ISSN: 1991-8763 937 Issue 11, Volume 3, November 2008

Fuzzy Programming Problem in the Weakly Structurable Dynamic System and Choice of Decisions

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Abstract: This work deals with the problems of the Weakly Structurable Continuous Dynamic System (WSCDS) optimal control and briefly discuss the results developed by G. Sirbiladze [17]. Sufficient and necessary conditions are presented for the existence of an extremal fuzzy optimal control processes, for which we use R. Bellman's optimality principle and take into consideration the gain-loss fuzzy process. A separate consideration is given to the case where an extremal fuzzy control process acting on the WSCDS (1) depends and (2) does not depend on an WSCDS state.
Applying Bellman's optimality principle and assuming that the gain-loss process exists for the WSCDS, a variant of the fuzzy integral representation of an optimal control is given for the WSCDS. This variant employs the instrument of extended extremal fuzzy composition measures constructed in [16].
The questions of defining a fuzzy gain relation for the WSCDS are considered, taking into account the available expert knowledge on the WSCDS subject-matter.
An example of constructing of the WSCDS optimal control is presented.

Key–Words: Fuzzy Optimal Control, Fuzzy Dynamic Systems, WSCDS, Extended Extremal Fuzzy Measures, Sugeno Type Extremal Integrals, Fuzzy Gain-Loss Process, Bellman's Optimality Principle

1 Introduction

In recent years, both the dynamics of fuzzy system and the modeling issue received an increased attention. Dynamics is an obvious problem in control; moreover, its interest goes far beyond control applications. Applications of the dynamics of fuzzy systems and of the modeling of dynamic systems by fuzzy systems range from physics to biology to economics to pattern recognition and to time series prediction.

Evidence exists that fuzzy models can explain cooperative processes, such as in biology, chemistry, material sciences, or in economy. Relationships between dynamics of fuzzy systems and the performance of decision support systems were found, and chaotic processes in various classes of fuzzy systems were shown as a powerful tool in analyzing complex, weakly structurable systems, as anomal and extremal processes.

To make the decision-making effective in the framework of computer systems supporting this process, we must solve analytic problems of state evaluation, model identification, complex dynamic system control, optimal control, filtering and so on.

It is well recognized that optimization and other decision support technologies have been playing an important role in improving almost every aspect of human society. Intensive study over the past several years has resulted in significant progress in both the theory and applications of optimization and decision science.

Optimization and decision-making problems are traditionally handled by either the deterministic or probabilistic approaches. The former provides an approximate solution, completely ignoring uncertainty, while the latter assumes that any uncertainty can be represented as a probability distribution. Of course, both approaches only partially capture reality uncertainty (such as stock price, commodity, cost, natural resource availability and so on) that indeed exist but not in the form of known probability distributions.

In alternative classical approaches to modeling and when working with complex systems the main accent is placed on the assumption of fuzziness. As the complexity of systems increases, our ability to define exactly their behaviour drops to a certain level, below which such characteristics of information as exactness and definiteness become mutually excluding. In such situations an exact quantitative analysis of real complex systems is apt to be not quite plausible. Hence, a conclusion comes to mind that problems of this kind should be solved by means of analytic-fundamental methods of fuzzy mathematics, while the system approach to constructing models of complex systems with fuzzy-statistical uncertainty guarantees the creation of computer-aided systems forming the instrumental basis of the solutions intelligent technology of expert-analytic problems. It is obvious that the source of fuzzy-statistical samples is the population of fuzzy characteristics of our knowledge. Fuzziness arises from observations of time moments as well as from other expert measurements.

In the Preface of the Journal of Fuzzy Optimization and Decision Making (vol. I, 2002, pp. 11–12) Professor L. A. Zadeh had said: “My 1970 paper with R.E. Bellman, “Decision-Making in a Fuzzy Environment” was intended to suggest a framework based on the theory of fuzzy sets for dealing with imprecision and partial truth in optimization and decision analysis. In the intervening years, a voluminous literature on applications of fuzzy logic to decision analysis has come into existence.”

Problems of making an optimal solution for systems with fuzzy uncertainty are difficult because it frequently happens that the controllable object possesses conflicting properties which might include:

1) imperfection of a control process due to information uncertainty;
2) unreliable elements of a control system;
3) nonuniqueness and the applicability of many criteria encountered in a control process;
4) restriction of possibilities (resources) of a control system;
5) loss of the ability of a control system to solve arising control problems.

Fuzzy programming problems have been discussed widely in literature ([2], [3], [6], [11], [17], [10] and so on) and applied in such various disciplines as operations research, economic management, business administration, engineering and so on. Liu B. (Liu [10], 2002) presents a brief review on fuzzy programming models, and classifies them into three broad classes: expected value models, chance-constrained programming and chance-dependent programming.

Our further study belongs to the first class, where we use the instrument of fuzzy measures ([4], [5], [16], [17], [18], [19], [20], [24]) or, speaking more exactly, extremal fuzzy measures and Sugeno integrals along with extremal fuzzy expected values.

Our attention is focused on the rapidly developing theory of fuzzy measures and integrals [4], [16], [19]. The application of fuzzy measures and integrals as an instrument of constructing of intelligent decision-making systems is not a novel idea [4], [16]. We employ the part of the theory of fuzzy measures which concerns extremal fuzzy measures [16] and which, in our opinion, is rather seldom used. We have constructed a new instrument of a fuzzy measure [16], the extension of which is based on Sugeno lower and upper integrals [24].

We will deal with the fuzzy control problems of fuzzy dynamic systems (WSCDS) [16], [17], where fuzzy uncertainty arises with time and time structures are monotone classes of measurable sets. On such structures the extremal fuzzy measures play the role of measures of uncertainty.

In the present paper, we continue to investigate the controllable extremal fuzzy processes defined in [16]. The subject/matter of our investigation is the existence of an optimal control for WSCDS’s. Section 2 contains some necessary preliminary concepts presented in [16]. Section 3 deals with problems of WSCDS optimization when the control parameter depends or does not depend on a state in which an WSCDS is. Questions of the existence of an optimal control are studied, and variants of their fuzzy integral representation are proposed. Section 4 contains analysis of the definition of the gain-loss fuzzy process for WSCDS’s, which is carried out using the available expert knowledge on the WSCDS subject/matter. Section 5 contains an example in which the WSCDS fuzzy optimal control process is constructed.

2 Preliminary Concepts

All definitions and results see in [16].

ISSN: 1991-8763 939 Issue 11, Volume 3, November 2008

2.1 On the space of extended extremal fuzzy measures

Definition 1 Let X be some nonempty set.

a) We call some class $\mathcal{B}^* \subset 2^X$ of subsets X an upper σ^*-monotone class if (i) $\emptyset, X \in \mathcal{B}^*$; (ii) $\forall A, B \in \mathcal{B}^* \Rightarrow A \cup B \in \mathcal{B}^*$; (iii) $\forall \{A_n\} \in \mathcal{B}^*$, $n = 1, 2, \ldots, A_n \uparrow A \Rightarrow A \in \mathcal{B}^*$.

b) We call some class $\mathcal{B}_* \subset 2^X$ of subsets X a lower σ_*-monotone class if (i) $\emptyset, X \in \mathcal{B}_*$; (ii) $\forall A, B \in \mathcal{B}_* \Rightarrow A \cap B \in \mathcal{B}_*$; (iii) $\forall \{A_n\} \in \mathcal{B}_*$, $n = 1, 2, \ldots, A_n \downarrow A \Rightarrow A \in \mathcal{B}_*$.

Definition 2 We call the classes $\mathcal{B}^*$ and $\mathcal{B}_*$ extremal if and only if

$$\forall A \in \mathcal{B}^* \Leftrightarrow \overline{A} \in \mathcal{B}_*.$$

Remark 1 Let $\mathcal{B} \subseteq 2^X$ be some σ-algebra. Then B is both a σ^*-monotone class and a σ_*-monotone class.

Definition 3 1) $(X, \mathcal{B}^*)$ is called an upper measurable space;

2) $(X, \mathcal{B}_*)$ is called a lower measurable space;

3) If $\mathcal{B}^*$ and $\mathcal{B}_*$ are extremal σ^*- and σ_*-monotone classes, then $(X, \mathcal{B}_*, \mathcal{B}^*)$ is called an extremal measurable space.

Example 1

$$\mathcal{B}_1^* \triangleq \{A \subset \mathbb{R}_0^+ \mid A = (\alpha; +\infty),\ \alpha \in \mathbb{R}_0^+\} \cup \cup \{\emptyset\} \cup \{\mathbb{R}_0^+\} \text{ is a } \sigma^*\text{-monotone class,}$$

$$\mathcal{B}_{1*} \triangleq \{A \subset \mathbb{R}_0^+ \mid A = [0; \alpha],\ \alpha \in \mathbb{R}_0^+\} \cup \cup \{\emptyset\} \cup \{\mathbb{R}_0^+\} \text{ is a } \sigma_*\text{-monotone class.}$$

$\mathcal{B}_1^*$ and $\mathcal{B}_{1*}$ are respectively called a Borel σ^*-monotone class and a Borel σ_*-monotone class of first kind. Clearly, $\mathcal{B}_1^*$ and $\mathcal{B}_{1*}$ are extremal.

Example 2

$$\mathcal{B}_2^* \triangleq \{A \subset \mathbb{R}_0^+ \mid A = [0; \alpha),\ \alpha \in \mathbb{R}_0^+\} \cup \cup \{\emptyset\} \cup \{\mathbb{R}_0^+\} \text{ is a } \sigma^*\text{-monotone class,}$$

$$\mathcal{B}_{2*} \triangleq \{A \subset \mathbb{R}_0^+ \mid A = [\alpha; +\infty),\ \alpha \in \mathbb{R}_0^+\} \cup \cup \{\emptyset\} \cup \{\mathbb{R}_0^+\} \text{ is a } \sigma_*\text{-monotone class.}$$

$\mathcal{B}_2^*$ and $\mathcal{B}_{2*}$ are respectively called a Borel σ^*- and a Borel σ_*-monotone class of second kind. It is obvious that $\mathcal{B}_2^*$ and $\mathcal{B}_{2*}$ are extremal.

Definition 4 Let $(X, \mathcal{B}^*)$ be some upper measurable space. A function $g^* : \mathcal{B}^* \to [0; 1]$ is called an upper fuzzy measure if: (i) $g^*(\emptyset) = 0$, $g^*(X) = 1$; (ii) $\forall A, B \in \mathcal{B}^*$, $A \subset B \Rightarrow g^*(A) \leq g^*(B)$; (iii) $\forall \{A_n\} \in \mathcal{B}^*$, $n = 1, 2, \ldots, A_n \uparrow A \Rightarrow g^*(A) = \lim_{n\to\infty} g^*(A_n)$.

Definition 5 Let $(X, \mathcal{B}_*)$ be some lower measurable space. A function $g_* : \mathcal{B}_* \to [0; 1]$ is called a lower fuzzy measure if: (i) $g_*(\emptyset) = 0$, $g_*(X) = 1$; (ii) $\forall A, B \in \mathcal{B}_*$, $A \subset B \Rightarrow g_*(A) \leq g_*(B)$; (iii) $\forall \{A_n\} \in \mathcal{B}_*$, $n = 1, 2, \ldots, A_n \downarrow A \Rightarrow g_*(A) = \lim_{n\to\infty} g_*(A_n)$.

Definition 6 Let $(X, \mathcal{B}_*, \mathcal{B}^*)$ be some extremal measurable space, g_* be a lower and g^* an upper fuzzy measure.

Then:

a) $g_* : \mathcal{B}_* \to [0; 1]$ and $g^* : \mathcal{B}^* \to [0; 1]$ is called extremal if and only if

$$\forall A \in \mathcal{B}_* \ :\ g_*(A) = 1 - g^*(\overline{A}).$$

b) $(X, \mathcal{B}_*, \mathcal{B}^*, g_*, g^*)$ is called a space of extremal fuzzy measures.

Definition 7 Let $(X_1, \mathcal{B}'_*, \mathcal{B}'^*)$ and $(X_2, \mathcal{B}''_*, \mathcal{B}''^*)$ be some extremal measurable spaces; $h : X_1 \to X_2$ is called measurable if

$$\forall A \in \mathcal{B}''^*,\ B \in \mathcal{B}''_* \ :\ h^{-1}(A) \in \mathcal{B}'^*,\ h^{-1}(B) \in \mathcal{B}'_*.$$

Definition 8 Let $(X, \mathcal{B}_*, \mathcal{B}^*)$ be some extremal measurable space. Then:

a) The function $h : X \to \mathbb{R}_0^*$ is called upper measurable if and only if h is measurable with respect to the spaces $(X, \mathcal{B}_*, \mathcal{B}^*)$ and $(\mathbb{R}_0^+, \mathcal{B}_{1*}, \mathcal{B}_1^*)$. Then

$$\forall \alpha \geq 0 \quad h^{-1}((\alpha; +\infty)) \in \mathcal{B}^*, \quad h^{-1}([0; \alpha]) \in \mathcal{B}_*.$$

b) The function $h : X \to \mathbb{R}_0^+$ is called lower measurable if and only if h is measurable with respect to the spaces $(X, \mathcal{B}_*, \mathcal{B}^*)$ and $(\mathbb{R}_0^+, \mathcal{B}_{2*}, \mathcal{B}_2^*)$. Then

$$\forall \alpha \geq 0 \quad h^{-1}([0; \alpha)) \in \mathcal{B}^*, \quad h^{-1}([\alpha; +\infty)) \in \mathcal{B}_*.$$

Definition 9 Let $(X, \mathcal{B}_*, \mathcal{B}^*)$ be some extremal measurable space.

a) The class of fuzzy subsets $\widetilde{A} \subset X$ with lower measurable compatibility functions

$$\begin{aligned}\widetilde{\mathcal{B}}_* &= \left\{\widetilde{A} \subset X \mid \mu_{\widetilde{A}} \text{ is lower measurable}\right\} = \\ &= \Big\{\widetilde{A} \in X \mid \forall\, 0 \leq \alpha \leq 1,\ \mu_{\widetilde{A}}^{-1}([0; \alpha)) \in \mathcal{B}^*, \\ &\qquad \mu_{\widetilde{A}}^{-1}([\alpha; +\infty)) \in \mathcal{B}^*\Big\}\end{aligned}$$

is called an extension of the σ_*-monotone class $\mathcal{B}_*$.

b) The class of fuzzy subsets $\widetilde{A} \subset X$ with upper measurable compatibility functions

$$\begin{aligned}\widetilde{\mathcal{B}}^* &= \left\{\widetilde{A} \subset X \mid \mu_{\widetilde{A}} \text{ is upper measurable}\right\} = \\ &= \Big\{\widetilde{A} \in X \mid \forall\, 0 \leq \alpha \leq 1,\ \mu_{\widetilde{A}}^{-1}([0; \alpha]) \in \mathcal{B}_*, \\ &\qquad \mu_{\widetilde{A}}^{-1}((\alpha; +\infty)) \in \mathcal{B}^*\Big\}\end{aligned}$$

is called an extension of the σ^*-monotone class $\mathcal{B}^*$.

