# A Solution to the Continuous and Discrete-Time Linear Quadratic Optimal Problems

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*Abstract*: The paper establishes a non-variational procedure in order to obtain the solution to an optimal control problem. The optimal control refers to a quadratic criterion with finite final time, regarding a time-variant linear system, for continuous and discrete time cases. The proposed solution is more convenient for implementation by comparison with the classical ones. The indicated optimal controller is advantageous especially in the time-invariant case. New solutions to Riccati differential / difference equations are also presented.

*Keywords:* optimal control, linear quadratic, finite final time, continuous and discrete time, Riccati equation

## 1 Introduction

The present paper refers to the linear quadratic (LQ) optimal control problem for time variant systems, with finite final time and free end-point. Both continuous and discrete time cases are discussed.

There are numerous books and papers devoted to this problem. We mention as references for continuous and discrete cases [1], [2], [3], [4], but many other books also contain the basic referring to this problem.

The paper proposes a non-variational method for solving the discussed problem. Generally, the solution to an optimal control problem can be obtained based on two main variational methods: the dynamic programming based on Bellman optimality principle (as it is presented in [2]), and the minimum principle. The last case is used in the form of the Pontryagin principle for constrained problems, or in a simpler form for unconstrained problems, when Hamilton or Euler-Lagrange equations are used (see [1]). Apart of these basic methods, certain non-variational procedures are used, as it is, for instance, the mathematical programming methods for discrete or discretised problems (see [3]). Also, in certain cases, the criterion can be written as a sum containing independent on control vector terms and one term depending on the control vector. The minimization of the last mentioned term leads to the solution of the problem. This approach is possible in the problems with quadratic performance index and it is used in the present paper.

Besides the use of an unconventional method, the paper presents the formula for the optimal vector control in an advantageous form, having a feedback component and a corrective one, depending on the initial state. A near idea in this direction is indicated in [5] for fixed-end point problem and in [6] for free end-point.

It is well known that the solution to the LQ problem is closely connected with the solving of a Riccati differential / difference equation (RDE) and there is a large bibliography in this direction. The methods for solving RDE can be grouped in the following main categories:

- Direct integration of RDE.
- Iterative solving of a simpler first order equation, accompanied by a supplementary relationship. We mention in this direction the use of Lyapunov differential equation, or the Bernoulli equation [7], or the Chandrasekhar method (extended for time-variant problems in [8]).
- Analytical, non recursive procedures: the most variants are based on the factorization of the solution in the form $Y(t)X^{-1}(t)$, where the *n*x*n* X and Y matrices (*n* is the order of the system) are obtained from a partitioned *2n*x*2n* transition matrix of the Hamilton matrix of the problem. The method was initially proposed in [9] for the time-invariant case and is presented for general case in [1]. A formula which uses only *n*x*n* transition matrices is proposed in [10] and straightforward verified. An analytical solution using only *n*x*n* matrices is indicated for time invariant problems in [6] and

extended for time variant ones in [11]. In the same category of non recursive solutions can be mentioned [12], [13], [14] and others.

Certainly, there are different other methods which cannot be included in the above categories and we mention here [15] and [16].

Different types of methods have some advantages, depending on problem (variant or not), on the dimensions, on certain particularities etc. A comparison of some methods is given in [7]. Many authors assert the advantages of the analytical methods, especially those that use only *n*x*n* transition matrices.

An important drawback of all known method is the fact that in the optimal control problems the RDE is solved in inverse time, starting from a final condition. The present paper presents a variant which solves RDE in direct time.

In the last years the researches were oriented to the extension of the possibilities of solving the RDE, especially for large scale systems, for the problems which introduce difficulties, for different generalized problems and also, different new procedures were proposed for problem solving, like those based on new mathematical programming methods [17], [18], [19], and [20].

Although there are not many papers dedicated to the basic procedures for LQ problem and RDE in the last time, the problem is not closed. Finding of efficient methods remains an interesting problem, because it simplifies the complexity of the implementation. We refer, for instance, to the optimal control of the electrical drives, where it is necessary to use sampling periods of milliseconds. Since the complexity of the algorithms for modern control method (like vector control) for electrical drives with AC motors is very high, the introducing of the optimal control is possible only if the corresponding algorithm is simple. The nowadays complexity of the optimal control algorithm explains the reluctance in the application of such a control, although it represent a doubtless advantage in the actual energy saving problem. Taking into account this justification, we appreciate that it is not lake of interest the study performed in this paper referring to the simplification of the solving of the LQ problem.

