One Type of Controller Design for Delayed Double Integrator System

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Abstract: - The aim of this paper is to present a constrained pole assignment controller that is based on the analysis of optimal settings for the double integrator system. It takes into account parasitic delays in the control structure and constraints imposed on the control signal. In addition, one can find here approximation procedure resulting to the models that were used at the controller design. The whole design is illustrated on two examples.

Key-Words: - Pole assignment control, constraints, saturation, double integrator model, delays, approximation.

1 Introduction

The pole placement method belongs to frequently used method for controller setting (e.g. [2], [3], [12], [13], [14]). Its aim is to specify the position of closed loop poles. Although the pole assignment control is considered as a typical linear technique, in [9] it was shown that the philosophy can easily be extended to the constrained systems using a new constrained pole assignment (*CPA*) controller. In the paper presented concept introduces dynamical classes of the *CPA* control, corresponding to the well-known Feldbaum's theorem [4] about *n*-interval of optimal control. However, the intervals of saturated pulses are separated by smooth control intervals and, eventually, not each control interval must reach a saturation limit.

The *CPA* control design for systems with the second order plant dynamics leads to a *PD* controller. In this way, the closed loop behavior is described by two poles α_1 and α_2. Since the control signal is constrained, only admissible reference signals can be achieved. Looking for analogies with the 1st order constrained pole assignment control, in [8] and [9] it was shown that the first pole α_1 ensures a regular decrease of the distance between an actual state and the required steady state (e.g. the origin). The distance decrease can be guaranteed only along a line specified by an eigenvector of the closed loop matrix corresponding to α_1. On the other side, the second pole α_2 ensures a regular decrease of the distance between an actual representative point and the already mentioned line. Two ordered combinations of the closed loop poles $[\alpha_1, \alpha_2]$ yield two different lines, i.e. two different transient responses. If the velocity of the transient response is the basic criterion of the controller design, the value of both poles should be the same. In the case that disturbance robustness is required, it is more suitable to choose the line with a smaller slope by the appropriate choice of two different poles.

As will be shown, the proposed algorithms can also be used in controlling higher order systems approximated by integral models.

2 *CPA PD* controller for I_2 system

Considering the pole assignment controller for a double integrator system one gets a control structure that is shown in Fig.1.

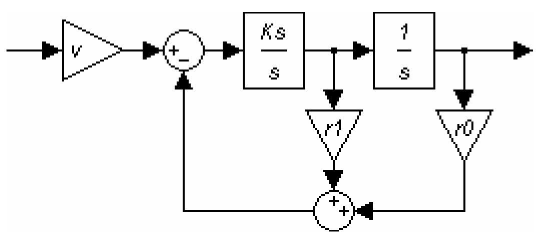

Fig. 1. Pole assignment controller for double integrator system

Its modification leads to the structure of a pole assignment *PD* controller (Fig.2).

Characteristic polynomial of both presented structures is

$$chp(s) = s^2 + K_S r_1 s + K_S r_0 \qquad (1)$$

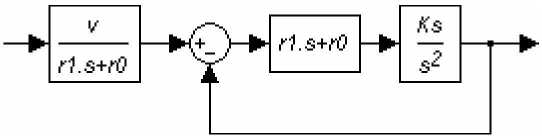

Fig. 2. Pole assignment *PD* controller

The aim of the control is to achieve aperiodical transient responses at the maximal possible controller gain. This can be ensured by double pole of the polynomial (1) that can be found solving a set of equations

$$chp(s)=0;\ \frac{d}{ds}chp(s)=0 \qquad (2)$$

where

$$\frac{d}{ds}chp(s)=2s+K_S r_1$$

Then, denoting a closed loop pole as α, there result

$$r_0=\frac{\alpha^2}{K_S}$$
$$r_1=-\frac{2\alpha}{K_S} \qquad (3)$$

whereby $\alpha\in(-\infty,0)$.

The choice of the closed loop pole α enables one to influence the velocity of the transient responses. However, as it will be shown later, the closed loop pole can also be used for a compensation of parasitic delays in the control structure.

