Control for stability and Positivity of 2-D linear discrete-time systems

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Abstract-This paper investigate the stabilizability of 2-D linear discrete-time systems described by the Roesser model with closed-loop positivity. Necessary and sufficient condition for the existence of desired state-feedback controllers guaranteeing the resultant closed-loop system to be asymptotically stable and positive is obtained. The synthesis of state-feedback controllers, including the requirement of positiveness of the controllers and its extension to uncertain plants are solved in terms of Linear Matrix Inequalities (LMIs) which can be easily verifies by using standard numerical software. Numerical examples are provided to illustrate the proposed conditions.

Key words-Positive Systems, 2-D Systems, Linear Matrix Inequalities (LMIs), Stability, Stabilization, Positive Control, Roesser Model.

1 Introduction

In last two decades, the two-dimensional (2-D) system theory has been payed a considerable attention by many researchers. The 2-D linear models were introduced in the seventies [3, 5] and have found many applications, such as in digital data filtering, image processing [13], modeling of partial differential equations [6], etc. In connection with Roesser [13] and Fornasini-Marchesini [4] models, some important problems such as realization, controllability, minimum energy control, has been extensively investigated (see for example [9]). On the other hand, the stabilization problem is not fully investigated and still not completely solved.

Recently, we observe a growing interest in theory and application of positive 2-D systems [8, 10, 12, 14]. Positive 2-D Roesser systems has been studied in [7] and more detailed description can be found in [10].

The stability of 2-D discrete linear systems can be reduced to checking the stability of 2-D characteristic polynomial [15, 16]. This appears to be difficult task for the control synthesis problem. In the literature, various types of easily checkable but only sufficient conditions for asymptotic stability and stabilization problems for 2-D discrete linear systems have been proposed [17, 18, 19, 20].

In the present paper, we first analyze the stability of positive 2-D Roesser model [7, 13] and obtain necessary and sufficient condition for its stability. On the other hand, we investigate the stabilization problem of positive 2-D linear discrete-time systems in Roesser Model. Instead of using algebraic techniques which have been widely employed for the analysis of positive system, our development is based on matrix inequalities. Based on the well-established results of Lyapunov stability theory and nonnegative matrix, equivalent conditions in terms of Linear Matrix Inequalities (LMIs) are obtained for the existence of stabilizing state-feedback controllers, including the requirement of positiveness of the controller and its extension to uncertain plants. A remarkable advantage of these conditions lies in the fact that they are not only necessary and sufficient, but also can be easily verifiable by using some standard numerical software. Moreover, these conditions readily construct a desired controller if its exists. To the authors' knowledge, this work represents the first LMI treatment on control synthesis for guaranteeing asymptotic stability and positivity. The remainder of the paper is structured as follows: In section 2 the problem is formulated and some preliminary results are given. Section 3 studies the stability problem. Section 4 present a necessary and sufficient condition for stabilization with positivity constraint. Robust stabilization problem is study in section 5. In section 6 numerical examples are given to illustrate the proposed results.

Notation: The following notation well be used throughout this paper; $\mathbf{N}$ denotes the set of integer numbers. R^n denote the n-dimensional Euclidean space; $R^{m\times n}$ denotes the set of all real matrices of dimension $m \times n$; $R_+^{m\times n}$ denotes the set of all $m \times n$ real matrices with nonnegative entries and $R_+^n \triangleq R_+^{1\times n}$; The notation $M > 0$ (resp. $M \geq 0$), where M is a real matrix (or a vector), means that all the components of M are strictly positive (resp. nonnegative); diag $\{...\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. For a complex number z, the quantity $|z|$ represents its modulus.

2 Problem formulation and Preliminaries

Consider the following 2-D system described by Roesser model [13]:

$$\left\{ \begin{array}{r} \left[\begin{array}{c} x^h(i+1,j) \\ x^v(i,j+1) \end{array} \right] = A \left[\begin{array}{c} x^h(i,j) \\ x^v(i,j) \end{array} \right] + Bu(i,j) \\ y(i,j) = C \left[\begin{array}{c} x^h(i,j) \\ x^v(i,j) \end{array} \right] + Du(i,j) \end{array} \right. \tag{1}$$

where $A = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right]$, $B = \left[\begin{array}{c} B_1 \\ B_2 \end{array} \right]$ and $C = \left[\begin{array}{cc} C_1 & C_2 \end{array} \right]$, $A_{11} \in R^{n_1\times n_1}$, $A_{12} \in R^{n_1\times n_2}$, $A_{21} \in R^{n_2\times n_1}$, $A_{22} \in R^{n_2\times n_2}$, $B_1 \in R^{n_1\times m}$, $B_2 \in R^{n_2\times m}$,$C_1 \in R^{l\times n_1}$, $C_2 \in R^{l\times n_2}$ and $D \in R^{l\times m}$ are given constant real matrices. The vectors $x^h(i,j) \in R^{n_1}$ and $x^v(i,j) \in R^{n_2}$ are, respectively, the horizontal and vertical states at the point (i,j) and the vectors $u(i,j) \in R^m$ and $y(i,j) \in R^l$ are respectively, the input and output signal of System (1). Boundary initial conditions for System (1) are given by two sequences (x_0^h) and (x_0^v) such that:

$$\left\{ \begin{array}{lcll} x^h(0,j) & = & x_0^h(j) & \forall j \in \mathbf{N}, \\ x^v(i,0) & = & x_0^v(i) & \forall i \in \mathbf{N}. \end{array} \right. \tag{2}$$

In the sequel, the following definition will be used.