## 2 LQ optimal control for continuous time systems

A linear time-variant system is considered

$$\begin{aligned}&\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x^0\\&x(t) \in \Re^n, u(t) \in \Re^m.\end{aligned} \tag{1}$$

The linear quadratic (LQ) optimal control problem refers to the system (1) and the quadratic criterion

$$\begin{aligned}I &= \frac{1}{2}x^T(t_f)Sx(t_f) +\\&+\frac{1}{2}\int_{t_0}^{t_f}\left[x^T(t)Q(t)x(t) + u^T(t)P(t)u(t)\right]dt,\end{aligned} \tag{2}$$

with $S \geq 0, Q(t) \geq 0, P(t) > 0$.

The problem referring to the system (1) and the criterion (2) has the solution [1], [2]

$$u^*(t) = -P^{-1}(t)B^T(t)\tilde{R}(t)x(t), \tag{3}$$

where $\tilde{R}(t)$ is a symmetrical *n*x*n* matrix, solution to the Riccati matriceal differential equation

$$\begin{aligned}\dot{\tilde{R}}(t) = \tilde{R}(t)N(t)\tilde{R}(t) - \tilde{R}(t)A(t)\\- A^T(t)\tilde{R}(t) - Q(t),\end{aligned} \tag{4}$$

where

$$N(t) = B(t)P^{-1}(t)B^T(t) \tag{5}$$

and

$$\tilde{R}(t_f) = S. \tag{6}$$

The minimum value of the criterion is

$$I^* = \frac{1}{2}x^T(t_0)\tilde{R}(t_0)x(t_0). \tag{7}$$

One can obtain the above indicated solution based on a non-variational method. This Section indicates a modified form of the classical method and also presents an analytical solution to the Riccati matriceal differential equation.

<u>Lemma1</u>: The control vector that minimizes the criterion (2) subject to the system (1) is

$$u^*(t) = -P^{-1}(t)B^T(t)[\bar{R}(t)x(t) + v(t)], \tag{8}$$

where the symmetric matrix $\bar{R}(t)$ is a particular solution to the equation (4), which satisfies a certain final condition

$$\bar{R}(t_f) = \bar{S}, \quad \bar{S} \geq 0, \tag{9}$$

and v(t) satisfies

$$\dot{v}(t) = -F(t)^T v(t), \tag{10}$$

with

$$F(t) = A(t) - N(t)\bar{R}(t) \tag{11}$$

and

$$v(t_f) = (S - \bar{S})x(t_f). \quad (12)$$

*Proof*: Let us consider the scalar function

$$\pi(t) = \frac{1}{2}x^T(t)\bar{R}(t)x(t) + x^T(t)v(t). \quad (13)$$

Tacking into account (1), one can write

$$\dot{\pi}(t) = \frac{1}{2}(2x^T A^T \bar{R}x + 2u^T B^T \bar{R}x + x^T \dot{\bar{R}}x) + \\ + u^T B^T v + x^T \bar{R}Nv$$

(the argument t was omitted). Adding to (2) the identity

$$\int_{t_0}^{t_f} \dot{\pi}(t)dt - \pi(t_f) + \pi(t_0) = 0,$$

yields

$$I = \pi(t_0) + \frac{1}{2}x^T(t_f)(S - \bar{S})x(t_f) - x^T(t_f)v(t_f) + \\ \int_{t_0}^{t_f} [\varphi(t) - \frac{1}{2}v^T(t)N(t)v(t)]dt, \quad (14)$$

where

$$\varphi(t) = \frac{1}{2}x^T \bar{R}N\bar{R}x + u^T B^T \bar{R}x + \frac{1}{2}u^T Pu + \\ + u^T B^T v + x^T \bar{R}Nv + \frac{1}{2}v^T Nv,$$

or

$$\varphi(t) = \frac{1}{2}(u + P^{-1}B^T\bar{R}x + P^{-1}B^Tv)^T P(u + P^{-1}B^T\bar{R}x + P^{-1}B^Tv). \quad (15)$$

The criterion (14) depends on u(t) only through the integral of φ(t), and this is a positive definite function. Therefore, the minimum value for I is obtained when φ(t) = 0. This condition is true only if u(t) satisfies (8).■

*Remark 1*: The classical solution (3), (4), (6) can be obtained in the same way if we adopt the scalar function

$$\pi(t) = \frac{1}{2}x^T(t)\tilde{R}(t)x(t). \quad (16)$$

Tacking into account that the LQ problem has a unique solution, the function $\pi(t)$ is the same in (13) and (16).