Definition 10 An extremal measurable space $(X, \widetilde{\mathcal{B}}_*, \widetilde{\mathcal{B}}^*)$ is called an extension of an extremal measurable space $(X, \mathcal{B}_*, \mathcal{B}^*)$.

Using the Sugeno integral, we next introduce the notion of extension of fuzzy extremal measures.

Definition 11 Let $(X, \mathcal{B}_*, \mathcal{B}^*, g_*, g^*)$ be some space of extremal fuzzy measures, and $(X, \widetilde{\mathcal{B}}_*, \widetilde{\mathcal{B}}^*)$ be an extension of the extremal measurable space $(X, \mathcal{B}_*, \mathcal{B}^*)$. Then:

a) the function

$$\widetilde{g}_*(\widetilde{A}) \equiv \int_{*X} \mu_{\widetilde{A}}(x) \circ g_*(\cdot) \triangleq$$
$$\triangleq \bigvee_{0<\alpha\leq 1} \left[\alpha \wedge g_*([\widetilde{A}]_{\bar{\alpha}})\right], \quad \forall \widetilde{A} \in \widetilde{\mathcal{B}}_*; \quad (1)$$

is called an extension of the fuzzy measure g_* on $\widetilde{\mathcal{B}}_*$;

b) the function

$$\widetilde{g}^*(\widetilde{A}) \equiv \int_X^* \mu_{\widetilde{A}}(x) \circ g^*(\cdot) \triangleq$$
$$\triangleq \bigwedge_{0<\alpha\leq 1} \left[\alpha \vee g^*([\widetilde{A}]_{\alpha})\right], \quad \forall \widetilde{A} \in \widetilde{\mathcal{B}}^*, \quad (2)$$

is called an extension of the fuzzy measure g^* on $\widetilde{\mathcal{B}}^*$.

Here $[\widetilde{A}]_\alpha = \{x \in X \mid \mu_{\widetilde{A}}(x) > \alpha\}$, $[\widetilde{A}]_{\bar{\alpha}} = \{x \in X \mid \mu_{\widetilde{A}}(x) \geq \alpha\}$, $0 < \alpha \leq 1$.

Definition 12 A space of extremal fuzzy measures $(X, \widetilde{\mathcal{B}}_*, \widetilde{\mathcal{B}}^*, \widetilde{g}_*, \widetilde{g}^*)$ is called an extension of the space $(X, \mathcal{B}_*, \mathcal{B}^*, g_*, g^*)$.

Let $(X, \mathcal{B}_*, \mathcal{B}^*, g_*, g^*)$ be some space of extremal fuzzy measures and $(X, \widetilde{\mathcal{B}}_*, \widetilde{\mathcal{B}}^*, \widetilde{g}_*, \widetilde{g}^*)$ be its extension.

Definition 13 a) Let $\widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{B}}_*$ be any fuzzy sets. Then the lower fuzzy Sugeno integral of the compatibility function $\mu_{\widetilde{B}}$ on the fuzzy set $\widetilde{A}$ is defined with respect to a lower fuzzy measure $\widetilde{g}_*$ by the formula

$$\int_{*\widetilde{A}} \mu_{\widetilde{B}}(x) \circ \widetilde{g}_*(\cdot) \triangleq \bigvee_{0<\alpha\leq 1} \left[\alpha \wedge \widetilde{g}_*(\widetilde{A} \cap [\widetilde{B}]_{\bar{\alpha}})\right]. \quad (3)$$

b) Let $\widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{B}}^*$ be any fuzzy sets. Then the upper fuzzy Sugeno integral of the compatibility function $\mu_{\widetilde{B}}$ on the fuzzy set $\widetilde{A}$ is defined with respect to a upper fuzzy measure $\widetilde{g}^*$ by the formula

$$\int_{\widetilde{A}}^* \mu_{\widetilde{B}}(x) \circ \widetilde{g}^*(\cdot) \triangleq \bigwedge_{0<\alpha\leq 1} \left[\alpha \vee \widetilde{g}^*(\widetilde{A} \cup [\widetilde{B}]_{\alpha})\right]. \quad (4)$$

Definition 14 Let $(X, \mathcal{B}_*, \mathcal{B}^*, g_*, g^*)$ be some space of extremal fuzzy measures.

a) Let $h \in \widetilde{\mathcal{B}}_*$ be some fuzzy set. The measure

$$\forall\, A \in \widetilde{\mathcal{B}}_* \;:\; \widetilde{g}_{h*}(\widetilde{A}) \triangleq \int_{*\widetilde{A}} \mu_h(x) \circ \widetilde{g}_*(\cdot) =$$
$$= \int_{*h} \mu_{\widetilde{A}}(x) \circ \widetilde{g}_*(\cdot) = \int_{*X} \mu_{h\cap\widetilde{A}}(x) \circ \widetilde{g}_*(\cdot) \quad (5)$$

is called the lower extension of g_* on $\widetilde{\mathcal{B}}_*$ with respect to h.

b) Let $h \in \widetilde{\mathcal{B}}^*$ be some fuzzy set. The measure

$$\forall\, A \in \widetilde{\mathcal{B}}^* \;:\; \widetilde{g}_h^*(\widetilde{A}) \triangleq \int_{\widetilde{A}}^* \mu_h(x) \circ \widetilde{g}^*(\cdot) =$$
$$= \int_h^* \mu_{\widetilde{A}}(x) \circ \widetilde{g}^*(\cdot) = \int_X^* \mu_{h\cup\widetilde{A}}(x) \circ \widetilde{g}^*(\cdot) \quad (6)$$

is called the upper extension of g^* on $\widetilde{\mathcal{B}}^*$ with respect to h.

Let $(X_1, \mathcal{B}'_*, \mathcal{B}'^*, g'_*, g'^*)$ and $(X_2, \mathcal{B}''_*, \mathcal{B}''^*, g''_*, g''^*)$ be any two spaces of extremal fuzzy measures.

We have constructed [16] the compositional space of extremal extended fuzzy measures $(X_1 \times X_2, \widetilde{\mathcal{B}}'_* \otimes \widetilde{\mathcal{B}}''_*, \widetilde{\mathcal{B}}^{*\prime} \otimes \widetilde{\mathcal{B}}^{*\prime\prime}, \widetilde{g}'_* \otimes \widetilde{g}''_*, \widetilde{g}^{*\prime} \otimes \widetilde{g}^{*\prime\prime})$.

2.2 On the algebraic structure of the fuzzy time intervals

A person who makes a decision always gives an “incomplete” prognosis about a time moment for extremal, crisis, anomalous and other situations that may occur in the future. The person (expert) who makes a decision connects all such situations with future fuzzy time moments and intervals. Clearly, his/her prognosis is of possibilistic character and the corresponding optimal decisions should be obtained by possibilistic-statistical analysis or, speaking more exactly, by analysis of monotone fuzzy time intervals, for which we need to construct a new fuzzy mathematical instrument.

Definition 15 a) Any fuzzy positive number $\widetilde{r} \equiv \widetilde{[0, \tau)} \in \widetilde{\mathcal{B}}_2^*$ is called an extended fuzzy current time interval.

b) Any fuzzy positive number $\widetilde{r} \equiv \widetilde{[\tau; +\infty)} \in \widetilde{\mathcal{B}}_{2*}$ is called an extended fuzzy future time interval.

ISSN: 1991-8763 Issue 11, Volume 3, November 2008

Definition 16 The class of current fuzzy time intervals is called the class of fuzzy nonnegative numbers $\widetilde{\mathcal{F}I}^*(T)$ with the properties ($\widetilde{r} \in \widetilde{\mathcal{F}I}^*(T)$):

(i) $\mu_{\widetilde{r}}(0) = 1$;

(ii) $\forall \tau_0 \geq 0$, $\mu_{\widetilde{r}}(\tau_0) = \bigvee_{\tau > \tau_0} \mu_{\widetilde{r}}(\tau)$;

(iii) $\mu_{\widetilde{r}}$ is nonincreasing on $T = \mathbb{R}_0^+$.

It is not difficult to verify that $\widetilde{\mathcal{F}I}^*(T)$ is a subclass of the space of extended fuzzy current time intervals $\widetilde{\mathcal{F}I}^*(T) \subset \widetilde{\mathcal{B}}_2^*$.

Analogously, we introduce the definition of the class $\widetilde{\mathcal{F}I}_*(T)$, which is extremal to $\widetilde{\mathcal{F}I}^*(T)$, i.e.,

$$\widetilde{r} \in \widetilde{\mathcal{F}I}_*(T) \subset \widetilde{\mathcal{B}}_{2*} \Leftrightarrow \overline{\overline{\widetilde{r}}} \in \widetilde{\mathcal{F}I}^*(T) \subset \widetilde{\mathcal{B}}_2^*.$$

Now let us consider the algebraic structures of the classes of extremal fuzzy time intervals $\langle \widetilde{\mathcal{F}I}^*(T), \widetilde{\mathcal{F}I}_*(T) \rangle$.

On the semilattice $\{\widetilde{\mathcal{F}I}^*(T), \preceq\}$ we introduce the algebraic sum operation $\widetilde{r}_1 \overset{*}{\oplus} \widetilde{r}_2$:

$$\mu_{\widetilde{r}_1 \overset{*}{\oplus} \widetilde{r}_2}(\tau) \triangleq \wedge \{\mu_{\widetilde{r}_1}(\tau_1) \vee \mu_{\widetilde{r}_2}(\tau_2) \mid \tau_1, \tau_2 \in T, \ \ \tau_1 + \tau_2 = \tau\}. \tag{7}$$

It is not difficult to verify that the structure $\{\widetilde{\mathcal{F}I}^*(T), \preceq, \overset{*}{\oplus}\}$ is a partially ordered commutative semigroup.

The algebraic sum operation $\overset{*}{\oplus}$ in $\widetilde{\mathcal{F}I}^*(T)$ induces in $\widetilde{\mathcal{F}I}_*(T)$ another operation (conjugate to $\overset{*}{\oplus}$) $\underset{*}{\oplus}$:

$$\forall \widetilde{r}_1, \widetilde{r}_2 \in \widetilde{\mathcal{F}I}_*(T) \ : \ \widetilde{r}_1 \underset{*}{\oplus} \widetilde{r}_2 = \overline{\overline{\widetilde{r}_1}} \overset{*}{\oplus} \overline{\overline{\widetilde{r}_2}}. \tag{8}$$

On $\widetilde{\mathcal{F}I}_*(T)$, the induced structure $\{\mathcal{F}I_*(T), \succeq, \underset{*}{\oplus}\}$ is a partially ordered commutative semigroup.

We call the pair of structures

$$\langle \{\widetilde{\mathcal{F}I}^*(T), \preceq, \overset{*}{\oplus}\}, \{\widetilde{\mathcal{F}I}_*(T), \succeq, \underset{*}{\oplus}\} \rangle \tag{9}$$

an extremal partially ordered commutative semigroup [16].

2.3 On the composition product of spaces of extremal fuzzy measures

Let $(X_1, \mathcal{B}'_*, \mathcal{B}'^*, g'_*, g'^*)$ and $(X_2, \mathcal{B}''_*, \mathcal{B}''^*, g''_*, g''^*)$ be any two spaces of extremal fuzzy measures.