Considering a constrained control signal $u\in\langle U_1,U_2\rangle$ the constrained pole assignment *PD* controller can be specified. Its concept was firstly introduced in [6] and later explained in [10] or [8].

Without loss of generality it is supposed that the required state has to reach zero position ($w=0$). Then, the continuous *PD* control algorithm (see also [10], [7]) can be described by the formula

$$u_r=-\frac{\alpha_1\alpha_2}{K_S}y+\frac{\alpha_1+\alpha_2}{K_S}\dot{y} \qquad (4)$$

for $\dot{y}\in\left(\frac{K_S U_2}{\alpha_1}\quad \frac{K_S U_1}{\alpha_1}\right)$ and by the formula

$$u_r=\left[1-\alpha_2\frac{y-\frac{1}{2}\left(\frac{\dot{y}^2}{K_S U_j}+\frac{K_S U_j}{\alpha_1^2}\right)}{\dot{y}}\right]U_j \qquad (5)$$

elsewhere, whereby if $y<0$ then $U_j=U_1$ else $U_j=U_2$.

In the last step the constraints are imposed on the control signal when

$$u=sat(u_r)=\begin{cases}U_1;\ u_r<U_1\\ u_r\\ U_2;\ u_r>U_2\end{cases} \qquad (6)$$

Considering the introduced control algorithm, the linear pole assignment control as well as the "bang-bang" minimum time control are involved as limit cases. For the closed loop poles tending to minus infinity, the algorithm tends to the relay minimum time one, and conversely. For the closed loop poles close to zero, the representative point does not leave the zone of linear pole assignment control.

3 *PD* controller for delayed systems

In the next section the influence of elementary parasitic time delays (transport delay – dead time or accumulated delay – time constant) on the controller settings is considered.

3.1 *PD* controller for I_2T_d system

Let us consider the control structure (Fig.3) with *PD* controller in the form

$$C(s)=K_C(T_D s+1) \qquad (7)$$

In the case of double integrator system combined with a dead time

$$F(s)=\frac{K_S}{s^2}e^{-T_d s} \qquad (8)$$

the characteristic equation of the close loop system is

$$chp(s)=s^2+K_C T_D K_S e^{-T_d s}s+K_C K_S e^{-T_d s} \qquad (9)$$

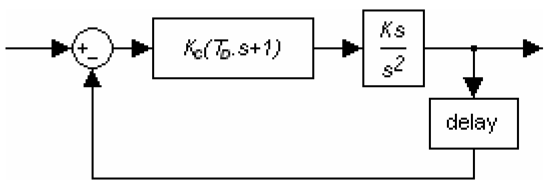

Fig. 3. *PD* controller with the double integrator system and a parasitic delay

The time optimal setting of *PD* controller follows the requirement of the fastest possible monotonic transient response. The controller parameters are to be found by solving equations

$$chp(s)=0;\ \frac{d}{ds}chp(s)=0;\ \frac{d^2}{ds^2}chp(s)=0 \qquad (10)$$

where

$$\frac{d}{ds}chp(s) = \left(2 - K_C T_D K_S T_d e^{-T_d s}\right)s +$$
$$+K_C T_D K_S e^{-T_d s} - K_C K_S T_d e^{-T_d s}$$
$$\frac{d^2}{ds^2}chp(s) = K_C T_D K_S T_d^2 e^{-T_d s} s - 2K_C T_D K_S T_d e^{-T_d s} +$$
$$+K_C K_S T_d^2 e^{-T_d s} + 2$$

Then, the optimal controller parameters can be expressed in the form

$$K_C = 2\frac{\left(-7+5\sqrt{2}\right)e^{-2+\sqrt{2}}}{K_S T_d^2} \doteq \frac{0.079}{K_S T_d^2} \quad (11)$$

$$T_D = \left(3+2\sqrt{2}\right)T_d \doteq 5.828 T_d \quad (12)$$

whereby the close loop pole is

$$s_{opt} = \frac{-2+\sqrt{2}}{T_d} \doteq -\frac{0.586}{T_d} \quad (13)$$

3.1.1 Equivalent pole

As it was mentioned before, the closed loop pole in the derived *CPA PD* controller can also be used for a compensation of parasitic delays in the control structure. For this purpose, it seems to be convenient to introduce a notion of equivalent pole that firstly was mentioned in [10].