The minimum values of the criterion results from (12), (14) and the condition φ(t) =0 and is

$$I^* = \pi(t_0) - \frac{1}{2}x^T(t_f)v(t_f) - \frac{1}{2}\int_{t_0}^{t_f} v^T(t)N(t)v(t)dt. \quad (17)$$

Farther we have to calculate v(t) in order to obtain the control vector u(t) with (8). The vector v(t) has to be expressed in terms of $x(t_0)$, which is the unique known terminal condition.

For this purpose we formulate the following

Lemma 2: The solution to the equation (10) is

$$v(t) = \Phi(t, t_0)v(t_0), \text{ or } v(t) = \Phi(t, t_f)v(t_f), \quad (18)$$

where $v(t_f)$ is given by (12), and

$$v(t_0) = \Phi(t_0, t_f)(S - \bar{S})M^{-1}(t_0, t_f)x(t_0). \quad (19)$$

In the above formulae, $\Phi(t, t_f)$ is the transition matrix for $-F^T$, and

$$M(t, t_f) = \Psi(t, t_f) + \Omega_{12}(t, t_f)(S - \bar{S}), \quad (20)$$

with $\Psi(t, t_f)$ the transition matrix for F and

$$\Omega_{12}(t, t_f) = \int_t^{t_f} \Psi(t, \tau)N(\tau)\Phi(\tau, t_f)d\tau. \quad (21)$$

*Proof*: The solution ($18_2$) is directly obtained from (10). Using (8), the equations (1) and (10) can be written in the form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = G(t)\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}, \quad G(t) = \begin{bmatrix} F(t) & -N(t) \\ 0 & -F^T(t) \end{bmatrix}. \quad (22)$$

The transition matrix for $G(t) \in \Re^{2nx2n}$ can be expressed as [6]

$$\Omega(t, t_f) = \begin{bmatrix} \Psi(t, t_f) & \Omega_{12}(t, t_f) \\ 0 & \Phi(t, t_f) \end{bmatrix}, \quad (23)$$

where the *n*x*n* matriceal blocks have the indicated meaning. The relation (23) can be immediately obtained based on the equations $\dot{\Omega}(t, t_f) = G(t)\Omega(t, t_f)$ and $\Omega(t_f, t_f) = I_{2n}$ ($I_{2n}$ is the identity matrix). Thus, the solution for (22) is

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \Omega(t, t_0)\begin{bmatrix} x(t_0) \\ v(t_0) \end{bmatrix} \text{ or } \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \Omega(t, t_f)\begin{bmatrix} x(t_f) \\ v(t_f) \end{bmatrix}. \quad (24)$$

From (12), (23) and second relation (22), yields

$$x(t) = M(t, t_f)x(t_f), \quad (25)$$

with $M(t, t_f)$ given by (20). One can prove that the matrix $M(t, t_f)$ is non-singular. This can be explained by the fact that the matrix $M(t, t_f)$ represents the transition from $x(t)$ to $x(t_f)$.

Using (12) and (24), one obtains the solution ($18_1$) with $v(t_0)$ given by (19).■

We are now in position to formulate supplementary remarks referring to the minimum value of the criterion.

Lemma 3: The minimum value of the performance index (2) is

$$I^* = \frac{1}{2} x^T(t_0)[\bar{R}(t_0) + \Phi(t_0, t_f)(S - \bar{S})M^{-1}(t_0, t_f)]x(t_0)\,. \quad (26)$$

*Proof*: Using (1), (8) and (10), one obtains

$$\frac{d}{dt}(v^T(t)x(t)) = -v^T(t)Nv(t) \quad (27)$$

and therefore, the minimum value of the performance index (17) becomes

$$I^* = \frac{1}{2}[x^T(t_0)\bar{R}(t_0)x(t_0) + x^T(t_0)v(t_0)] \quad (28)$$

and replacing (19) in the last equation, one obtains (26). ■

We can formulate now the following

Theorem 1: The solution of the optimal control problem referring to the criterion (2) and the linear system (1) is

$$u^*(t) = u_f(t) + u_c(t)\,, \quad (29)$$

where $u_f(t)$ is the feedback component

$$u_f(t) = -P^{-1}B^T(t)\bar{R}(t)x(t) \quad (30)$$

and

$$u_c(t) = -P^{-1}B^T v(t) \quad (31)$$

is a corrective component. The corrective vector v(t) can be computed with (18), where $v(t_0)$ is given by (19). The minimum value of the criterion is (26).