Definition 17 Let some subset $H \subset X_1 \times X_2$ be a binary relation. We introduce the following mappings $\forall x_0 \in X_1$ and $\forall y_0 \in X_2$:

$$\begin{aligned} E_H(x_0, \cdot) &\triangleq \{y \in X_2 \mid (x_0, y) \in H\}, \\ E_H(\cdot, y_0) &\triangleq \{x \in X_1 \mid (x, y_0) \in H\}. \end{aligned} \tag{10}$$

a) A binary relation $H \subset X_1 \times X_2$ is called lower measurable if $\forall A \in \mathcal{B}''_*$ and $\forall B \in \mathcal{B}'_*$ there exist sequences $\{x_n\}_{n \in N} \subset B$, $\{y_n\}_{n \in N} \subset A$ such that $E_H(x_n, \cdot) \supset E_H(x_{n+1})$, $E_H(\cdot, y_n) \supset E_H(\cdot, y_{n+1})$, $n = 1, 2, \ldots$. We have

$$\begin{aligned} \Gamma_{H*}(A) &\triangleq \{x \in X_1 \mid \forall y \in A : (x, y) \in H\} \equiv \\ &\equiv \bigcap_{y \in A} E_H(\cdot, y) = \bigcap_{n=1}^{\infty} E_H(\cdot, y_n) \in \mathcal{B}'_* \end{aligned} \tag{11}$$

and

$$\begin{aligned} \Gamma'_{H*}(B) &\triangleq \{y \in X_2 \mid \forall x \in B : (x, y) \in H\} \equiv \\ &\equiv \bigcap_{x \in B} E_H(x, \cdot) = \bigcap_{n=1}^{\infty} E_H(x_n, \cdot) \in \mathcal{B}''_*. \end{aligned} \tag{12}$$

b) Denote by $\mathcal{B}'_* \otimes \mathcal{B}''_*$ the set of all binary lower measurable relations from $X_1 \times X_2$ and call it the composition product of measurable spaces $\mathcal{B}'_*$ and $\mathcal{B}''_*$.

a′) A binary relation $H \subset X_1 \times X_2$ is called upper measurable if $\forall A \in \mathcal{B}''^*$ and $\forall B \in \mathcal{B}'^*$ there exist sequences $\{x_n\}_{n \in N} \subset B$, $\{y_n\}_{n \in N} \subset A$ such that $E_H(x_n, \cdot) \subset E_H(x_{n+1})$, $E_H(\cdot, y_n) \subset E_H(\cdot, y_{n+1})$, $n = 1, 2, \ldots$. We have

$$\begin{aligned} \Gamma^*_H(A) &\triangleq \{x \in X_1 \mid \exists y \in A : (x, y) \in H\} \equiv \\ &\equiv \bigcup_{y \in A} E_H(\cdot, y) = \bigcup_{n=1}^{\infty} E_H(\cdot, y_n) \in \mathcal{B}'^* \end{aligned} \tag{13}$$

and

$$\begin{aligned} \Gamma'^*_H(B) &\triangleq \{y \in X_2 \mid \exists x \in B : (x, y) \in H\} \equiv \\ &\equiv \bigcup_{x \in B} E_H(x, \cdot) = \bigcup_{n=1}^{\infty} E_H(x_n, \cdot) \in \mathcal{B}''^*. \end{aligned} \tag{14}$$

b′) Denote by $\mathcal{B}'^* \otimes \mathcal{B}''^*$ the set of all binary upper measurable relations from $X_1 \times X_2$ and call it the composition product of measurable spaces $\mathcal{B}'^*$ and $\mathcal{B}''^*$.

It is not difficult to verify that $\mathcal{B}'_* \otimes \mathcal{B}''_*$ is a lower σ_*-monotone class and $\mathcal{B}'^* \otimes \mathcal{B}''^*$ is a upper σ^*-monotone class.

ISSN: 1991-8763 942 Issue 11, Volume 3, November 2008

Theorem 1 Let $(X_1, \mathcal{B}'_*, g'_*)$ and $(X_2, \mathcal{B}''_*, g''_*)$ be two spaces of lower fuzzy measures. Then on the composition lower measurable space $(X_1 \times X_2, \mathcal{B}'_* \otimes \mathcal{B}''_*)$ the measure $g_* : \forall H \in \mathcal{B}'_* \otimes \mathcal{B}'_*$ defined by

$$\begin{aligned} g_*(H) &\equiv g'_* \otimes g''_*(H) \triangleq \\ &\triangleq \bigvee_{E \in \mathcal{B}'_*} \{g'_*(E) \wedge g''_*(\Gamma'_{H*}(E))\} \equiv \\ &\equiv \bigvee_{F \in \mathcal{B}''_*} \{g'_*(\Gamma_{H*}(F)) \wedge g''_*(F)\} \end{aligned} \quad (15)$$

is a lower fuzzy measure.

Theorem 2 Let $(X_1, \mathcal{B}'^*, g'^*)$ and $(X_2, \mathcal{B}''^*, g''^*)$ be two spaces of upper fuzzy measures. Then, on the composition upper measurable space $(X_1 \times X_2, \mathcal{B}'^* \otimes \mathcal{B}''^*)$, the measure $g^* : \forall H \in \mathcal{B}'^* \otimes \mathcal{B}''^*$ defined by

$$\begin{aligned} g^*(H) &\equiv g'^* \otimes g''^*(H) \triangleq \\ &\triangleq \bigwedge_{E \in \mathcal{B}'^*} \{g'^*(E) \vee g''^*(\Gamma'^*_H(E))\} = \\ &= \bigwedge_{F \in \mathcal{B}''^*} \{g'^*(\Gamma^*_H(F)) \vee g''^*(F)\} \end{aligned} \quad (16)$$

is an upper fuzzy measure.

Theorem 3 a) Let $H \in \mathcal{B}'_* \otimes \mathcal{B}''_*$ be some binary lower measurable relation ($H \subset X_1 \times X_2$). Then the value of the measure $g'_* \otimes g''_*$ on H is represented through g'_* and g''_* as the following composition:

$$\begin{aligned} g'_* \otimes g''_*(H) &= \int_{*X_2} g'_*(E_H(\cdot, y)) \circ g''_*(\cdot) = \\ &= \int_{*X_1} g''_*(E_H(x, \cdot)) \circ g'_*(\cdot); \end{aligned} \quad (17)$$

b) Let $H \in \mathcal{B}'^* \otimes \mathcal{B}''^*$ be some binary upper measurable relation. Then the value of the measure $g'^* \otimes g''^*$ on H is represented through g'^* and g''^* as the following composition:

$$\begin{aligned} g'^* \otimes g''^*(H) &= \int^*_{X_2} g'^*(E_H(\cdot, y)) \circ g''^*(\cdot) = \\ &= \int^*_{X_1} g''^*(E_H(x, \cdot)) \circ g'^*(\cdot). \end{aligned} \quad (18)$$

Now let us proceed to defining fuzzy binary relations on $X_1 \times X_2$.

Definition 18 a) A fuzzy set $\widetilde{H} \subset X_1 \times X_2$ is called a lower fuzzy binary relation if the compatibility function $\mu_{\widetilde{H}} : X_1 \times X_2 \to [0;1]$ is lower measurable;

b) A fuzzy set $\widetilde{H} \subset X_1 \times X_2$ is called an upper fuzzy binary relation if the compatibility function $\mu_{\widetilde{H}}$ is upper measurable.

We have constructed the compositional space of extremal extended fuzzy measures $(X_1 \times X_2, \widetilde{\mathcal{B}}'_* \otimes \widetilde{\mathcal{B}}''_*, \widetilde{\mathcal{B}}^{*\prime} \otimes \widetilde{\mathcal{B}}^{*\prime\prime}, \widetilde{g}'_* \otimes \widetilde{g}''_*, \widetilde{g}^{*\prime} \otimes \widetilde{g}^{*\prime\prime})$.

2.4 On the weakly structurable continuous dynamic system

In [16], [21] we described objects of a fuzzy dynamic system. Let X ($X \neq \varnothing$) be the set of states of some system to be investigated. Let $(X, \mathcal{B}, g)$ be the space of a fuzzy measure on the measurable space $(X, \mathcal{B})$, where $\mathcal{B}$ is a σ-algebra in X (fuzzy restrictions on states).

Let the time structure of fuzzy dynamic systems is represented by some space of extended extremal fuzzy measures

$$\langle T, \widetilde{\mathcal{F}I}_*(T), \widetilde{\mathcal{F}I}^*(T), \widetilde{g}_{T*}, \widetilde{g}^*_T \rangle, \quad T = \mathbb{R}^*_0, \quad (19)$$

where $\widetilde{g}_{T*}$ and $\widetilde{g}^*_T$ are some extremal fuzzy measures on $\widetilde{\mathcal{B}}_{T*} \equiv \widetilde{\mathcal{B}}_{2*}$ and $\widetilde{\mathcal{B}}^*_T \equiv \widetilde{\mathcal{B}}^*_2$, respectively.

Definition 19 a) A family $\{\widetilde{r}^*_\tau\}_{\tau \geq 0}$, $\widetilde{r}^*_\tau \in \widetilde{\mathcal{B}}^*_T$, $\tau \geq 0$, of monotonically increasing upper fuzzy time intervals, i.e.,

$$\forall \tau_2 > \tau_1 \geq 0, \quad \widetilde{r}^*_{\tau_1} \preceq \widetilde{r}^*_{\tau_2}$$

is called a process of current fuzzy time intervals.

b) A family $\{\widetilde{r}_{\tau*}\}_{\tau \geq 0}$, $\widetilde{r}_{\tau*} \in \widetilde{\mathcal{B}}_{T*}$, $\tau \geq 0$, of monotonically decreasing lower fuzzy time intervals, i.e.,

$$\forall \tau_2 > \tau_1 \geq 0, \quad \widetilde{r}_{\tau_1 *} \succeq \widetilde{r}_{\tau_2 *}$$

is called a process of future fuzzy time intervals.

c) A pair of processes of future and current fuzzy time intervals $\{\widetilde{r}_{\tau*}, \widetilde{r}^*_\tau\}_{\tau \geq 0}$ is called a process of extremal fuzzy time intervals.

Definition 20 A process of extremal fuzzy time intervals $(\widetilde{r}_{\tau*}, \widetilde{r}^*_\tau)$ is called ergodic if there exist the limits

$$\lim_{\tau \to +\infty} \widetilde{r}_{\tau*} = \widetilde{r}_{\infty*} \in \widetilde{\mathcal{B}}_{T*}, \quad \lim_{\tau \to +\infty} \widetilde{r}^*_\tau = \widetilde{r}^*_\infty \in \widetilde{\mathcal{B}}^*_T.$$

A relation between the spaces $(X, \mathcal{B}, \mathcal{B}, g, g^*)$ and $(T, \mathcal{B}_{T*}, \mathcal{B}^*_T, g_{T*}, g^*_T)$ and their extensions through conditional measures can be represented as

ISSN: 1991-8763   943   Issue 11, Volume 3, November 2008

follows: $\forall r_* \in \mathcal{B}_{T*}$, $r^* \in \mathcal{B}_T^*$, $\widetilde{r}_* \in \widetilde{\mathcal{B}}_{T*}$, $\widetilde{r}^* \in \widetilde{\mathcal{B}}_T^*$

$$\begin{aligned} g_{T*}(r_*) &= \int\!\!\!\!\!-_X g_{t*}(r_* \mid x) \circ g(\cdot), \\ g_T^*(r^*) &= \int\!\!\!\!\!-_X g_t^*(r^* \mid x) \circ g^*(\cdot), \\ \widetilde{g}_{T*}(\widetilde{r}_*) &= \int\!\!\!\!\!-_X \widetilde{g}_{t*}(\widetilde{r}_* \mid x) \circ g(\cdot), \\ \widetilde{g}_T^*(\widetilde{r}^*) &= \int\!\!\!\!\!-_X \widetilde{g}_t^*(\widetilde{r}^* \mid x) \circ g^*(\cdot), \end{aligned} \tag{20}$$

For any lower and upper fuzzy time intervals $\widetilde{r}_* \in \widetilde{\mathcal{B}}_{T*}$ and $\widetilde{r}^* \in \widetilde{\mathcal{B}}_T^*$ there exist fuzzy sets $\widetilde{A}_{\widetilde{r}_*} \in \widetilde{\mathcal{B}}$, $\widetilde{A}_{\widetilde{r}^*} \in \widetilde{\mathcal{B}}$ such that $\forall x \in X$

$$\mu_{\widetilde{A}_{\widetilde{r}_*}}(x) = \widetilde{g}_{t*}(\widetilde{r}_* \mid x), \quad \mu_{\widetilde{A}_{\widetilde{r}^*}}(x) = \widetilde{g}_t^*(\widetilde{r}^* \mid x). \tag{21}$$

Definition 21 For extremal fuzzy time intervals $(\widetilde{r}_*, \widetilde{r}^*)$ the fuzzy sets $\widetilde{A}_{\widetilde{r}_*}$ and $\widetilde{A}_{\widetilde{r}^*} \in \widetilde{\mathcal{B}}$ from the extended measurable space of system states are called the expert knowledge reflections of extremal fuzzy time intervals with respect to extended extremal conditional fuzzy measures $\widetilde{g}_{t*}(\cdot \mid x)$ and $\widetilde{g}_t^*(\cdot \mid x)$.

Let us formulate a theorem that describes the ergodicity of a reflection process in an ergodic process of extremal fuzzy time intervals.

Theorem 4 An ergodic process $(\widetilde{r}_{\tau*}, \widetilde{r}_\tau^*)_{\tau \geq 0}$ of extremal fuzzy time intervals on the measurable space $(X, \widetilde{\mathcal{B}})$ of states of the system induces an ergodic reflection process $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*) \equiv (\widetilde{A}_{\widetilde{r}_{\tau*}}, \widetilde{A}_{\widetilde{r}_\tau^*})_{\tau \geq 0}$.

Let the initial time moment $t = 0$ the fuzzy state of the fuzzy dynamic system be represented by a pair of fuzzy sets $\widetilde{A}_{0*}, \widetilde{A}_0^* \in \widetilde{\mathcal{B}}$.

Now assume that the Fuzzy Dynamic System is represented by some operator $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ describing the system state change dynamics.