The equivalent pole is the closed loop pole that after substitution in the pole assignment control law, deduced for an ideal double integrator system, gives the same controller setting as would be calculated for the control structures considering parasitic delays.

It means, in the structures with parasitic delays equivalent poles will yield the same controller setting as closed loop poles in the ideal control structure (Fig.1). Considering the control structure (Fig.3) with *PD* controller (7) the equivalent pole can be found by fulfilling two requirements:

$$K_C \overset{!}{=} r_0$$
$$K_C T_D \overset{!}{=} r_1 \quad (14)$$

whereby r_0 and r_1 are given by (3). In this way there arises a set of two equations for one variable α. Considering 2 various poles instead of one double pole one gets

$$K_C \overset{!}{=} \frac{\alpha_1 \alpha_2}{K_S}$$
$$K_C T_D \overset{!}{=} -\frac{(\alpha_1+\alpha_2)}{K_S} \quad (15)$$

Hence

$$\alpha_{1,2} = -\frac{1}{2}K_S K_C T_D \pm \frac{1}{2}\sqrt{K_S^2 K_C^2 T_D^2 - 4K_S K_C}$$

Since a pole assignment controller is designed only for the case of real poles, the complex pair has to be approximated by the real part, or by the module. Both approximations give relatively close solutions and therefore it is sufficient to consider the simpler approximation by the real part. Then

$$\alpha_e = -\frac{1}{2}K_S K_C T_D \quad (16)$$

Solving (15) for I_2T_d system one receives

$$\alpha_{1,2} = \frac{\left(1-\sqrt{2}\right)e^{-2+\sqrt{2}}}{T_d} \pm$$
$$\pm\, i\frac{\sqrt{\left[\left(2\sqrt{2}-3\right)e^{-2+\sqrt{2}} + 10\sqrt{2} - 14\right]e^{-2+\sqrt{2}}}}{T_d}$$

and following (16) the equivalent pole of the continuous controller (*CPA PD* controller) is

$$\alpha_e = -\frac{0.231}{T_d} \quad (17)$$

In Fig.4 it is possible to compare transient responses of the I_2T_d system that is controlled by two sets of the continuous *PD* controller. The first one is done according to the optimal setting given by (11, 12) and the second one is calculated on the base of equivalent pole approximation (17). There are also presented corresponding control signals.

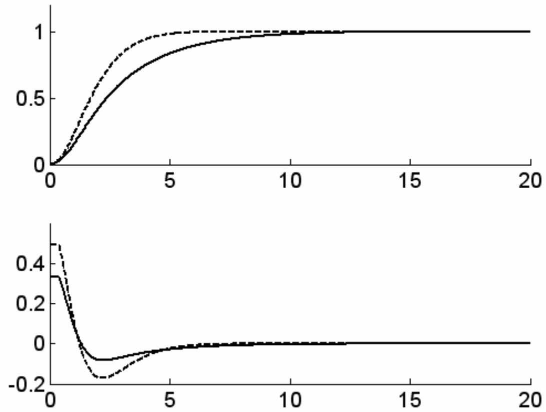

Fig. 4. Transient responses and control signals of I_2T_d system (K_S=1; T_d=0.4) that is controlled by the optimal *PD* controller (dashed line) and by the *PD* controller set according to the approximation by equivalent poles (solid line)

It is evident that with the equivalent pole approximation the dynamics of the structure is slowed down and there seems to be no sense in using it. However, when the constrained control signal is used, the optimal setting can lead to overshoot as it is evident from Fig.5.