The *proof* can be directly obtained from lemmas 1 and 2. ■

*Remark 2*: The above presented solution has advantages by comparison with the classical methods if one can find a particular solution to the matriceal differential Riccati equation.

# 3 LQ optimal control for discrete time systems

The above formulated properties can be translated for the discrete time case. The system equation is

$$x(k+1) = A(k)x(k) + B(k)u(k)\,, \quad (32)$$

$x(k) \in \Re^n, u(k) \in \Re^m$, x $(k_0)$ =$x^0$, k having integer values. The performance index is

$$\begin{aligned} I = &\frac{1}{2} x^T(k_f)Sx(k_f) + \\ &+ \frac{1}{2}\sum_{k=k_0}^{k_f-1}[x^T(k)Q(k)x(k) + u^T(k)P(k)u(k)], \end{aligned} \quad (33)$$

It is well known [3], [4] that the optimal control vector is

$$u^*(k) = -\bar{P}^{-1}(k)B^T(k)\tilde{R}(k+1)A(k)x(k)\,, \quad (34)$$

where

$$\bar{P}(k) = P(k) + N(k) \text{ and } N(k) = B^T(k)\bar{R}(k+1)B(k) \quad (35)$$

and $\tilde{R}(k)$ is the solution to the discrete Riccati equation

$$\begin{aligned} \tilde{R}(k) = &Q(k) + A^T(k)[\tilde{R}^{-1}(k+1) \\ &+ B(k)P^{-1}(k)B^T(k)]^{-1}A(k), \end{aligned} \quad (36)$$

with $\tilde{R}(k_f) = S$ and the minimum value for index I is

$$I^* = \frac{1}{2} x^T(k_0)\tilde{R}(k_0)x(k_0)\,. \quad (37)$$

In the previous equations, the inverse matrices exist because they are positive defined. In fact, one can verify that all inverse matrices which appear in the sequel exist.

Lemma 4: The control vector that minimizes the criterion (2) subject to the system (1) is

$$u^*(k) = K(k)x(k) - P^{-1}(k)B^T(k)v(k+1)\,, \quad (38)$$

where $\bar{R}(k)$ is a certain solution to (36), which satisfies a final condition $\bar{R}(k_f) = \bar{S} \geq 0$,

$$K(k) = -\bar{P}^{-1}(k)B^T(k)\bar{R}(k+1)A(k) \quad (39)$$

and the vector v(k) is given by the recurrence

$$v(k) = Z(k)v(k+1)\,, \quad (40)$$

with

$$Z(k) = A^T(k)[I_n - \bar{R}(k+1)B(k)\bar{P}^{-1}(k)B^T(k)] \quad (41)$$

and

$$v(k_f) = (S - \bar{S})x(k_f)\,. \quad (42)$$

*Proof:* The proof is similar with Lemma 1, starting from the scalar function

$$\pi(k) = \frac{1}{2} x^T(k)\bar{R}(k)x(k) + x^T(k)v(k)\,. \quad (43)$$

Adding (33) with the identity

$$\sum_{k=k_0}^{k_f-1}[\pi(k+1)-\pi(k)]-\pi(k_f)+\pi(t_0)=0\,, \tag{44}$$

and considering (32) and (36), one obtains

$$\begin{aligned} I=\pi(k_0)+\frac{1}{2}x^T(k_f)(S-\bar{S})x(k_f)-x^T(k_f)v(k_f)+ \\ \sum_{k=k_0}^{k_f-1}[\varphi(k)-\frac{1}{2}v^T(k+1)B(k)P^{-1}(k)B^T(k)v(k+1)], \end{aligned} \tag{45}$$

where φ(k) can be written in the form

$$\begin{aligned} \varphi(k)=\frac{1}{2}[u(k)-K(k)x(k)+\bar{P}^{-1}(k)B^T(k)v(k+1)]^T \\ \bar{P}(k)[u(k)-K(k)x(k)+\bar{P}^{-1}(k)B^T(k)v(k+1)]. \end{aligned} \tag{46}$$