Definition 22 *a)* Let $(\widetilde{r}_{\tau*}, \widetilde{r}_\tau^*)_{\tau \geq 0}$ be some process of extremal fuzzy time intervals. A pair $(\widetilde{Q}_*, \widetilde{Q}^*)$ of lower and upper measurable binary relations $\widetilde{Q}_* \in \widetilde{\mathcal{B} \otimes \mathcal{B}_{T*}}$ and $\widetilde{Q}^* \in \widetilde{\mathcal{B} \otimes \mathcal{B}_T^*}$ is called a fuzzy process describing the system state dynamics in the process of extremal fuzzy time intervals $(\widetilde{r}_{\tau*}, \widetilde{r}_\tau^*)_{\tau \geq 0}$ if the following representation holds $\forall (x, \tau) \in X \times T$:

$$\mu_{\widetilde{Q}_*}(x,\tau) \overset{\Delta}{=} \int\!\!\!\!\!-_{*\,\widetilde{r}_{\tau*}} \left[\int\!\!\!\!\!-_{\widetilde{A}_{0*}} \mu_{\widetilde{\rho}_*}(x, x', t) \circ g(\cdot) \right] \circ \widetilde{g}_{T*}(\cdot) \equiv$$

$$\equiv \int\!\!\!\!\!-_{*\,\widetilde{r}_{\tau*}} \mu_{\widetilde{\rho}'_*}(x,t) \circ \widetilde{g}_{T*}(\cdot), \tag{22}$$

$$\mu_{\widetilde{Q}^*}(x,\tau) \overset{\Delta}{=} \int\!\!\!\!\!-_{\widetilde{r}_\tau^*}^{*} \left[\int\!\!\!\!\!-_{\widetilde{A}_0^*} \mu_{\widetilde{\rho}^*}(x, x', t) \circ g^*(\cdot) \right] \circ \widetilde{g}_T^*(\cdot) \equiv$$

$$\equiv \int\!\!\!\!\!-_{\widetilde{r}_\tau^*}^{*} \mu_{\widetilde{\rho}'^*}(x,t) \circ \widetilde{g}_T^*(\cdot).$$

b) Fuzzy Dynamic System represented by the following cortege

$$\langle X, \mathcal{B}, g, T, \{\widetilde{r}_{\tau*}\}, \{\widetilde{r}_\tau^*\}, (\widetilde{\rho}_*, \widetilde{\rho}^*), (\widetilde{g}_{T*}, \widetilde{g}_T^*), (\widetilde{R}_*, \widetilde{R}^*), (\widetilde{Q}_*, \widetilde{Q}^*) \rangle$$

is called the weakly structurable continuous dynamic system if extremal processes are ergodic.

Now let us consider an important theorem, in which a relation between the processes $(\widetilde{Q}_*, \widetilde{Q}^*)$ and $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$ is defined.

Theorem 5 Let $(\widetilde{r}_{\tau*}, \widetilde{r}_\tau^*)_{\tau \geq 0}$ be some ergodic process of extremal fuzzy time intervals, and $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$ be a process of expert knowledge reflection of extremal fuzzy time intervals on $(X, \widetilde{\mathcal{B}})$ with respect to some extended extremal conditional fuzzy measures $\widetilde{g}_{t*}(\cdot \mid x)$ and $\widetilde{g}_t^*(\cdot \mid x)$. Then

a) the process $(\widetilde{Q}_*, \widetilde{Q}^*)$ describing the WSCDS state dynamics in the process of extremal fuzzy time intervals $(\widetilde{r}_{\tau*}, \widetilde{r}_\tau^*)_{\tau \geq 0}$ has the following representation: $\forall (x, \tau) \in X \times T$

$$\begin{aligned} \mu_{\widetilde{Q}_*}(x,\tau) &= \int\!\!\!\!\!-_{*\,T} \mu_{\widetilde{\rho}'_*}(x,t) \circ \widetilde{g}_{E_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(\cdot), \\ \mu_{\widetilde{Q}^*}(x,\tau) &= \int\!\!\!\!\!-_{T}^{*} \mu_{\widetilde{\rho}'^*}(x,t) \circ \widetilde{g}_{E_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(\cdot), \end{aligned} \tag{23}$$

where on the right-hand sides of lower and upper Sugeno integrals the integration measures are the extremal fuzzy measures extended with respect to the process $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$ on the measurable spaces $\widetilde{\mathcal{B}}_{T*}$ and $\widetilde{\mathcal{B}}_T^*$, respectively.

b) The process $(\widetilde{Q}_*, \widetilde{Q}^*)$ is ergodic.

3 The Fuzzy Dynamic Programming Problem in WSCDS

3.1 Case when a fuzzy control does not depend on the WSCDS state

All definitions and results see in [17], [21].

In alternative classical approaches to modeling and when working with the weakly structurable systems the main accent is placed on the assumption of fuzzyness [2], [3], [6], 1[5], [21]. We will deal with fuzzy dynamic systems (WSCDS), where fuzzy uncertainty arises with time and time structures are monotone classes of measurable sets [16].

We start describing objects of a controllable WSCDS. Let X ($X \neq \emptyset$) be the set of states of some system (WSCDS) to be investigated. Let $(X, \mathcal{B}, g)$ be the space of a fuzzy measure on the measurable space $(X, \mathcal{B})$, where $\mathcal{B}$ is a σ-algebra in X (fuzzy restrictions on states).

Let the time structure of a WSCDS is represented by (9) and some space of extended extremal fuzzy measures

$$(T, \widetilde{\mathcal{B}}_{T*}, \widetilde{\mathcal{B}}_T^*, \widetilde{g}_{T*}, \widetilde{g}_T^*), \quad T = \mathbb{R}_0^*,$$

where $\widetilde{g}_{T*}$ and $\widetilde{g}_T^*$ are some extremal fuzzy measures on $\widetilde{\mathcal{B}}_{T*} \equiv \widetilde{\mathcal{B}}_{2*}$ and $\widetilde{\mathcal{B}}_T^* \equiv \widetilde{\mathcal{B}}_2^*$, respectively.

Let U ($U \neq \emptyset$) be the set of all admissible controls (of external factors) acting on the WSCDS. Assume that controls are subjected to restrictions of uncertain character in the form of some space of a fuzzy measure $(U, \mathcal{B}_U, g_U)$, where $\mathcal{B}_U$ is the measurable space of controls, while the fuzzy measure g_U describes the restrictions imposed on controls.

We consider the optimization problems of a controllable WSCDS when the model of the continuous extremal fuzzy process is described by the system of fuzzy integral equations [16], [21]:

$$\mu_{\widetilde{Q}_*}(x,\tau) = \int\!\!\!\!\!\!\!-_{*}\limits_{U\times T} \Big\{ \mu_{\mathbb{E}_{\widetilde{u}_*}(\cdot,\tau)}(u) \wedge \mu_{\mathbb{E}_{\widetilde{\rho}'_*}(x,\cdot,\cdot)}(u,t) \Big\} \circ \circ\, \widetilde{g_U \otimes g_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}}(\cdot), \tag{24}$$

$$\mu_{\widetilde{Q}^*}(x,\tau) = \int\!\!\!\!\!\!\!-^{*}\limits_{U\times T} \Big\{ \mu_{\mathbb{E}_{\widetilde{u}^*}(\cdot,\tau)}(u) \vee \mu_{\mathbb{E}_{\widetilde{\rho}'^*}(x,\cdot,\cdot)}(u,t) \Big\} \circ \circ\, \widetilde{g_U^* \otimes g_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}}(\cdot),$$

where $(\widetilde{Q}_*, \widetilde{Q}^*)$ is a fuzzy extremal process describing the system state dynamics; $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$ is an extremal fuzzy process of expert reflections in extremal fuzzy time intervals (the expert reflections on the states of WSCDS in the extremal fuzzy time intervals); $(\widetilde{\rho}_*, \widetilde{\rho}^*)$ is the operator of the WSCDS states change dynamics; on right-hand sides of Sugeno extended lower and upper integrals the integration measures are the extremal compositional fuzzy measures extended with respect to the process $(\widetilde{\mathbb{R}}_*, \widetilde{\mathbb{R}}^*)$ (Definition 21); $\mathbb{E}$ is a symbol of projector of Galois indexing mapping.

We say that the effectiveness of WSCDS control is defined by some set of Criteria K, on which fuzzy restrictions are given for measurable subsets of K, i.e. the fuzzy measure space $(K, \mathcal{B}_K, g_K)$ (fuzzy restriction on the criteria) is defined on K [17].

Let $\widetilde{L} \in \widetilde{\mathcal{B}_K \otimes \mathcal{B}_U}$ be some fuzzy binary relation of "losses" with respect to each of the criteria $v \in K$ in the choice of control $u \in U$. Note that $\mu_{\widetilde{L}}$ is a $\mathcal{B}_K \otimes \mathcal{B}_U$-measurable compatibility function

$$\mu_{\widetilde{L}}(v,u) : K \times U \to [0,1]. \tag{25}$$

Then the complement $\overline{\overline{\widetilde{L}}}$ is called the fuzzy relation of WSCDS "gain" and the values

$$\mu_{\overline{\overline{\widetilde{L}}}}(v,u) = 1 - \mu_{\widetilde{L}}(v,u) \tag{26}$$

define the measure of gain in the choice of control $u \in U$ for a criterion $v \in K$.

Definition 23 a) Given all criteria, a $\mathcal{B}_U \otimes \mathcal{B}_T^*$-measurable function: $\forall (u,t) \in U \times T$

$$\mathbf{P}_{\widetilde{u}^*}^K(u,t) \triangleq \triangleq \int\!\!\!\!\!\!\!-\limits_{K} \left\{ \mu_{\mathbb{E}_{\widetilde{u}^*}(\cdot,t)}(u) \vee \mu_0^*(u) \vee \mu_{\overline{\overline{\widetilde{L}}}}(v,u) \right\} \circ \circ\, \widetilde{g}_K^*(\cdot), \tag{27}$$

where the extended fuzzy measure $\widetilde{g}_K^* : \mathcal{B}_K \to [0,1]$ is the dual fuzzy measure of $\widetilde{g}_K$ ($\forall \widetilde{S} \in \widetilde{\mathcal{B}}_K : \widetilde{g}_K^*(\widetilde{S}) = 1 - \widetilde{g}_K(\overline{\overline{\widetilde{S}}})$), is called a gain with respect to a current (upper) fuzzy control process $\widetilde{u}^* \in \widetilde{\mathcal{B}_U \otimes \mathcal{B}_T^*}$ with respect to the initial fuzzy control $\mu_{\mathbb{E}_{\widetilde{u}^*}(\cdot,\tau_0)}(u) \equiv \mu_0^*(u)$.

b) Given all criteria, a $\mathcal{B}_U \otimes \mathcal{B}_{T*}$-measurable function: $\forall (u,t) \in U \times T$

$$\mathbf{q}_{\widetilde{u}_*}^K(u,t) \triangleq \triangleq \int\!\!\!\!\!\!\!-\limits_{K} \Big\{ \mu_{\mathbb{E}_{\widetilde{u}_*}(\cdot,t)}(u) \wedge \mu_{0*}(u) \wedge \mu_{\widetilde{L}}(v,u) \Big\} \circ \circ\, \widetilde{g}_K(\cdot) \tag{28}$$

is called a loss with respect to a future (lower) fuzzy control process $\widetilde{u}_* \in \widetilde{\mathcal{B}_U \otimes \mathcal{B}_{T*}}$ with respect to the initial fuzzy control $\mu_{\mathbb{E}_{\widetilde{u}_*}(\cdot,\tau_0)}(u) \equiv \mu_{0*}(u)$.

Definition 24 a) A $\mathcal{B} \otimes \mathcal{B}_T^*$-measurable function: $\forall (u,\tau) \in U \times T$

$$I_{\widetilde{u}^*}(u,\tau) \triangleq \int\!\!\!\!\!\!\!-^{*}\limits_{T} \mathbf{P}_{\widetilde{u}^*}^K(u,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(\cdot) \tag{29}$$

is called an integral current gain with respect to a current (upper) fuzzy control process $\widetilde{u}^* \in \widetilde{\mathcal{B}_U \otimes \mathcal{B}_T^*}$ on a current fuzzy time interval $\widetilde{r}_\tau^* \in \widetilde{\mathcal{B}}_T^*$.

b) A $\mathcal{B}_U \otimes \mathcal{B}_{T*}$-measurable function: $\forall(u,\tau) \in U \times T$

$$J_{\widetilde{u}_*}(u,\tau) \triangleq \fint_{T}^{}{}_{*} \mathbf{q}^K_{\widetilde{u}_*}(u,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(\cdot) \qquad (30)$$

is called an integral future loss with respect to a future (upper) fuzzy control process $\widetilde{u}_* \in \widetilde{\mathcal{B}_U \otimes \mathcal{B}_{T*}}$ on a future fuzzy time interval $\widetilde{r}_{\tau*} \in \widetilde{\mathcal{B}}_{T*}$.