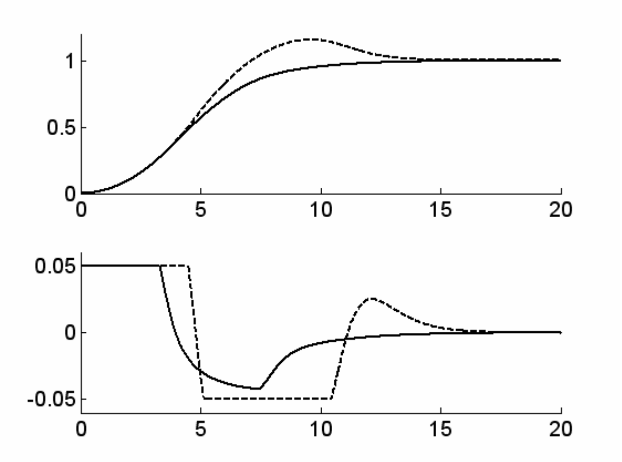

Fig. 5. Transient responses and constrained control signals of I_2T_d system (K_S=1; T_d=0.4) that is controlled by the optimal *PD* controller (dashed line) and by the *CPA PD* controller set according to the approximation by equivalent poles (solid line)

3.2 *PD* controller for I_2T_1 system

Now, let us consider a double integrator system with one time constant that can represent a parasitic delay in the feedback (Fig.3).

$$F(s) = \frac{K_S}{s^2(T_1 s + 1)} \tag{18}$$

After introducing the extended state vector $x = \begin{pmatrix} y & \dot{y} & u_r \end{pmatrix}^T$ it can be written in the form

$$\frac{d}{dt}\begin{pmatrix} y \\ \dot{y} \\ u_r \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & K_S \\ 0 & 0 & -\frac{1}{T_1} \end{pmatrix}\begin{pmatrix} y \\ \dot{y} \\ u_r \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{pmatrix} u \tag{19}$$

The control structure with the *PD* controller

$$u = -rx = -\begin{pmatrix} K_C & K_C T_D & 0 \end{pmatrix} x$$

leads to the closed loop matrix

$$A_r = A + br = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & K_S \\ -K_C T_1 & -K_C T_D T_1 & -\frac{1}{T_1} \end{pmatrix}$$

The aim of the controller design is to achieve a well-balanced dynamics, when the characteristic polynomial $chp(s)$ has a triple pole. This can be fulfilled following conditions (10) where

$$chp(s) = \det(sI - A_r) =$$
$$= s^3 + \frac{1}{T_1}s^2 + \frac{K_C T_D}{T_1}K_S s + \frac{K_C}{T_1}K_S$$
$$\frac{dchp(s)}{ds} = 3s^2 + \frac{2}{T_1}s + \frac{K_C T_D}{T_1}K_S$$
$$\frac{d^2 chp(s)}{ds^2} = 6s + \frac{2}{T_1}$$

Hence, one receives

$$K_{Copt} = \frac{1}{27 K_S T_1^2} \tag{20}$$

$$T_{Dopt} = 9T_1 \tag{21}$$

$$s_{opt} = -\frac{1}{3T_1} \tag{22}$$

The equivalent pole can be found solving the set of equations (15) and taking the real part of the solution:

$$\alpha_{1,2} = \frac{-3 \pm i\sqrt{3}}{18T_1} \tag{23}$$

or directly according to the relation (16) when

$$\alpha_e = -\frac{1}{6T_1} \tag{24}$$

4 Process approximation

The design of control algorithms introduced in previous sections follows the predefined simplified I_2T_d and I_2T_1 models of the "real" systems. The approximation of systems can be done in various ways. Process dynamics are very often determined from a step response experiment. This requires, however, that the system is at rest before the input is applied, and that there are no measurement errors [2]. In practice it is difficult to ensure such conditions. So, the step response method is limited to the determination of simple models. Models obtained from a step experiment are, however, often sufficient for the controller tuning.

One of the first approximations of the system by an integrator based model was described in [15]. The model corresponded to an integrator with dead time (I_1T_d model) and it was characterized by two parameters. The process gain R and the dead time L can easily be determined graphically from the step response (see Fig.6).