This positive defined function is zero and therefore I has a minimal value if u(k) satisfies (38). ■

Lemma 5: The solution to the equation (40) is

$$v(k)=Z^{k_0-k}v(k_0)\,,\ \text{or}\ \ v(k)=Z^{k_f-k}v(k_f)\,, \tag{47}$$

where $v(k_f)$ is given by (42), and

$$v(k_0)=\frac{1}{2}Z^{k_f-k_0}(S-\bar{R})M^{-1}x(k_0) \tag{48}$$

(M is indicated below). The formulation and the proof of this lemma have similar forms in time variant case, but they are indicated only for time invariant problems, because M has a complicated form in the first case.

*Proof*: The solution ($47_2$) is directly obtained from (40). If the control vector given by (38) is replaced in (a1), the system equation becomes:

$$x(k+1)=Fx(k)-B\bar{P}^{-1}B^T v(k+1)\,, \tag{49}$$

with

$$F=A+BK\,. \tag{50}$$

The solution to the discrete equation (49) is

$$\begin{aligned} x(k)=F^{-(k_f-k)}x(k_f)- \\ -\sum_{j=1}^{k_f-k}F^{-(k_f-k-j+1)}B\bar{P}^{-1}B^T v(k_f-j+1). \end{aligned} \tag{51}$$

From (51) for $k=k_0$, yields

$$\begin{aligned} x(k_0)=F^{-(k_f-k_0)}x(k_f)- \\ -F^{-(k_f-k_0)}\sum_{j=1}^{k_f-k_0}F^{j-1}B\bar{P}^{-1}B^T Z^{-1}v(k_f-j). \end{aligned} \tag{52}$$

Replacing $v(k_f-j)$ from ($47_2$) in (52) and considering $M = M(k_0,k_f)$, where

$$\begin{aligned} M(k,k_f)=F^{-(k_f-k)}- \\ -F^{-(k_f-k)}\frac{1}{2}\sum_{j=1}^{k_f-k}F^{j-1}B\bar{P}^{-1}B^T Z^{j-1}(S-\bar{R}), \end{aligned} \tag{53}$$

yields

$$x(k_f)=M^{-1}x(k_0)\,. \tag{54}$$

Taking into account the above relations, the final form for $v(k_0)$ is (48). ■

Lemma 6: The minimum value of the performance index (33) is

$$I^*=\frac{1}{2}[x^T(k_0)\bar{R}(k_0)x(k_0)+x^T(k_0)v(k_0)] \tag{55}$$

or

$$I^*=\frac{1}{2}x^T(k_0)[\bar{R}(k_0)+Z^{k_k-k_0}(S-\bar{R}(k_0)M^{-1}]x(k_0)\,. \tag{56}$$

*Proof* is similar with Lemma 3: using (32), (38), (40) and (41), one can write

$$\begin{aligned} -\sum_{k=k_0}^{k_f-1}v^T(k+1)B(k)P^{-1}(k)B^T(k)v(k+1)= \\ =x^T(k_f)v(k_f)-x^T(k_0)v(k_0). \end{aligned} \tag{57}$$

Then one obtains (55) from (45) and from the condition φ(k)=0, introduced in Lemma 1. The form (56) results from (55) and Lemma 5. ■

Theorem 2: The solution to the discrete optimal control problem referring to the linear system (32) and the criterion (33) is

$$u^*(k)=u_f(k)+u_c(k)\,, \tag{58}$$

where $u_f(t)$ is the feedback component

$$u_f(k)=-\bar{P}^{-1}(k)B^T(k)\bar{R}(k)A(k)x(k) \tag{59}$$

and

$$u_c(k)=-\bar{P}^{-1}(k)B^T(k)Z^{-1}(k)v(k) \tag{60}$$

is the corrective component, with v(k) given by Lemma 5.

*Proof* results immediately from lemmas 4 and 5 ■

# 4 The solution to the Riccati differential and difference equations

An analytical formula for the solution to the Riccati differential equation can be established starting from the previous results.