We have thus defined, on U, an extremal fuzzy "gain-loss" process $(I_{u^*}, \dot{J}_{u_*})$. Further, for model (24) we will consider, in terms of (29) and (30), the problem of formation of an optimal control (in the sense of minimization of the future loss and maximization of the current gain) of an extremal process: $\forall(u,t) \in U \times T$

$$\begin{aligned} &\fint_T^{*} \mathbf{P}^K_{\widetilde{u}^*}(u,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(\cdot) \Rightarrow \max_{\widetilde{u}^*}, \\ &\fint_{T}{}_{*} \mathbf{q}^K_{\widetilde{u}_*}(u,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(\cdot) \Rightarrow \min_{\widetilde{u}_*}. \end{aligned} \qquad (31)$$

Functional equations by means of which we can define an extremal fuzzy optimal control in the sense of extremalization of criteria (31) can be written in the following form, $\forall(u,\tau') \in U \times [\tau_0,\tau]$:

$$\begin{aligned} &\dot{J}_{\overset{\circ}{\widetilde{u}}_*}(u,\tau') = \bigwedge_{\widetilde{u}_* \in \widetilde{\mathcal{B}_U \otimes \mathcal{B}_{T*}}} \dot{J}_{\widetilde{u}_*}(u,\tau') = \\ &= \bigwedge_{\widetilde{u}_* \in \widetilde{\mathcal{B}_U \otimes \mathcal{B}_{T*}}} \fint_T{}_{*} \mathbf{q}^K_{\widetilde{u}_*}(u,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau')}(\cdot), \\ &I_{\overset{\circ}{\widetilde{u}}^*}(u,\tau') = \bigvee_{\widetilde{u}_* \in \widetilde{\mathcal{B}_U \otimes \mathcal{B}^*_T}} I_{\widetilde{u}^*}(u,\tau') = \\ &= \bigvee_{\widetilde{u}^* \in \widetilde{\mathcal{B}_U \otimes \mathcal{B}^*_T}} \fint_T^{*} \mathbf{P}^K_{\widetilde{u}^*}(u,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau')}(\cdot), \end{aligned} \qquad (32)$$

with the initial control conditions

$$\begin{aligned} &\mathbb{E}_{\overset{\circ}{\widetilde{u}}_*}(\cdot,\tau_0) \equiv \widetilde{u}_{0*} \in \mathcal{B}_U, \\ &\mathbb{E}_{\overset{\circ}{\widetilde{u}}^*}(\cdot,\tau_0) \equiv \widetilde{u}^*_0 \in \mathcal{B}_U \end{aligned} \qquad (33)$$

and the WSCDS initial states $\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_0)$ and $\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_0)$.

Definition 25 An extremal fuzzy control process $(\overset{\circ}{\widetilde{u}}_*, \overset{\circ}{\widetilde{u}}^*)$, $\tau_0 \le \tau' \le \tau$, with the initial conditions (33) is called an optimal for WSCDS (24) in the sense of Bellman's optimality principle if criterion (32) is satisfied.

The following theorem which gives the optimality condition (an analogue of Bellman's equation [1]) is valid.

Theorem 6 Let a controllable WSCDS be described by system (24). Then an extremal fuzzy control process $(\overset{\circ}{\widetilde{u}}_*, \overset{\circ}{\widetilde{u}}^*)$, $\tau_0 \le \tau' \le \tau$, is optimal in the sense of criterion (32) if and only if the following inequalities are fulfilled: $\forall(u,\tau') \in U \times [\tau_0,\tau]$

$$\begin{aligned} \dot{J}_{\overset{\circ}{\widetilde{u}}_*}(u,\tau') &\le \Big(\fint_K \mu_{\widetilde{L}}(v,u) \circ \widetilde{g}_K(\cdot)\Big) \wedge \\ &\qquad \wedge \mu_{\mathbb{E}_{\overset{\circ}{\widetilde{u}}_*}(\cdot,\tau_0)}(u), \\ I_{\overset{\circ}{\widetilde{u}}^*}(u,\tau') &\ge \Big(\fint_K \mu_{\overline{\widetilde{L}}}(v,u) \circ \widetilde{g}^*_K(\cdot)\Big) \vee \\ &\qquad \vee \mu_{\mathbb{E}_{\overset{\circ}{\widetilde{u}}^*}(\cdot,\tau_0)}(u); \end{aligned} \qquad (34)$$

Theorem 7 An extremal fuzzy optimal control process $(\overset{\circ}{\widetilde{u}}_*, \overset{\circ}{\widetilde{u}}^*)$ for the WSCDS (24) in the sense of criterion (32) not depending on a WSCDS state can be defined by the following system of fuzzy-integral equations: $\forall(u,\tau') \in U \times [\tau_0,\tau]$

$$\begin{aligned} &\mu_{\overset{\circ}{\widetilde{u}}_*}(u,\tau') = \mu_{\overset{\circ}{\widetilde{u}}_*}(u,\tau_0) \wedge \\ &\wedge \Big(\fint_K \mu_{\widetilde{L}}(v,u) \circ \widetilde{g}_K(\cdot)\Big) \wedge \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\Delta(\tau_0,\tau'))}(T), \\ &\mu_{\overset{\circ}{\widetilde{u}}^*}(u,\tau') = \mu_{\overset{\circ}{\widetilde{u}}^*}(u,\tau_0) \vee \\ &\vee \Big(\fint_K \mu_{\overline{\widetilde{L}}}(v,u) \circ \widetilde{g}^*_K(\cdot)\Big) \vee \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\Delta(\tau_0,\tau'))}(T). \end{aligned} \qquad (35)$$

Remark 2 Expressions in (35) of an extremal optimal fuzzy control process $(\overset{\circ}{\widetilde{u}}_*, \overset{\circ}{\widetilde{u}}^*)$, $\tau_0 \le \tau' \le \tau$, are a variant of the solution of inequalities (34), but this fuzzy-integral representation of an optimal control gives a good analogue of the solution of the problem of stochastic dynamic programming, where the expression of an optimal control contains "direct" analogues to (35): $\fint_K \mu_{\widetilde{L}}(v,u) \circ g_K(\cdot)$ is the Bellman functional which is an analogue of the kernel in the representation of a stochastic optimal control or, more exactly, an analogue of the signal of a stochastic model or its deterministic part, while the values of the extended fuzzy measures $\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\Delta(\tau_0,\tau'))}(T)$ and $\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\Delta(\tau_0,\tau'))}(T)$ are analogues of stochastic measure in the representation of stochastic optimal controls.

It is studied the case when a fuzzy control of WSCDS depends not only on time $\tau' \in [0,\tau]$ but also on a WSCDS state $x \in X$.

ISSN: 1991-8763 946 Issue 11, Volume 3, November 2008

3.2 Case when a fuzzy control depends on the WSCDS state

Now let us consider a more difficult case, where a fuzzy control of a controllable WSCDS depends not only on time $\tau' \in [\tau_0, \tau]$, but also on a WSCDS state $x \in X$. Then the fuzzy control is considered as $\widetilde{u}' \in \widetilde{\mathcal{B}_U \otimes \mathcal{B}}$, a future fuzzy control process as $\widetilde{u}'_* \in (\widetilde{\mathcal{B}_U \otimes \mathcal{B}}) \otimes \widetilde{\mathcal{B}}_{T*}$, a current fuzzy control process as $\widetilde{u}'^* \in (\widetilde{\mathcal{B}_U \otimes \mathcal{B}}) \otimes \widetilde{\mathcal{B}}^*_T$ or $\mu_{\widetilde{u}'} : U \times X \to [0,1]$, $\mu_{\widetilde{u}'_*} : (U \times X) \times T \to [0,1]$, $\mu_{\widetilde{u}'^*} : (U \times X) \times T \to [0,1]$.

In this situation WSCDS model (24) changes and we obtain: $\forall (x,\tau) \in X \times T$

$$\begin{aligned} &\mu_{\widetilde{Q}_*}(x,\tau) = \\ &= \int_{U\times T *} \Big\{ \mu_{\mathbb{E}_{\widetilde{u}'_*}(\cdot,x,\tau)}(u) \wedge \mu_{\mathbb{E}_{\widetilde{\rho}'_*}(x,\cdot,\cdot)}(u,t) \Big\} \circ \\ &\quad \circ \widetilde{g_U \otimes g_{\mathbb{E}_{\widetilde{R}_*}(\cdot,\tau)}}(\cdot), \\ &\mu_{\widetilde{Q}^*}(x,\tau) = \\ &= \int^*_{U\times T} \Big\{ \mu_{\mathbb{E}_{\widetilde{u}'^*}(\cdot,x,\tau)}(u) \vee \mu_{\mathbb{E}_{\widetilde{\rho}'^*}(x,\cdot,\cdot)}(u,t) \Big\} \circ \\ &\quad \circ \widetilde{g_U \otimes g_{\mathbb{E}_{\widetilde{R}^*}(\cdot,\tau)}}(\cdot). \end{aligned} \tag{36}$$

The maximal gain and the minimal loss for a fuzzy control $\widetilde{u}' \in \widetilde{\mathcal{B}_U \otimes \mathcal{B}}$ change as follows: $\forall (v,x) \in K \times X$

$$\begin{aligned} I_{\widetilde{u}'}(v,x) &\triangleq \bigvee_{u\in U} \left\{ \mu_{\mathbb{E}_{\widetilde{u}}(\cdot,x)}(u) \wedge \mu_{\overline{\widetilde{L}}'}(v,u,x) \right\}, \\ \dot{J}_{\widetilde{u}'}(v,x) &\triangleq \bigwedge_{u\in U} \left\{ \mu_{\mathbb{E}_{\overline{\widetilde{u}}}(\cdot,x)}(u) \wedge \mu_{\widetilde{L}'}(v,u,x) \right\}. \end{aligned} \tag{37}$$

where $\widetilde{L}'$ is a fuzzy loss taking into account the state $x \in X$.

a) a maximal gain $\mathbf{P}_{\widetilde{u}'^*}$ is calculated by

$$\begin{aligned} &\mathbf{P}_{\widetilde{u}'^*}(v,x,t) \triangleq \\ &\triangleq \bigvee_{u\in U} \Big\{ \big(\mu_{\mathbb{E}_{\widetilde{u}'^*}(\cdot,x,t)}(u) \vee \mu_{\mathbb{E}_{\widetilde{u}'^*}(\cdot,x,\tau_0)}(u) \big) \wedge \\ &\qquad \wedge \mu_{\overline{\widetilde{L}}'}(v,u,x) \Big\}. \end{aligned} \tag{38}$$

and b) a minimal loss $\mathbf{q}_{\widetilde{u}'_*}$ is calculated by

$$\begin{aligned} &\mathbf{q}_{\widetilde{u}'_*}(v,x,t) \triangleq \\ &\triangleq \bigwedge_{u\in U} \Big\{ \Big(\mu_{\mathbb{E}_{\widetilde{u}'_*}(\cdot,x,t)}(u) \wedge \mu_{\mathbb{E}_{\widetilde{u}'_*}(\cdot,x,\tau_0)}(u) \Big) \vee \\ &\qquad \vee \mu_{\widetilde{L}'}(v,u,x) \Big\}. \end{aligned} \tag{39}$$

In Definition 23, a gain and a loss for all criteria are calculated as follows: $\forall (u,x,t) \in U \times X \times [\tau_0, \tau]$

$$\begin{cases} \mathbf{P}^K_{\widetilde{u}'^*}(u,x,t) \triangleq \\ \triangleq \int_K \Big[\mu_{\mathbb{E}_{\widetilde{u}'^*}(\cdot,x,t)}(u) \vee \mu_{\mathbb{E}_{\widetilde{u}'^*}(\cdot,x,\tau_0)}(u) \vee \\ \qquad \vee \mu_{\overline{\widetilde{L}}'}(v,u,x) \Big] \circ \widetilde{g}^*_K(\cdot), \\ \mathbf{q}^K_{\widetilde{u}'_*}(u,x,t) \triangleq \\ \triangleq \int_K \Big[\mu_{\mathbb{E}_{\widetilde{u}'_*}(\cdot,x,t)}(u) \wedge \mu_{\mathbb{E}_{\widetilde{u}'_*}(\cdot,x,\tau_0)}(u) \wedge \\ \qquad \wedge \mu_{\overline{\widetilde{L}}'}(v,u,x) \Big] \circ \widetilde{g}_K(\cdot). \end{cases} \tag{40}$$

In Definition 24, the integral current gain $I_{\widetilde{u}'^*}$ and the future loss $\dot{J}_{\widetilde{u}'_*}$ are calculated as follows: $\forall (u,x,t) \in U \times X \times [\tau_0, \tau]$

$$\begin{cases} I_{\widetilde{u}'^*}(u,x,\tau) \triangleq \int^*_T \mathbf{P}^K_{\widetilde{u}'^*}(u,x,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{R}^*}(\cdot,\tau)}(\cdot), \\ J_{\widetilde{u}'_*}(u,x,\tau) \triangleq \int_{T *} \mathbf{q}^K_{\widetilde{u}'_*}(u,x,t) \circ \widetilde{g}_{\mathbb{E}_{\widetilde{R}_*}(\cdot,\tau)}(\cdot). \end{cases} \tag{41}$$

The optimization problem (31) can be now rewritten as

$$\begin{cases} \underset{\widetilde{u}'^* \in (\widetilde{\mathcal{B}_U \otimes \mathcal{B}}) \otimes \widetilde{\mathcal{B}}^*_T}{\mathrm{Opt}^*(\widetilde{u}'^*)} \overset{\max}{:\Longrightarrow} \\ \int^*_T \mathbf{P}^K_{\widetilde{u}'^*}(u,x,t) \circ \int^*_T \mu_{\widetilde{\Gamma}^*_\tau}(t') \circ \widetilde{g}^*_T(\cdot), \\ \underset{\widetilde{u}'_* \in (\widetilde{\mathcal{B}_U \otimes \mathcal{B}}) \otimes \widetilde{\mathcal{B}_{T*}}}{\mathrm{Opt}_*(\widetilde{u}'_*)} \overset{\min}{:\Longrightarrow} \\ \int_{T *} \mathbf{q}^K_{\widetilde{u}'_*}(u,x,t) \circ \int_{T *} \mu_{\widetilde{\Gamma}_{\tau *}}(t') \circ \widetilde{g}_{T*}(\cdot). \end{cases} \tag{42}$$

with the initial control conditions

$$\begin{aligned} \mathbb{E}_{\widetilde{\widetilde{u}}'_*}(\cdot,\tau_0) &\equiv \widetilde{u}_{0*} \in \widetilde{\mathcal{B}}_U \otimes \widetilde{\mathcal{B}}, \\ \mathbb{E}_{\widetilde{u}'^*}(\cdot,\tau_0) &\equiv \widetilde{u}^*_0 \in \widetilde{\mathcal{B}}_U \otimes \widetilde{\mathcal{B}}, \end{aligned} \tag{43$'$}$$

and the initial state conditions

$$\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau_0) \ \text{ and } \ \mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau_0) \in \widetilde{\mathcal{B}}. \tag{43$''$}$$

We can reformulate Definition 25 as follows.