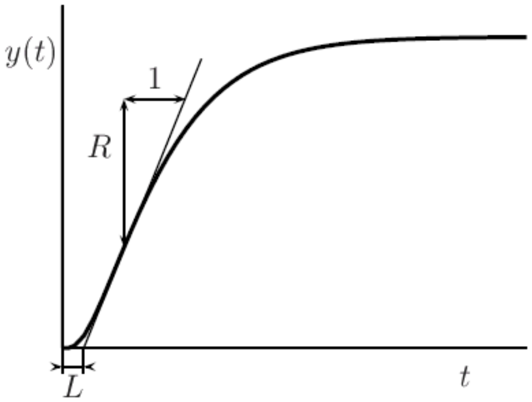

Fig.6. Typical process reaction curve

The symbol R was used for the slope of a tangent drawn to the point of inflection of the unit step response. Parameter L was defined as the time at which this tangent cuts the time axis. The model is the basis for the Ziegler-Nichols tuning procedure discussed in [15] and it can also be fitted to an unstable process. The Ziegler-Nichols approximation of the process may also be accepted as an approximation for I_1T_1 models. Other methods can be found e.g. in [11].

The mentioned I_1T_d and I_1T_1 models can be estimated in different ways and after generalization the proposed methods can also be used for determination of approximate models of higher order.

One solution is to approximate the initial phase of a process reaction curve given by points y_i by I_2T_d model using the least squares method. Then, the initial part of the curve can be described by

$$\hat{y}_i = K_S \frac{(iT - T_d)^2}{2} \tag{25}$$

The procedure is based on a successive increase of dead time T_d from the starting value $T_d = 0$ using the step T until the square of difference between the process and model outputs

$$S = \sum_{i=0}^{m} (y_i - \hat{y}_i)^2 \tag{26}$$

reaches a minimum value. For I_2T_d model

$$S = \sum_{i=0}^{m} \left[y_i - K_S \frac{(iT - T_d)^2}{2} \right]^2 = \\ = \sum_{i=0}^{m} \left[y_i - K_S T^2 \frac{(i-k)^2}{2} \right]^2 \overset{!}{=} \min_{k,m} \tag{27}$$

The interval of approximation is specified by the parameter m which is gradually increased. Then, the gain of the process model K_S is determined as the maximum gain calculated for various values of m. Another possibility is to find the maximum value of the transport delay $T_d = kT$. However, in both cases the same results are achieved.

The reason for looking for the maximum values of either model gain K_S or dead time T_d is that they lead to the minimal overshoot of the process output.

The identification algorithm for the system approximation by I_2T_d model is shown in Fig.7.