Theorem 3: The solution to the Riccati differential equation (4) can be written in the forms

$$\tilde{R}(t) = \bar{R}(t) + \Phi(t,t_f)(S - \bar{S})M^{-1}(t,t_f) \quad (61)$$

or

$$\tilde{R}(t) = \bar{R}(t) + \Phi(t,t_0)WM_0^{-1}(t,t_0), \quad (62)$$

where

$$M_0(t,t_0) = \Psi(t,t_0) + \Omega_{12}(t,t_0)[\tilde{R}(t_0) - \bar{R}(t_0)] \quad (63)$$

and the constant matrix W is given by

$$W = \Phi(t_0,t_f)(S - \bar{S})M^{-1}(t_0,t_f). \quad (64)$$

*Proof*: Using a similar way as in Lemma 2, the corrective vector v(t) can be expressed in terms of x(t) as in (19), replacing $t_0$ with $t$. The scalar function π(t) can be written from (13), (18) and (19)

$$\pi(t) = \frac{1}{2}x^T(t)[\bar{R}(t) + \\ + \Phi(t,t_f)(S - \bar{S})M^{-1}(t,t_f)]x(t).$$

Comparing with (16), one obtains (61).

In the same way, the solution can be expressed in terms of initial values

$$\tilde{R}(t) = \bar{R}(t) + \Phi(t,t_0)[\tilde{R}(t_0) - \bar{R}(t_0)]M_0^{-1}(t,t_0). \quad (65)$$

The unknown matrix $\tilde{R}(t_0)$ can be established from (61), for t = $t_0$ and, replacing in (65), one can reach (62). ■

*Remark 3*: The proposed analytical solutions for the matrix differential Riccati equation use only *n*x*n* transition matrices, unlike the known analytical solution [1] which implies a *2n*x*2n* transition matrix.

The procedure involves the knowledge of a particular solution of this equation and the solving of a linear differential equation. This is an extension of the well known classical method for the scalar Riccati equation.

The minimum values of the criterion (26) is the same with one obtained from (16) for t = $t_0$.

*Remark 4:* The form (62) of the solution allows the real time computation by comparison with classical methods that solve the Riccati equation in inverse time, starting from the final condition. The proposed solution can be obtained in direct time. It is possible a recursive computing, since the matrices $\Phi(t,t_0)$ and $M_0(t,t_0)$ can be recurrently computed with initial values $\Phi(t_0,t_0) = I_n$ and $M_0(t_0,t_0) = I_n$ (the identity matrix).

A similar theorem can be formulated for the discrete problem.

<u>Theorem 4</u>: The solution to the Riccati difference equation can be expressed in the forms

$$\tilde{R}(k) = \bar{R}(k) + Z^{k_f - k}(S - \bar{S})M^{-1}(k,k_f) \quad (66)$$

or

$$\tilde{R}(k) = \bar{R}(k) + Z^{-(k-k_0)}(\tilde{R}(k_0) - \bar{R}(k_0))M_0^{-1}(k,k_0). \quad (67)$$

In the above relations $\tilde{R}(k_0)$ is obtained from (66) for k=$k_0$ and $M_0^{-1}(k,k_0)$ from (53) replacing $k_f$ with $k_0$. The proof is similar with one for the Theorem 3, using the corresponding equations for the discrete time case. Also, the Remark 4 is valid in this case.

*Remark 5*: An advantage of the analytical solutions is the fact that some computing difficulties can be avoided in certain concrete problems. Of course, some ill-posed sub-problems can arise, especially referring to the computing of the inverse matrices $M^{-1}$ and $M_0^{-1}$ for both continuous and discrete case. There are also other inverse matrices in the discrete problems, with the specification that they appear also in the classical procedure.

## 5 Time invariant problems

The time invariant linear quadratic problem, when all the matrices from the system equation and from the criterion are constant, represents a particular important and frequently meet case. The previous lemmas and theorems can be applied replacing the particular solution $\bar{R}(t)$ or $\bar{R}(k)$, respectively, with the constant matrix R. Since R is a constant matrix, it satisfies the algebraic Riccati equation (RAE)

$$RNR - RA - A^TR - Q = 0 \quad (68)$$

(for continuous time problems) and

$$R = Q + A^TRA - A^TRB\bar{P}^{-1}B^TRA \quad (69)$$

(for discrete time case). Obviously, we have to replace $\bar{S}$ with R too.