Definition 26 If the fuzzy control depends on a WSCDS state $x \in X$, then an extremal fuzzy control process $(\overset{\circ}{\widetilde{u}}_*, \overset{\circ}{\widetilde{u}}^*)$, $\tau_0 \le \tau' \le \tau$, with the initial conditions (43) is called an optimal control (an extremal optimal control process) in the sense of Bellman's optimality principle provided that criteria (42) are satisfied.

Suppose we are given some fuzzy conditional measure $g_x(\cdot \mid v)$ that connects the fuzzy measure spaces $(X, \mathcal{B}, g)$ and $(K, \mathcal{B}_K, g_K)$:

$$g(A) = \fint_K g_x(A \mid v) \circ g_K(\cdot), \quad \forall A \in \mathcal{B}, \tag{44}$$

where $\forall v \in K$, $g_x(\cdot \mid v) : \mathcal{B} \to [0,1]$ is a fuzzy measure and $\forall A \in \mathcal{B}$, $g_x(A \mid \cdot) : K \to [0,1]$ is a $\mathcal{B}_K$-measurable function. Note that the conditional fuzzy measure $g_x(\cdot \mid v)$ takes into account the influence of a WSCDS state $x \in X$ in terms of an estimate by a criterion $v \in K$.

We introduce the following definitions.

Definition 27 The process $(\overset{\circ}{\widetilde{u}}_*, \overset{\circ}{\widetilde{u}}^*)$, $\overset{\circ}{\widetilde{u}}_* \in \widetilde{\mathcal{B}_U \otimes \mathcal{B}_{T*}}$, $\overset{\circ}{\widetilde{u}}^* \in \widetilde{\mathcal{B}_U \otimes \mathcal{B}^*_T}$ defined as $\forall u \in U$

$$\begin{cases} \mu_{\overset{\circ}{\widetilde{u}}_*}(u, \tau') = \\ = \fint_X \Big\{ \mu_{\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau')}(x) \wedge \mu_{\mathbb{E}_{\overset{\circ}{\widetilde{u}}'_*}(u,\cdot,\tau')}(x) \wedge \\ \quad \wedge\, \mu_{\mathbb{E}_{\overset{\circ}{\widetilde{u}}'_*}(u,\cdot,\tau_0)}(x) \Big\} \circ g(\cdot), \\ \mu_{\overset{\circ}{\widetilde{u}}^*}(u, \tau') = \\ = \fint_X \Big\{ \mu_{\mathbb{E}_{\widetilde{Q}^*}(\cdot,\tau')}(x) \vee \mu_{\mathbb{E}_{\overset{\circ}{\widetilde{u}}'^*}(u,\cdot,\tau')}(x) \vee \\ \quad \vee\, \mu_{\mathbb{E}_{\overset{\circ}{\widetilde{u}}'^*}(u,\cdot,\tau_0)}(x) \Big\} \circ g^*(\cdot). \end{cases} \tag{45}$$

is called an extremal fuzzy WSCDS control process in fuzzy extremal states $(\mathbb{E}_{\widetilde{Q}_*}(\cdot, \tau'), \mathbb{E}_{\widetilde{Q}^*}(\cdot, \tau'))$ $(\tau_0 \le \tau' \le \tau)$.

Definition 28 A fuzzy loss $\widetilde{L}$ is defined with a WSCDS state taken into account through the conditional fuzzy measure

$$\mu_{\widetilde{L}}(v, u) \triangleq \fint_X \mu_{\widetilde{L}'}(v, u, x) \circ g_x(\cdot \mid v), \tag{46}$$

where $\widetilde{L}' \in \widetilde{\mathcal{B}_K} \otimes \widetilde{\mathcal{B}_U} \otimes \widetilde{\mathcal{B}}$ is a fuzzy WSCDS loss for a choice $u \in U$ with a strategy $v \in K$ in a state $x \in X$.

The following theorem is true.

Theorem 8 An optimal fuzzy extremal control process $(\overset{\circ}{\widetilde{u}}_*, \overset{\circ}{\widetilde{u}}^*)$ for WSCDS (36) in the sense of criterion (42) depending on a WSCDS state, can be defined by the system of fuzzy-integral equations

$$\begin{aligned} &\mu_{\overset{\circ}{\widetilde{u}}_*}(u, \tau') = \\ &= \mu_{\overset{\circ}{\widetilde{u}}_*}(u, \tau_0) \wedge \fint_{X^2 \times K} \mu_{\mathbb{E}_{\widetilde{\rho}''_*}(\cdot,\cdot,\cdot,u,\tau')}(x', x'', v) \circ \\ &\quad \circ \left(\widetilde{g_x(\cdot \mid v) \otimes g(\cdot) \otimes g_K(\cdot)} \right) \wedge \\ &\quad \wedge\, \widetilde{g}_{\mathbb{E}_{\widetilde{R}_*}(\cdot,\Delta(\tau_0,\tau))}(T), \\ &\mu_{\overset{\circ}{\widetilde{u}}^*}(u, \tau') = \\ &= \mu_{\overset{\circ}{\widetilde{u}}^*}(u, \tau_0) \vee \fint_{X^2 \times K} \mu_{\mathbb{E}_{\widetilde{\rho}''^*}(\cdot,\cdot,\cdot,u,\tau')}(x', x'', v) \circ \\ &\quad \circ \left(\widetilde{g^*_x(\cdot \mid v) \otimes g^*(\cdot) \otimes g^*_K(\cdot)} \right) \vee \\ &\quad \vee\, \widetilde{g}_{\mathbb{E}_{\widetilde{R}^*}(\cdot,\Delta(\tau_0,\tau))}(T), \end{aligned} \tag{47}$$

where the fuzzy relations $\widetilde{\rho}''_*$ and $\widetilde{\rho}''^*$ are defined as follows:

$$\begin{aligned} &\widetilde{\rho}''_* = \widetilde{Q}'_* \cap \widetilde{L}',\ \widetilde{\rho}''_* \in \widetilde{\mathcal{B}^2 \otimes \mathcal{B}_K} \otimes (\widetilde{\mathcal{B}_U \otimes \mathcal{B}_{T*}}), \\ &\widetilde{\rho}''^* = \widetilde{Q}'^* \cap \overline{\widetilde{L}}',\ \widetilde{\rho}''^* \in \widetilde{\mathcal{B}^2 \otimes \mathcal{B}_K} \otimes (\widetilde{\mathcal{B}_U \otimes \mathcal{B}^*_T}), \end{aligned} \tag{48}$$

while the fuzzy relations $\widetilde{Q}'_*$ and $\widetilde{Q}'^*$ are cylindrical continuations of the relations $(\widetilde{Q}_*, \widetilde{Q}^*)$. The process $(\widetilde{Q}_*, \widetilde{Q}^*)$ describes WSCDS state dynamics on $\widetilde{\mathcal{B}}_K$ and $\forall \tau' \in [\tau_0, \tau]$, $\forall (v, x) \in K \times X$

$$\mu_{\mathbb{E}_{\widetilde{Q}'_*}(\cdot,\cdot,v)}(x, \tau') \triangleq \mu_{\widetilde{Q}_*}(x, \tau'), \mu_{\mathbb{E}_{\widetilde{Q}'^*}(\cdot,\cdot,v)}(x, \tau') \triangleq \\ \triangleq \mu_{\widetilde{Q}^*}(x, \tau').$$

Proof. Using certain properties of the extremal Sugeno integrals and the composition properties of extremal extended fuzzy measures (see [16]), we obtain: $\forall (u, \tau') \in U \times [\tau_0, \tau]$

$$\begin{aligned} &\mu_{\overset{\circ}{\widetilde{u}}_*}(u, \tau') = \fint_X \Big[\mu_{\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau')}(x') \wedge \mu_{\mathbb{E}_{\overset{\circ}{\widetilde{u}}_*}(\cdot,\tau_0)}(u) \wedge \\ &\wedge \Big\{ \fint_K \Big[\fint_X \mu_{\mathbb{E}_{\widetilde{L}'}(\cdot,u,\cdot)}(v, x'') \circ \\ &\circ\, g_x(\cdot \mid v) \Big] \circ \widetilde{g}_K(\cdot) \Big\} \Big] \circ g(\cdot) \wedge \widetilde{g}_{\mathbb{E}_{\widetilde{R}_*}(\cdot,\Delta(\tau_0,\tau'))}(T) = \\ &= \mu_{\overset{\circ}{\widetilde{u}}_*}(u, \tau_0) \wedge \fint_X \Big\{ \mu_{\mathbb{E}_{\widetilde{Q}_*}(\cdot,\tau')}(x') \wedge \end{aligned}$$

ISSN: 1991-8763 948 Issue 11, Volume 3, November 2008

$$\wedge \left[\oint_{X\times K} \mu_{\mathbb{E}_{\widetilde{L}'}(\cdot,u,\cdot)}(v,x'') \circ \left(\widetilde{g_x(\cdot \mid v)} \otimes \widetilde{g_K(\cdot)} \right) \right] \Bigg\} \circ$$

$$\circ\, g(\cdot) \wedge \widetilde{g}_{\mathbb{E}_{\widetilde{R}_*}(\cdot,\Delta(\tau_0,\tau'))}(T) = \mu_{\overset{\circ}{\widetilde{u}}_*}(u,\tau_0) \wedge$$

$$\wedge \oint_X \Bigg\{ \oint_{X\times K} \left[\mu_{\mathbb{E}_{\widetilde{Q}'_*}(\cdot,\tau',\cdot)}(x',v) \wedge \mu_{\mathbb{E}_{\widetilde{L}'}(\cdot,u,\cdot)}(x'',v) \right] \circ$$

$$\circ \left(\widetilde{g_x(\cdot \mid v)} \otimes \widetilde{g_K(\cdot)} \right) \Bigg\} \circ g(\cdot) \wedge$$

$$\wedge\, \widetilde{g}_{\mathbb{E}_{\widetilde{R}_*}(\cdot,\Delta(\tau_0,\tau'))}(T) = \mu_{\overset{\circ}{\widetilde{u}}_*}(u,\tau_0) \wedge$$

$$\wedge \oint_{X^2\times K} \left[\mu_{\mathbb{E}_{\widetilde{Q}'_*}(\cdot,\tau',\cdot)}(x',v) \wedge \mu_{\mathbb{E}_{\widetilde{L}'}(\cdot,u,\cdot)}(x'',v) \right] \circ$$

$$\circ \left(\widetilde{g_x(\cdot \mid v)} \otimes \widetilde{g_K(\cdot)} \otimes \widetilde{g(\cdot)} \right) \wedge \widetilde{g}_{\mathbb{E}_{\widetilde{R}_*}(\cdot,\Delta(\tau_0,\tau'))}(T) =$$

$$= \mu_{\overset{\circ}{\widetilde{u}}_*}(u,\tau_0) \wedge \oint_{X^2\times K} \left[\mu_{\mathbb{E}_{\widetilde{\rho}''_*}(\cdot,\cdot,\cdot,u,\tau')}(x',x'',v) \right] \circ$$

$$\circ \left(\widetilde{g_x(\cdot \mid v)} \otimes \widetilde{g(\cdot)} \otimes \widetilde{g_K(\cdot)} \right) \wedge \widetilde{g}_{\mathbb{E}_{\widetilde{R}_*}(\cdot,\Delta(\tau_0,\tau'))}(T),$$

where

$$\mathbb{E}_{\widetilde{\rho}''_*}(\cdot,\cdot,\cdot,u,\tau') = \mathbb{E}_{\widetilde{Q}'_*}(\cdot,\tau',\cdot) \cap \mathbb{E}_{\widetilde{L}'}(\cdot,u,\cdot).$$

We have thereby proved the first equality in (47). The second equality in (47) is easy to prove by applying the properties of the extremal Sugeno integrals for complementary fuzzy relations and dual fuzzy measures.

The theorem is proved. □

4 Definition of Fuzzy Relations of a Future Loss and a Current Gain in the WSCDS Optimization Problem

Proceeding from the results obtained in the preceding section on an optimal WSCDS control, we see that this control is defined by fuzzy relations of a future loss and a current gain (Theorems 6–8). Their compatibility functions are analogues of Bellman's functions in the classical dynamic programming method. Thus we need to obtain a fuzzy relation of a loss $\widetilde{L}$ (or $\widetilde{\widetilde{L}}$) taking into account a state in which the controlled WSCDS is. Here we will consider the case only for $\widetilde{L}$, using the processed available information on the WSCDS structure and characteristics. The set K defines the set of criterion estimates of the WSCDS, while U is the set of all possible controlling influences on the system. To obtain the compatibility function $\mu_{\widetilde{L}}(v,u)$ we should additionally consider the set of WSCDS characteristics Ω (for instance, of WSCDS state characteristics, external disturbances, additional restrictions and so on). Each characteristic $\omega \in \Omega$ takes its own values in some universal set A_ω (most frequently, A_ω is a numerical set). It is assumed that these characteristics may be fuzzy or have fuzzy values in A_ω, the distribution of which is known.