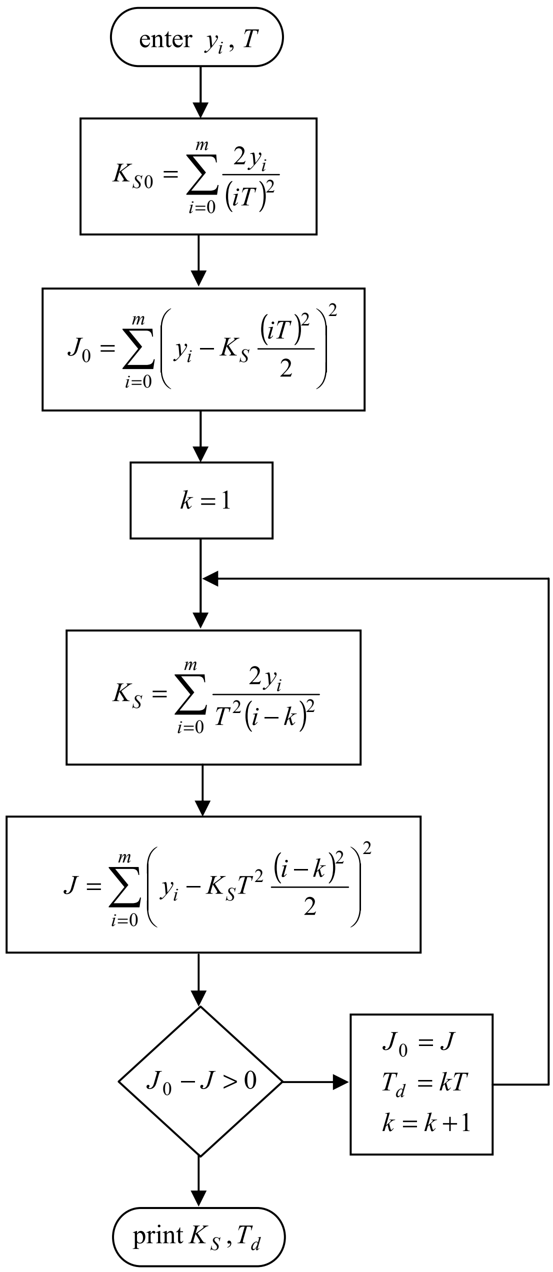

Fig.7. Identification algorithm for the system approximation by I_2T_d model

Approximation of the process reaction curve by I_2T_1 model can be accomplished using the modified Prony's method (see e.g. [5]). Since the structure of the model is known, there is no problem to find characteristic polynomial of discrete transfer function and to determine parameter T_1 from the related equation. Precision in the calculation of the parameter T_1 can be increased by solving the predetermined system of equations for example by means of the least squares method. The gain of the approximated model K_S is computed from the analytical form of transient response

$$y_i = K_S\left[iT\left(\frac{iT}{2}-T_1\right)+T_1^{\,2}\left(1-D^i\right)\right];\ \ i\in\langle 0,m\rangle \tag{28}$$

whereby also in this case the predetermined system of equations can be solved. The approximation interval is chosen to guarantee a maximal possible gain K_S (time constant T_1).

5 Examples

In following two examples let us illustrate both the system approximation and the controller design as well.

5.1 Example 1

Let's consider the fourth order system given by the transfer function

$$F(s)=\frac{1}{(s+1)(\alpha s+1)(\alpha^2 s+1)(\alpha^3 s+1)};\ \alpha=0.2$$

The system has 4 poles whose spacing is determined by the parameter α. It is to be controlled by *CPA PD* controller described in this article. The control signal is limited to $U_{\min}=-2$ and $U_{\max}=2$. The pole of the controller is designed according to the approximation of the system by I_2T_d and I_2T_1 models. The effort of both approximations is to copy the beginning of the transient response (Fig.8). In this way one can receive models in Table 1.

Model	Transfer function
I_2T_d	$\frac{3.8}{s^2}e^{-0.05s}$
I_2T_1	$\frac{3.05}{s^2(0.03s+1)}$

Table 1. Approximation models of the system in Ex.1

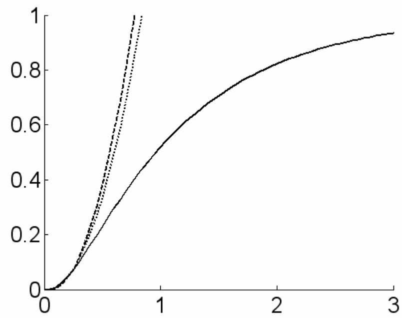

Fig. 8. Example 1: approximation of the system step response (solid line) by I_2T_d (dashed line) and by I_2T_1 model (dotted line).

The equivalent pole of the controller that should compensate a parasitic delay in the control loop (modeled by the transport or time constant delay), can be computed according to (17) or (24) respectively. For I_2T_d model it is $\alpha_e=-4.6116$ and for I_2T_1 model $\alpha_e=-5.5556$.

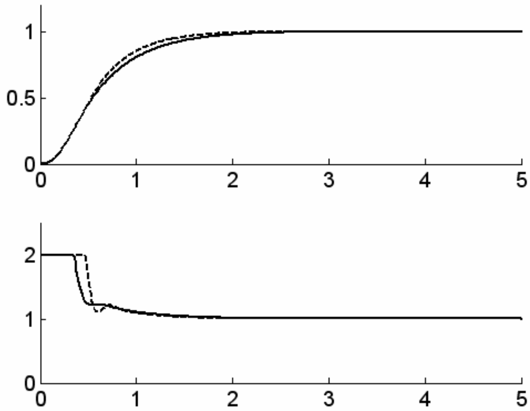

Fig. 9. Transient responses and control signals for the first benchmark example controlled by *CPA PD* controller. The system is approximated by I_2T_d (solid line) and by I_2T_1 model (dashed line).