Referring to the Theorems 1 and 2, it results that the control vector $u$ has a feedback component $u_f$ with a constant matrix depending on R and a corrective component $u_c$, which can be recurrently computed. A positive defined matrix R ensures the system stability after the final moment if the corrective component vanishes. In the Theorems 3 and 4, the analytical solutions to RDEs are expressed in terms of the solution to RAE and contain supplementary terms which can be recurrently computed. These aspects represent important advantages in implementation of the optimal control and for the obtaining the solution to RDE by comparison with other methods.

These advantages are not so evident for time variant problem, except the cases when we easy obtain a particular solution to RDE. This is the reason for what we do not insist on the discrete time variant case. It should be noticed that the presented relations have the same form in time variant problems as in invariant ones, but the matrix M has a more complicated form.

## 6 Algorithms for optimal control

We shall refer to the time invariant problems, when the proposed methods have significant advantages, by comparison with the classical procedures. The extension to the time variant case is immediately. In order to simplify the presentation, we shall consider in the sequel only continuous time systems, but, of course, the discrete time case is similar.

In the classical methods, the optimal controller (3) is time variant even in the case of an invariant problem. This fact introduces difficulties in the controller implementation, taking into account also that the RDE must be solved in inverse time.

Based on the previous results, one can establish a more efficient for implementation algorithms:

(a) A first possibility is to use the implementation based on relation (3), but using also (61) in order to compute the matrix $\tilde{R}(t)$, with $\bar{R}(t) = R$. Note that (61) offers a simpler analytical solution by comparison with other known solutions.

(b) Another way is similar with (a) but using (62) in order to compute the solution to the matrix differential Riccati equation. In this case, the solution is obtained in direct time and all the variant terms from (62) can be recurrently computed.

(c) A more convenient way is based on expressions offered by the Theorem 2. Let us note that, in the invariant case, the feedback component $u_f(t)$ given by (28) is a usual feedback one and is identical with the one obtained in the similar optimization problem with infinite final time. The corrective component $u_c(t)$ given by (29) ensures the coincidence with the unique solution obtained in the finite final time problem. The vector v(t) can be recurrently computed with ($18_1$), with initializations (19) that depends on $x(t_0)$. This procedure has more advantages than others methods since the proposed controller is carried out only with invariant blocks.

*Remark 6*: The formulas used in the previous algorithms are rather complicated, but the most part of the computing is performed off-line, in the controller design stage. Important is the fact that the on-line control can be easy implemented since it implies to compute a limited number of time-variant elements and these elements (vectors or matrices) can be recurrently computed. This advantage appears especially for the third possibility of implementation in the time-invariant problems.

*An example* is presented in the sequel in order to illustrate the behaviour of the optimal system. Among the performed tests, a $4^{th}$ order system with two control variables was selected

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & 4 & -1 & 0 \\ 4 & 2 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 4 & 0 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u(t).$$

The final time is $t_f = 0.3s$ and the matrices in (2) are chosen as follows: S=diag(10,10,10,10), Q=diag(1,1,1,1), p=diag(1,1). The initial state vector is $x(0) = \begin{bmatrix} -5 & 5 & -8 & -4 \end{bmatrix}^T$. All the above methods were used for both continuous and discrete case (of course, after the discretisation of the problem). The differences among these cases are insignificant, so that only one figure for the behaviour of the optimal system is presented.

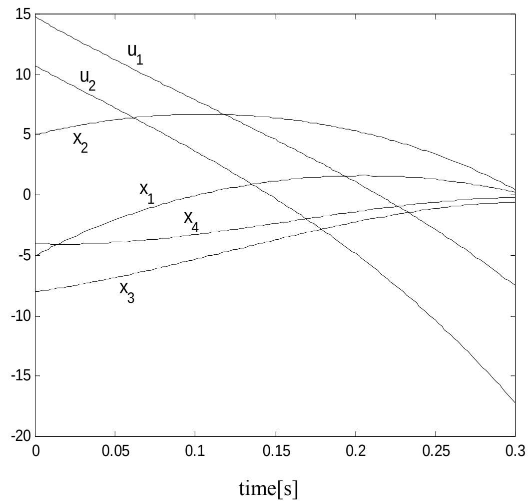


Fig. 1 – The behaviour of the optimal system

## 6 Conclusions

A non-variational procedure for continuous and discrete time LQ problem is presented.

The proposed solution is more convenient for implementation by comparison with the classical solution.

Two analytical formulas for the solution to the matrix differential / difference Riccati equation are also indicated. These formulas have advantages by comparison with usual methods.

