(t)U(t,a)\Delta_s^+(U(a,s))B(s)\Delta^+u(s)- \\ -\sum_{a<s\leq t} C(t)U(t,a)\Delta_s^-(U(a,s))B(s)\Delta^-u(s)+ \\ +D(t)u(t).$$

If Σ is completely observable, then for $x_{01} \neq x_{02}$ since $y_1 \neq y_2$ we have

$$0 < \int_a^b \|y_1(t) - y_2(t)\|^2 \mathrm{d}t = \\ = \int_a^b (y_1^T(t) - y_2^T(t))(y_1(t) - y_2(t))\mathrm{d}t = \\ = (x_{01}^T - x_{02}^T)\times \\ \times \left(\int_a^b U(t,a)^T C(t)^T C(t) U(t,a)\mathrm{d}t\right)(x_{01} - x_{02}),$$

hence $x^T \mathcal{O} x > 0$ for any $x \in \mathbf{R}^n$, $x \neq 0$.

Conversely, if $\mathcal{O} > 0$ and the system Σ is not completely observable, then there exist $x_{01} \neq x_{02}$ such that $y_1(t) \equiv y_2(t)$, hence $C(t)U(t,a)(x_{01} - x_{02}) = 0$ for any $t \in [a,b]$. It follows that

$$(x_{01} - x_{02})^T \left(\int_a^b U(t,a)^T C(t)^T C(t) U(t,a)\mathrm{d}t\right) \times \\ \times (x_{01} - x_{02}) = 0;$$

this contradicts the hypothesis that $\mathcal{O} > 0$. □

From Theorem 14 we get

Corollary 4. *The following statements are equivalent:*

i) The system Σ is completely observable over $[a,b]$.

ii) rank$\mathcal{O}(a,b) = n$.

iii) $\mathcal{O}(a,b) > 0$.

The concept of observability is connected to the problem of recovering the initial state of the system from the corresponding exterior data, i.e. the control u and the output y. The following result clarifies this property.

Theorem 15. *Assume that the system Σ is completely observable over $[a,b]$. If the control u produces the output y, then the initial state of the system is*

$$x_0 = \mathcal{O}(a,b)^{-1} \int_a^b U(t,a)^T C(t)^T \tilde{y}(t) dt \tag{36}$$

where

$$\tilde{y}(t) = y(t) - \int_a^t C(t)U(t,s)B(s)du(s)- \\ -\sum_{a\leq s<t} C(t)U(t,a)\Delta_s^+(U(a,s))B(s)\Delta^+u(s)+ \\ +\sum_{a<s\leq t} C(t)U(t,a)\Delta_s^-(U(a,s))B(s)\Delta^-u(s)- \\ -D(t)u(t). \tag{37}$$

Proof. It results from (21) and (37) that

$$\tilde{y}(t) = C(t)U(t,a)x_0. \tag{38}$$

We premultiply (38) by $U(t,a)^T C(t)^T$ and we consider the Perron-Stieltjes integral on $[a,b]$. By (35) and (38) one obtains $\mathcal{O}(a,b)x_0 = \int_a^b U(t,a)^T C(t)^T C(t)U(t,a)x_0 \mathrm{d}t = \int_a^b U(t,a)^T C(t)^T \tilde{y}(t)dt$. Since Σ is completely observable rank $\mathcal{O}(a,b) = n$, hence the matrix $\mathcal{O}(a,b)$ is nonsingular and (36) results by the multiplication of the above equality by $\mathcal{O}(a,b)^{-1}$. □

Obviously, the initial state x_0 determined by (36) and the control u give the whole trajectory $x(t)$ of the system (see (22)).

Definition 9. The system $\Sigma^d = (A^d, B^d, C^d, D^d)$ is said to be the *dual* of the GLS $\Sigma = (A,B,C,D)$ if

$$A^d = -A^T,\ B^d = C^T,\ C^d = B^T,\ D^d = D^T. \tag{39}$$

Theorem 16. *The system $\Sigma = (A,B,C,D)$ is completely observable if and only if its dual Σ^d is completely controllable.*

Proof. In [9, §III 4] it is shown that if (20) holds then $V(s,t) = U(s,t)^T$ is the fundamental matrix of the equation $dx = d[-A^t]x$. It results that the controllability Gramian of Σ^d on $[a,b]$ denoted by $\mathcal{C}_{\Sigma^d}(a,b)$ equals the observability Gramian of Σ since by (39) we have

$$\mathcal{C}_{\Sigma^d}(a,b) = \int_a^b V(a,r)B^d(r)B^d(t)^T V(a,r)^T dr =$$

$$= \int_a^b U(r,a)^T C(r)^T C(r)U(r,a)dr = \mathcal{O}_\Sigma(a,b).$$

Therefore rank $\mathcal{C}_{\Sigma^d}(a,b)$ = rank $\mathcal{O}_\Sigma(a,b)$ and the proof is complete by Corollary 4 and by Theorem 10. □

Definition 10. A state x_0 is *unobservable* over the time interval $[a,b]$ if the initial states x_0 and 0 provide the same output y for any admissible input u.