The simulation results are presented in Fig.9. Both transient responses are monotonic. The controller based on I_2T_d approximation is a little bit slower than the second one but the advantage is that its control signal has a flatter character.

5.2 Example 2

The *CPA PD* controller can also be used for the system with damped oscillatory character (2 complex poles). Let's consider the system with transfer function

$$F(s)=\frac{1}{(0.2s+1)(s^2+0.1s+1)}$$

Following the beginning of the transient response it can be approximated by the integral models in Table 2. As it can be seen in Fig.10 both approximation methods give similar results.

Model	Transfer function
I_2T_d	$\frac{0.76}{s^2}e^{-0.1s}$
I_2T_1	$\frac{0.795}{s^2\left(0.128s+1\right)}$

Table 2. Approximation models of the system in Example 2

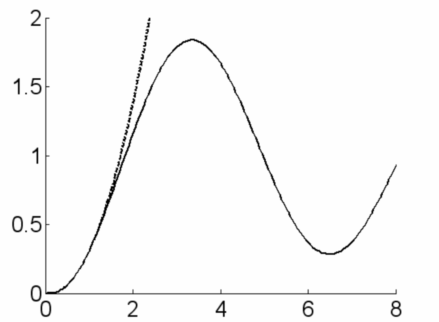

Fig. 10. Example 2: approximation of the system step response (solid line) by I_2T_d (dashed line) and by I_2T_1 model (dotted line).

The equivalent pole of *CPA PD* controller is set to $\alpha_e = -2.3058$ for I_2T_d and $\alpha_e = -1.3021$ for I_2T_1 approximation. The computed control signal is limited to $U_{\min} = -1.5$ and $U_{\max} = 1.5$. The simulation results can be seen in Fig.11.

It is evident that both approximations are good enough to be used as a starting point for the whole controller design. The resulting transient responses have a monotonic character and the control signal is also acceptable.

6 Conclusion

The introduced controllers, presented here as constrained pole assignment controllers (*CPA* controllers), are derived for the double integrator plant models. The controller design enables to consider the influence of elementary parasitic time delays (transport delay - dead time or accumulative delay - time constant) on controller setting.

The poles of the controller are adapted according to delays present in the control loop. The controller analysis of optimal setting enables one to determine the so-called equivalent poles that account for the present delay and can be substituted into the pole assignment control law computed for a delay free system. Since the controller structure remains fixed, i.e. it doesn't change its form and order due to the identified time delays, the controllers can be associated with the well-known group of parameter-optimized controllers.

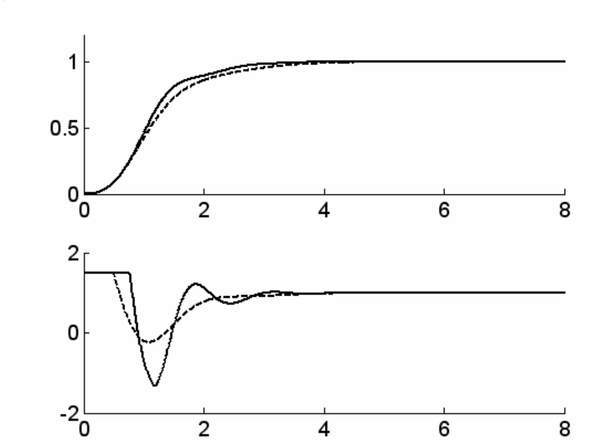

Fig. 11. Transient responses and control signals for the second benchmark example controlled by *CPA PD* controller. The system is approximated by I_2T_d (solid line) and by I_2T_1 model (dashed line).

7 Acknowledgments

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