One of the proposed algorithms is based on the direct time solving of the Riccati equation.

It is also indicated an efficient possibility of implementation for the optimal controller, using a usual feedback and a corrective component, depending on the initial state. This optimal controller is advantageous especially in the time-invariant case.

*References*:


[1] M. Athans, P.L. Falb., *Optimal Control*, Mc Graw Hill, New York, 1966.

[2] B.D.O. Anderson, J.B. Moore, *Optimal Control. Linear Quadratic Methods*, Prentice Hall, 1990.

[3] A.P. Sage, C.C. III White, *Optimal System Control*, Prentice Hall, 1977.

[4] B.C. Kuo, *Digital Control System*, Saunders College Publishing, Philadelphia, 1992.

[5] T. Totani, K. Nonami, A New Approach to Realize Linear Optimal Control by Means of Compensation Input, *IEEE Trans. on Automatic Control*, Vol.AC-29, No.9, Sept. 1984, pp. 832- 834.

[6] C. Botan, The optimal output regulator problem for linear systems, *6th Internat. Conf. on Control Syst. And Computer Science*, Bucharest, Romania, May, 1985 pp. 90 – 94.

[7] C.S. Kenney, R.B. Leipnik, Numerical Integration of the Differential Matrix Riccati Equation, *IEEE Trans. on Automatic Control*, Vol.AC-30, No.10, Oct. 1985, pp. 962 – 970.

[8] D.G. Lainiotis, Generalized Chandrasekar algorithms: Time-varying models, *IEEE Trans. Automat. Contr*., Vol.21, Oct. 1976, pp. 728 – 732.

[9] E.J. Davison, M.C. Maki, The numerical solution of the matrix Riccati differential equation, *IEEE Trans. Automat. Contr.*, Vol.18, Feb. 1973, pp. 71 – 73.

[10] I. Rusnak, Almost analytic representation for the solution of the differential Riccati equation, *IEEE Trans. Automat. Contr.*, Vol.33, No.2, Feb. 1988, pp. 191 – 193.

[11] C. Botan, On the solution of the Riccati differential matrix equation, *Symposium on Computational System Analyses*, Elsever Science Publishers, Berlin, 1992 pp. 141- 146.

[12] F. Incertis, On closed Form Solutions for the Differential Matrix Riccati Equation Problem, *IEEE Trans. Automat. Contr*., Vol.AC-28,No.8, 1983, pp. 845 – 848.

[13] C.H. Choi, A.J. Laub, Efficient Matrix-Valued Algorithms for Solving Stiff Riccati Differential Equations, *IEEE Trans. Automat. Contr*., Vol.AC-35, No.7, 1990, pp.770 – 776.

[14] D.R. Vaughan, A Nonrecursive Algebraic Solution for the Discrete Riccati Equation, *IEEE Trans. Automat. Contr*., Oct. 1970, pp. 597 – 599.

[15] M. Sorine, P. Winternitz, Superposition Laws for Solutions of Differential Matrix Riccati Equations Arising in Control Theory, *IEEE Trans. Automat. Contr.*, Vol.AC-30, No.3, 1985, pp. 266 – 272.

[16] L.Z. Liao, D. Li, Successive Method for General Multiple Linear-Quadratic Control Problem in Discrete Time, *IEEE Trans. Automat. Contr*., Vol.AC-45, No.7, July 2000, pp. 1380 – 1385.

[17] C.H. Lee, An Improved Lower Matrix Bound of the Solution of the Unified Coupled Riccati Equation, *IEEE Trans. on Automatic Control*, Vol.AC-50, No.8, Aug. 2005, pp. 1221 – 1223.

[18] N.E. Barabanov, R. Ortega, On the solvability of extended Riccati equations, *IEEE Trans. Automat. Contr.*, Vol.AC-49, No.4, Apr. 2004, pp.598 -602.

[19] A. Hansson, A primal-dual interior-point method for robust optimal control of linear discrete-time systems, *IEEE Trans. Automat. Contr.*, Vol.AC-45, No.8, Aug. 2000, pp. 1532-1536.

[20] D.D. Yao, S. Zhang, X.Y. Zhou, A primal-dual semidefinite programming approach to linear quadratic control, *IEEE Trans. Automat. Contr.*, Vol.AC-46, No.9, Sept. 2001, pp. 962 – 970.