Proposition 4. *The state x_0 is unobservable over $[a,b]$ if and only if*

$$C(t)U(t,a)x_0 = 0, \quad \forall t \in [a,b]. \tag{40}$$

Proof. Equality (21) gives the output $y_{x_0}(t)$ produced by x_0,

$$\begin{aligned} y_{x_0}(t) &= C(t)U(t,a)x_0 + \\ &+ \int_a^t C(t)U(t,s)B(s)\mathrm{d}u(s) + \\ &\sum_{a\le s<t} C(t)U(t,a)\Delta_s^+(U(a,s))B(s)\Delta^+u(s) - \\ &\sum_{a<s\le t} C(t)U(t,a)\Delta_s^-(U(a,s))B(s)\Delta^-u(s) + \\ &+ D(t)u(t) \end{aligned} \tag{41}$$

and by replacing x_0 by 0 we get the output $y_0(t)$

$$\begin{aligned} y_0(t) &= \int_a^t C(t)U(t,s)B(s)\mathrm{d}u(s) + \\ &\sum_{a\le s<t} C(t)U(t,a)\Delta_s^+(U(a,s))B(s)\Delta^+u(s) - \\ &\sum_{a<s\le t} C(t)U(t,a)\Delta_s^-(U(a,s))B(s)\Delta^-u(s) + \\ &+ D(t)u(t). \end{aligned} \tag{42}$$

Then x_0 is unobservable over $[a,b]$ iff $y_{x_0}(t) = y_0(t)$, hence by (41) and (42) iff $C(t)U(t,a)x_0 = 0$, $\forall t \in [a,b]$. □

By paraphrase, we get:

Corollary 5. *The set of all the states of Σ which are unobservable over $[a,b]$ is the subspace*

$$X_{uo} = \bigcap_{t\in[,b]} \text{Ker}\ (C(t)U(t,a)).$$

Since a system is completely observable over $[a,b]$ iff $X_{uo} = \{0\}$ we get

Corollary 6. *The system Σ is completely observable over $[a,b]$ if and only if*

$$\bigcap_{t\in[a,b]} \text{Ker}\ (C(t)U(t,a)) = \{0\}.$$

Corollary 7. $X_{uo} = \text{Ker}\ \mathcal{O}(a,b)$.

Proof. If $C(t)U(t,a)x = 0$, $\forall t \in [a,b]$, then $\mathcal{O}(a,b)x = \int_a^b U(t,a)^T C(t)^T C(t)U(t,a)x\mathrm{d}t = 0$. Conversely, if $\mathcal{O}(a,b)x = 0$ then $x^T\mathcal{O}(a,b)x = 0$, hence

$$0 = \int_a^b x^T U(t,a)^T C(t)^T C(t)U(t,a)xdt =$$

$$= \int_a^b ||C(t)U(t,a)x||^2 dt.$$

Then $C(t)U(t,a)x = 0$ a.e. on $[a,b]$.

$\square$

Canonical forms from the point of view of the concept of observability, similar to (32) as well as the reachability and observability together can be derived.

Let us consider the subspaces of the controllable states $X_c = R(C(a,b))$ and of the unobservable state X_{uo}. By taking the direct sum decomposition of the state space $X = \mathbf{R}^n = X_1 \oplus X_2 \oplus X_3 \oplus X_4$, where $X_1 = X_c \cap X_{uo}$, $X_c = X_1 \oplus X_2$, $X_{uo} = X_1 \oplus X_3$, one can obtain Kalman's canonical form.

Theorem 17. *Any GLS* $\Sigma = (A,B,C,D)$ *is isomorphic to a GLS* $\tilde{\Sigma} = (\tilde{A},\tilde{B},\tilde{C},\tilde{D})$, *where*

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{C} = [0 \quad C_2 \quad 0 \quad C_4].$$

The subsystem $\Sigma_2 = (A_{22}, B_2, C_2, \cdot)$ *is completely controllable and completely observable.*

Conclusion. The state space representation studied in this paper have the coefficient matrices of bounded variation and the controls are vectors over spaces of regulated functions. This approach seems to be the most general framework in which the linear control systems can be studied.

Basic concepts as controlability, observability and minimum energy control are analysed in this framework and necessary and sufficient conditions of complete controllability or complete observability are provided. An optimal control is obtained which solves the problem of the minimum energy transfer for completely controllable systems. In the case of completely observable systems a formula is obtained which recovers the initial state from the exterior data. The duality between the concepts of controllability and observability is emphasized.

This study can be continued in many directions such as stability, positivity, 2D generalized systems, linear quadratic optimal control etc.

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