It is advisable to choose such characteristics of the set Ω that define to a maximal extent an estimate obtained by a criterion $v \in K$. To define the values $\mu_{\mathbb{E}_{\widetilde{L}}(\cdot,v)}(u)$, we first restore by various methods (expert evaluation, processing of available observation data and so on (see, e.g., [18])) the conditional fuzzy measure $g_\omega(\cdot \mid v)$ that preassigns "a degree of desirability" of a value $a \in A_\omega$ of a characteristic $\omega \in \Omega$, and b) $g_\Omega(\cdot \mid v)$ that defines "the importance of taking into account the value" of a characteristic $\omega \in \Omega$ for an estimate obtained by a criterion $v \in K$. It is assumed that the fuzzy measure spaces $\langle \Omega, \mathcal{B}_\Omega, g_\Omega(\cdot \mid v)\rangle$, $\langle A_\omega, \mathcal{B}_{A_\omega}, g_\omega(\cdot \mid v)\rangle$ are given ones, and also that the fuzzy measures $g_\omega(\cdot \mid v)$ and $g_\Omega(\cdot \mid v)$ define some knowledge base on the WSCDS subject-matter. The WSCDS knowledge base is defined by the train

$$\langle K, U, \Omega, \{A_\omega\}_{\omega\in\Omega}, g_\omega(\cdot \mid v), g_\Omega(\cdot \mid v)\rangle. \qquad (49)$$

All components of the WSCDS knowledge base are assumed to be described a priori. To describe the function $\mu_{\mathbb{E}_{\widetilde{L}}(\cdot,v)}(u)$ it is necessary to measure (or to prognose or estimate) possible (most probably fuzzy) values of the characteristics of Ω for a choice of control $u \in \Omega$.

Suppose that as a result of measurements we have, for a concrete $u \in \Omega$, some compatibility function

$$h_\omega : X \times A_\omega \times U \to [0,1], \qquad (50)$$

that defines possible values $a \in A_\omega$ of a characteristic $\omega \in \Omega$ for a chosen control. Then the fuzzy gain function $\mu_{\mathbb{E}_{\widetilde{\widetilde{L}}}(\cdot,v)}(u)$ is defined by a double fuzzy integral of the form

$$\mu_{\mathbb{E}_{\widetilde{\widetilde{L}}}(\cdot,v)}(u) =$$
$$= \oint_\Omega \left[\oint_{A_\omega} h(\omega,a,u) \circ g_\omega(\cdot \mid v) \right] \circ g_\Omega(\cdot,\mid v). \qquad (51)$$

This integral is interpreted as follows: after taking the first integral, for a fixed criterion $v \in K$ of a control choice we have a gain $u \in U$ for each characteristic from Ω. The second integral defines a generalized gain degree of a choice of control $u \in U$ for each criterion $v \in K$.

The use of (51) in the case of (35) for defining an optimal WSCDS control allows us to solve "static" problems of an optimal choice in the possibility uncertainty conditions.

If an expert group $\mathbb{E}$ takes part in the estimation of WSCDS states and the possibility distribution of experts' competence is $\pi_{\mathbb{E}}(e)$, $e \in \mathbb{E}$, then for each expert $e \in E$ the function h naturally depends on e so that the integral definition (51) can be replaced by

$$\mu_{\mathbb{E}_{\widetilde{L}}(\cdot,v)}(u) = \fint_{\mathbb{E}} \left\{ \fint_{\Omega} \left[\fint_{A_\omega} h(\omega, a, u, e) \circ g_\omega(\cdot \mid v) \right] \circ \circ g_\Omega(\cdot, \mid v) \right\} \circ \mathbf{P}\text{OSS}(\cdot). \quad (52)$$

where $\mathbf{P}$OSS is a fuzzy possibility measure on $(\mathbb{E}, 2^{\mathbb{E}})$ with the possibility distribution $\pi_{\mathbb{E}}(\cdot)$.

5 Example

Let the set of WSCDS states be finite, $X = \{1, 2, 3, 4\}$; $g^* : 2^X \to [0, 1]$ be the possibility measure with the possibility distribution on X

$$\Pi(i) \triangleq \frac{i}{4}, \quad i = 1, 2, 3, 4$$

$$\Big(\forall B \in 2^X : g^*(A) = \vee_{i \in A} \pi(i)\Big).$$

Let the WSCDS be subjected to the influence of an external control factor with the finite set $U = \{u_1, u_2\}$ (for example, $u_1 \triangleq$"$+1$", $u_2 \triangleq$"-1"). Let the uniform probability distribution play the role of the fuzzy measure $g_U : 2^U \to [0, 1]$, i.e. $g_U(\{u_1\}) = g_U(\{u_2\}) = \frac{1}{2}$. The two-element set $K = \{v_1, v_2\}$ is taken as the set of chosen criteria, while the uniform probability distribution $g_K(\{v_1\}) = g_K(\{v_2\}) = \frac{1}{2}$ is considered as playing the role of the fuzzy measure $g_K : 2^K \to [0, 1]$. Thus we have the fuzzy measure spaces $(X, 2^X, g)$, $(K, 2^K, g_K)$ and $(U, 2^U, g_U)$. The dual measure g^* on 2^X is the necessity measure $g(A) = 1 - \vee_{i \notin A} \pi(i)$. Since the fuzzy measures g_U and g_K are the probability ones, we know they are autodual and

$$g_U^* = g_U, \quad g_K^* = g_K.$$

It is assumed that the initial moment of WSCDS observation is $\tau_0 \equiv 0$. Let the initial extremal fuzzy distributions of an optimal control be

$$\mu_{\overset{\circ}{\widetilde{u}}_*}(u_1, 0) = \frac{1}{2} = \mu_{\overset{\circ}{\widetilde{u}}^*}(u_1, 0);$$

$$\mu_{\overset{\circ}{\widetilde{u}}_*}(u_2, 0) = \frac{1}{4} = \mu_{\overset{\circ}{\widetilde{u}}^*}(u_2, 0).$$

Let the binary fuzzy loss relation $\widetilde{L}$ on $U \times K$ be defined as follows:

$$\mu_{\widetilde{L}}(u_1, v_1) = \mu_{\widetilde{L}}(u_2, v_2) = \frac{1}{2},$$

$$\mu_{\widetilde{L}}(u_1, v_2) = \mu_{\widetilde{L}}(u_2, v_1) = \frac{1}{4}.$$

The distributions of extremal fuzzy time intervals are given as

$$\mu_{\widetilde{r}_{\tau *}}(t) = \begin{cases} 0, & 0 \le t \le \tau, \\ 1 - \frac{\tau}{t}, & t > \tau, \end{cases}$$
$$\mu_{\widetilde{r}^*_\tau}(t) = \begin{cases} 1, & 0 \le t < \tau, \\ \frac{\tau}{t}, & t \ge \tau. \end{cases} \quad (53)$$

Let the initial distribution ($\tau_0 \equiv 0$) of the WSCDS state description process look like

$$\widetilde{A}_{0*} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$
$$\widetilde{A}_0^* \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \quad (54)$$

We consider the example of the space $(T, \widetilde{\mathcal{B}}_{T*}, \widetilde{\mathcal{B}}^*_T, \widetilde{g}^*_T, \widetilde{g}^*_T)$ where

$$g_{T*}([t, +\infty)) \triangleq \frac{1}{1+t}, \quad [t, +\infty) \in \mathcal{B}_{T*},$$
$$g_T^*([0, t)) \triangleq \frac{t}{1+t}, \quad [0, t) \in \mathcal{B}^*_T, \quad t > 0. \quad (55)$$

Further, we introduce the conditional fuzzy measures on $\mathcal{B}_{T*}$ and $\mathcal{B}^*_T$ with respect to the set $X = \{1, 2, 3, 4\}$:

$$g_{t*}(r_{\tau *} \mid i) = \frac{1}{1+i\tau}, \text{ where } i \in X,\ r_{\tau *} \in \mathcal{B}_{T*},$$
$$g_t^*(r^*_\tau \mid i) = \frac{i\tau}{1+i\tau}, \text{ where } i \in X,\ r^*_\tau \in \mathcal{B}^*_T. \quad (56)$$

Thus the WSCDS state description process can be represented as follows:

$$\mu_{\widetilde{Q}_*}(x, \tau) = \fint_{U \times T}^{\;} {}_* \Big\{ \mu_{\mathbb{E}_{\widetilde{u}*}(\cdot,\tau)}(u) \wedge \mu_{\mathbb{E}_{\widetilde{\rho}'_*}(x,\cdot,\cdot)}(u, t) \Big\} \circ \circ \widetilde{g_U} \otimes \widetilde{g_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}}(\cdot),$$

$$\mu_{\widetilde{Q}^*}(x, \tau) = \fint_{U \times T}^{*} \Big\{ \mu_{\mathbb{E}_{\widetilde{u}^*}(\cdot,\tau)}(u) \vee \mu_{\mathbb{E}_{\widetilde{\rho}'^*}(x,\cdot,\cdot)}(u, t) \Big\} \circ \circ \widetilde{g_U} \otimes \widetilde{g_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}}(\cdot),$$

(57)

where $\widetilde{A}_{0*} \equiv \mathbb{E}_{\widetilde{Q}_*}(\cdot,0)$, $\widetilde{A}_0^* \equiv \mathbb{E}_{\widetilde{Q}^*}(\cdot,0)$, $(\widetilde{\mathbb{R}}_*,\widetilde{\mathbb{R}}^*)$ is the extremal fuzzy reflection process (see [16]), $\forall(x,\tau)\in X\times T$, $\forall(x,t)\in U\times T$:

$$\begin{cases} \mu_{\widetilde{\mathbb{R}}_*}(x,\tau) \triangleq \widetilde{g}_{t*}(\widetilde{r}_{\tau*} \mid x) = \mu_{\widetilde{A}_{\tau*}}(x), \\ \mu_{\widetilde{\mathbb{R}}^*}(x,\tau) \triangleq \widetilde{g}_t^*(\widetilde{r}_\tau^* \mid x) = \mu_{\widetilde{A}_\tau^*}(x), \end{cases} \tag{58}$$

and

$$\begin{cases} \mu_{\widetilde{\rho}'_*}(x,u,t) \triangleq \displaystyle\int_{\widetilde{A}_{0*}} \mu_{\widetilde{\rho}_*}(x,u,x',t)\circ\widetilde{g}(\cdot), \\ \mu_{\widetilde{\rho}'^*}(x,u,t) \triangleq \displaystyle\int_{\widetilde{A}_0^*} \mu_{\widetilde{\rho}^*}(x,u,x',t)\circ\widetilde{g}^*(\cdot), \end{cases} \tag{59}$$

where $\widetilde{A}_{\tau*}\in\widetilde{\mathcal{B}}$ and $\widetilde{A}_\tau^*\in\mathcal{B}$ are expert reflections on the WSCDS states in the fuzzy extremal intervals $\widetilde{r}_{\tau*}\in\widetilde{\mathcal{B}}_{T*}$ and $\widetilde{r}_\tau^*\in\widetilde{\mathcal{B}}_T^*$, respectively; $(\widetilde{\rho}_*,\widetilde{\rho}^*)$ is the WSCDS input-output operator (see [16]). As is known the operator $(\widetilde{\rho}'_*,\widetilde{\rho}'^*)$ is restored from the experimental-expert knowledge base on the WSCDS so that if we fix some admissible extremal control process $(\widetilde{u}_*,\widetilde{u}^*)$ (including an optimal control too), then, using the calculation procedure for Sugeno extremal integrals [16], we can write expressions for the process $(\widetilde{Q}_*,\widetilde{Q}^*)$. However we pursue a different aim here: using WSCDS data, we are to construct the extremal optimal control process $(\overset{\circ}{\widetilde{u}}_*,\overset{\circ}{\widetilde{u}}{}^*)$ by formulas (35).

Since the sets X,U,K are finite, it is not difficult to establish that the conditions (34) for an optimal extremal control process to exist are satisfied. By virtue of the results of Theorems 6 and 7, we can write one of the variants for an extremal optimal fuzzy control process as follows: $\forall(u,\tau)\in(X,T)$

$$\begin{aligned} \mu_{\overset{\circ}{\widetilde{u}}_*}(u,\tau) &= \mu_{\overset{\circ}{\widetilde{u}}_*}(u,0)\wedge\left(\int_K \mu_{\widetilde{L}}(u,v)\circ\widetilde{g}_K(\cdot)\right)\wedge \\ &\wedge\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T), \\ \mu_{\overset{\circ}{\widetilde{u}}{}^*}(u,\tau) &= \mu_{\overset{\circ}{\widetilde{u}}{}^*}(u,0)\vee\left(\int_K \mu_{\overline{\widetilde{L}}}(u,v)\circ\widetilde{g}_K^*(\cdot)\right)\vee \\ &\vee\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T), \end{aligned} \tag{60}$$

where $u\in\{$“+1”, “−1”$\}$, $v\in\{v_1,v_2\}$; $\mu_{\overset{\circ}{\widetilde{u}}_*}(u,0)$ and $\mu_{\overset{\circ}{\widetilde{u}}{}^*}(u,0)$ are already defined, while the extended extremal fuzzy measures are defined in the form [16]

$$\begin{cases} \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T) = \displaystyle\int_{*T} \mu_{\widetilde{r}_{\tau*}}(t)\circ\widetilde{g}_{T*}(\cdot) \triangleq \\ \qquad \triangleq \displaystyle\int_{*T} \mu_{\widetilde{r}_{\tau*}}(t)\circ\int_X g_{t*}(\cdot\mid x)\circ g(\cdot), \\ \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T) = \displaystyle\int_T^* \mu_{\widetilde{r}_\tau^*}(t)\circ\widetilde{g}_T^*(\cdot) \triangleq \\ \triangleq \displaystyle\int_T^* \mu_{\widetilde{r}_\tau^*}(t)\circ\int_X g_t^*(\cdot\mid x)\circ g^*(\cdot). \end{cases} \tag{61}$$

Now we are to calculate the Sugeno integrals in formulas (60) and the values of extremal fuzzy measures (61).

Let us calculate the values of $\int_K \mu_{\widetilde{L}}(u,v)\circ\widetilde{g}_K(\cdot)$:

1) $u=u_1\equiv$“+1”:

$$\begin{aligned} &\int_K \mu_{\widetilde{L}}(u_1,v)\circ\widetilde{g}_K(\cdot) = \\ &= \bigwedge_{0<\alpha\le 1}\left\{\alpha\vee g_K(v\in K\mid \mu_{\widetilde{L}}(u_1,v)\ge\alpha\right\} = \\ &= \Big[\bigwedge_{0\le\alpha\le\frac14}(\alpha\vee g_K(K))\Big]\wedge \\ &\wedge\Big[\bigwedge_{\frac14\le\alpha\le\frac12}(\alpha\vee g_K(\{v_2\}))\Big]\wedge \\ &\wedge\Big[\bigwedge_{\frac12<\alpha\le1}(\alpha\vee g_K(\emptyset))\Big] = 1\wedge\frac12\wedge\frac12=\frac12. \end{aligned}$$

2) $u=u_2=$“−1”:

$$\begin{aligned} &\int_K \mu_{\widetilde{L}}(u_2,v)\circ\widetilde{g}_K(\cdot) = \\ &= \bigwedge_{0<\alpha\le 1}\left\{\alpha\vee g_K(v\in K\mid \mu_{\widetilde{L}}(u_2,v)\ge\alpha\right\} = \\ &= \Big[\bigwedge_{0\le\alpha\le\frac14}(\alpha\vee g_K(K))\Big]\wedge \\ &\wedge\Big[\bigwedge_{\frac14\le\alpha\le\frac12}(\alpha\vee g_K(\{v_1\}))\Big]\wedge \\ &\wedge\Big[\bigwedge_{\frac12<\alpha\le1}(\alpha\vee g_K(\emptyset))\Big] = \\ &= 1\wedge\Big[\bigwedge_{\frac14\le\alpha\le\frac12}\Big(\alpha\vee\frac12\Big)\Big]\wedge\Big[\bigwedge_{\frac12\le\alpha<1}(\alpha)\Big] = \\ &= 1\wedge\frac12\wedge\frac12=\frac12. \end{aligned}$$

Since

$$\int_K \mu_{\overline{\widetilde{L}}}(u,v)\circ\widetilde{g}_K^*(\cdot) = 1-\int_K \mu_{\widetilde{L}}(u,v)\circ\widetilde{g}_K(\cdot),$$

we have

$$\oint_K \mu_{\widetilde{L}}(u_1,v)\circ \widetilde{g}^*_K(\cdot) = \oint_K \mu_{\widetilde{L}}(u_2,v)\circ \widetilde{g}_K(\cdot) = \frac{1}{2}\,.$$

Therefore $\forall \tau > 0$

$$\begin{aligned}
\mu_{\tilde{\tilde{u}}_*}(u_1,\tau) &= \tfrac{1}{2}\wedge\tfrac{1}{2}\wedge \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T) = \tfrac{1}{2}\wedge \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T),\\
\mu_{\tilde{\tilde{u}}_*}(u_2,\tau) &= \tfrac{1}{4}\wedge\tfrac{1}{2}\wedge \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T) = \tfrac{1}{4}\wedge \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T),\\
\mu_{\tilde{\tilde{u}}^*}(u_1,\tau) &= \tfrac{1}{2}\vee\tfrac{1}{2}\vee \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T) = \tfrac{1}{2}\vee \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T),\\
\mu_{\tilde{\tilde{u}}^*}(u_2,\tau) &= \tfrac{1}{4}\vee\tfrac{1}{2}\vee \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T) = \tfrac{1}{2}\vee \widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T).
\end{aligned}$$

Now we are to calculate the values of the so-called extremal fuzzy "white noise" (61):

$$\begin{aligned}
\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T) &= \oint_T{}_{*}\, \mu_{\widetilde{r}_{\tau *}}(t)\circ \oint_X g_{t*}(\cdot\mid x)\circ g(\cdot) =\\
&= \bigvee_{0<\alpha\le 1}\{\alpha\wedge \widetilde{g}_{T*}([\widetilde{r}_{\tau *}]_{\overline{\alpha}})\} =\\
&= \bigvee_{0<\alpha\le 1}\left\{\alpha\wedge \oint_X \widetilde{g}_{t*}([\widetilde{r}_{\tau *}]_{\overline{\alpha}}\mid x)\circ g(\cdot)\right\}.
\end{aligned}$$

From (53) we obtain the expression for an α-cut for $\widetilde{r}_{\tau *}$:

$$[\widetilde{r}_{\tau *}]_{\overline{\alpha}} = \left\{\begin{array}{ll} T & \text{if } \alpha = 0,\\ \left[\frac{\tau}{1-\alpha}, +\infty\right) & \text{if } 0<\alpha<1,\\ \emptyset & \text{if } \alpha = 1, \end{array}\right\} \in \mathcal{B}_{T*}.$$

Now (56) implies

$$\widetilde{g}_{t*}([\widetilde{r}_{\tau *}]_{\overline{\alpha}}\mid i) = \left\{\begin{array}{ll} 1 & \text{if } \alpha = 0,\\ \frac{1}{1+i\,\frac{\tau}{1-\alpha}} & \text{if } 0<\alpha<1,\\ \emptyset & \text{if } \alpha=1, \end{array}\right.$$

$\forall i\in X$ and

$$\begin{aligned}
&\oint_X \widetilde{g}_t([\widetilde{r}_{\tau *}]_{\overline{\alpha}}\mid i)\circ g(\cdot) =\\
&= \bigvee_{0<\beta\le 1}\left\{\beta\wedge g\left(\left\{i\in X \mid \frac{1}{1+i\,\frac{\tau}{1-\alpha}}\ge\beta\right\}\right)\right\}.
\end{aligned}$$

It is not difficult to verify that $(0<\alpha<1, \tau>0)$

$$\begin{aligned}
&\left\{i\in X\mid \frac{1}{1+i\,\frac{\tau}{1-\alpha}}\ge\beta\right\} =\\
&= \left\{\begin{array}{ll} \emptyset & \text{if } 1\ge\beta>\frac{1-\alpha}{1-\alpha+\tau}\,,\\ \{1\} & \text{if } \frac{1-\alpha}{1-\alpha+\tau}\ge\beta>\frac{1-\alpha}{1-\alpha+2\tau}\,,\\ \{1,2\} & \text{if } \frac{1-\alpha}{1-\alpha+2\tau}\ge\beta>\frac{1-\alpha}{1-\alpha+3\tau}\,,\\ \{1,2,3\} & \text{if } \frac{1-\alpha}{1-\alpha+3\tau}\ge\beta>\frac{1-\alpha}{1-\alpha+4\tau}\,,\\ X & \text{if } \frac{1-\alpha}{1-\alpha+4\tau}\ge\beta>0. \end{array}\right.
\end{aligned}$$

Denote $B_0 \equiv \left(\frac{1-\alpha}{1-\alpha+\tau};1\right]$, $B_1 \equiv \left(\frac{1-\alpha}{1-\alpha+2\tau};\frac{1-\alpha}{1-\alpha+\tau}\right]$, $B_2 \equiv \left(\frac{1-\alpha}{1-\alpha+3\tau};\frac{1-\alpha}{1-\alpha+2\tau}\right]$, $B_3 \equiv \left(\frac{1-\alpha}{1-\alpha+4\tau};\frac{1-\alpha}{1-\alpha+3\tau}\right]$, $B_4\equiv\left(0;\frac{1-\alpha}{1-\alpha+4\tau}\right]$.
Then

$$\begin{aligned}
&\oint_X \widetilde{g}_{t*}([\widetilde{r}_{\tau *}]_{\overline{\alpha}}\mid x)\circ g(\cdot) = \Big[\bigvee_{\beta\in B_0}(\beta\wedge g(\emptyset))\Big]\vee\\
&\vee\Big[\bigvee_{\beta\in B_1}(\beta\wedge g(\{1\})\Big]\vee\Big[\bigvee_{\beta\in B_2}(\beta\vee g(\{1,2\}))\Big]\vee\\
&\vee\Big[\bigvee_{\beta\in B_3}(\beta\vee g(\{1,2,3\}))\Big]\vee\Big[\bigvee_{\beta\in B_4}(\beta\wedge g(X))\Big] =\\
&= 0\vee\Big[\bigvee_{\beta\in B_1}(\beta\wedge 0)\Big]\vee\Big[\bigvee_{\beta\in B_2}(\beta\wedge 0)\Big]\vee\\
&\vee\Big[\bigvee_{\beta\in B_3}(\beta\wedge 0)\Big]\vee\Big[\bigvee_{\beta\in B_4}(\beta\wedge 1)\Big] =\\
&= \bigvee_{\beta\in B_4}\beta = \frac{1-\alpha}{1-\alpha+4\tau}\,.
\end{aligned}$$

We finally obtain

$$\begin{aligned}
&\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T) =\\
&= \bigvee_{0<\alpha<1}\left\{\alpha\wedge\oint_X \widetilde{g}_{t*}([\widetilde{r}_{\tau *}]_{\overline{\alpha}}\mid x)\circ g(\cdot)\right\} =\\
&= \bigvee_{0<\alpha<1}\left\{\alpha\wedge\frac{1-\alpha}{1-\alpha+4\tau}\right\}.
\end{aligned}$$

After studying the function in the braces with respect to α, we can continue calculations:

$$\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(T) = \begin{cases} \bigvee_{0<\alpha<1}\{\alpha\} = 1 \quad \text{if } 0<\tau\le 1,\\ \bigvee_{\alpha\in[1;2\tau-1-2\sqrt{\tau(\tau-1)}]}\{\alpha\} =\\ \quad = 2\tau-1-2\sqrt{\tau(\tau-1)} \ \text{ if } \tau>1. \end{cases}$$

Since $\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}_*}(\cdot,\tau)}(\cdot)$ and $\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(\cdot)$ are extended extremal measures, we have

$$\widetilde{g}_{\mathbb{E}_{\widetilde{\mathbb{R}}^*}(\cdot,\tau)}(T) = \begin{cases} 0 & \text{if } 0<\tau\le 1,\\ 2+2\sqrt{\tau(\tau-1)} & \text{if } \tau>1. \end{cases}$$

For an optimal control we obtain the following expressions:

$$\mu_{\tilde{\tilde{u}}_*}(u_1,\tau) = \begin{cases} \frac{1}{2}\,, \quad 0<\tau\le 1,\\ \frac{1}{2}\wedge(2\tau-1-2\sqrt{\tau(\tau-1)}),\\ \quad \tau>1, \end{cases}$$

$$\mu_{\tilde{\tilde{u}}_*}(u_2,\tau) = \begin{cases} \frac{1}{4}\,, \quad 0<\tau\le 1,\\ \frac{1}{4}\wedge(2\tau-1-2\sqrt{\tau(\tau-1)}),\\ \quad \tau>1, \end{cases}$$

$$\mu_{\underset{u^*}{\tilde{\delta}}}(u_1,\tau) = \begin{cases} \frac{1}{2}, \quad 0<\tau\le 1, \\ \frac{1}{2} \vee (2+2\sqrt{\tau(\tau-1)}-2\tau), \\ \quad \tau>1 \end{cases}$$
$$= \mu_{\underset{u^*}{\tilde{\delta}}}(u_2,\tau).$$

Note that when $\tau \to +\infty$ a current description process of fuzzy time intervals extends unlimitedly, while a future description process of fuzzy time intervals vanishes. The latter fact is reflected in the expressions for the fuzzy optimal extremal controls:

$$\begin{cases} \lim\limits_{\tau\to\infty} \mu_{\underset{u^*}{\tilde{\delta}}}(u,\tau) \to 1, \quad u \in U = \{u_1,u_2\}, \\ \lim\limits_{\tau\to\infty} \mu_{\underset{u_*}{\tilde{\delta}}}(u,\tau) \to 0, \quad u \in U = \{u_1,u_2\}. \end{cases}$$

i.e. the uncertainty for a current fuzzy control process vanishes, while a future fuzzy optimal control process is not considered.

We have thereby finished the consideration of the example.

6 Conclusion

Problems of optimization of a continuous controllable extremal fuzzy process are considered using R. Bellman's optimality principle. An extremal fuzzy "gain-loss" process is defined, which plays the role of Bellman's function in the classical variant of the dynamic programming problem. Theorems 6–8 allow one to write variants of an optimal control for the WSCDS. A fuzzy gain relation is defined using the expert knowledge base on the WSCDS subject-matter.

A practical example is given to illustrate the results obtained.

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