Definition 2.1 *System (1) with zeros input $u = 0$, is called positive if for any given nonnegative boundary conditions $x_0^h(j) \geq 0$ and $x_0^v(i) \geq 0$, the resulting states and the output are also nonnegative $x(i,j) \geq 0$ and $y(i,j) \geq 0$.*

The following result shows how one can check the positiveness of System (1) (see [10]).

Proposition 2.1 *System (1) is positive if and only if $A \in R_+^{(n_1+n_2)\times(n_1+n_2)}$, $B \in R_+^{(n_1+n_2)\times m}$, $C \in R_+^{l\times(n_1+n_2)}$ and $D \in R_+^{l\times m}$.*

Asymptotic stability for general Roesser model [13] has been extensively studied in the literature. A well-known necessary and sufficient frequency condition for asymptotic stability is stated in the following.

Lemma 2.1 *Let $A_{11} \in R^{n_1 \times n_1}$, $A_{12} \in R^{n_1 \times n_2}$, $A_{21} \in R^{n_2 \times n_1}$, $A_{22} \in R^{n_2 \times n_2}$ be given constant real matrices. Then, 2-D system described by the Rosser model (1) with zeros input $u = 0$, is asymptotically stable if and only if the following condition holds*

$$\begin{cases} \mathbf{det}(\begin{bmatrix} I_{n_1} - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_{n_2} - z_2 A_{22} \end{bmatrix}) \neq 0, \\ \forall (z_1, z_2) \in \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\} \end{cases} \tag{3}$$

In the sequel, our purpose is to investigate the existence of state-feedback control laws

$$u(i,j) = K \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + v(t) \tag{4}$$

where $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ such that the resulting closed-loop system given by:

$$\begin{cases} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \bar{A} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Bv(i,j) \\ y(i,j) = \bar{C} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Dv(i,j) \end{cases} \tag{5}$$

is positive and asymptotically stable. where $K \triangleq [k_{ij}] \in R^{l \times (n_1 + n2)}$ is the controller gain to be determined and

$$\bar{A} \triangleq [\bar{a}_{ij}] = A + BK, \ \bar{C} \triangleq [\bar{a}_{ij}] = C + DK. \tag{6}$$

Of course, if one utilizes directly the results of Lemma 2.1 and Proposition 2.1, one can have the following necessary and sufficient condition for the closed-loop system to be positive and asymptotically stable:

$$\begin{bmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1 & A_{22} + B_2 K_2 \end{bmatrix} \geq 0,$$

$$det\{\begin{bmatrix} I_{n_1} - z_1(A_{11} + B_1 K_1) & -z_1(A_{12} + B_1 K_2) \\ -z_2(A_{21} + B_2 K_1) & I_{n_2} - z_2(A_{22} + B_2 K_2) \end{bmatrix}\} \neq 0 \tag{7}$$

$$\forall (z_1, z_2) \in \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}.$$

However, this is a very complicated formulation which leads to a very hard problem to solve, since we have a linear constraint (positivity constraint) but also mixed with the very highly nonlinear infinite dimensional constraint (asymptotic stabilizability constraint as stated above). A significant contribution of this paper is reflected by its simplicity and completeness. Effectively, all the provided main results involve easily checkable necessary and sufficient conditions. It will be shown, in the sequel, how we can still completely solve problem (7) in term of Linear Matrix Inequalities which avoids unnecessary computational burden.

3 Stability Analysis

This section provides preliminary stability results for the free linear 2-D system described by the Roesser model:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}. \tag{8}$$

In fact, it will be shown that the asymptotic stability of System (8) (under the positivity constraint) is equivalent to the following 1-D discrete-time system:

$$x(k+1) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x(k). \tag{9}$$

We also use the following definition for 1D discrete-time system (9).

Definition 3.1 *System (9) is called positive if for any given nonnegative initial conditions $x(0) \geq 0$, the resulting states are also nonnegative $x(i) \geq 0$.*

Next, recall that the spectral radius $\rho(M)$ of a matrix $M \in R^{n \times n}$ is defined as:

$$\rho(M) = \max\{|\lambda_1|, \ldots, |\lambda_n|\},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of M. Also, for a complex matrix $N = [n_{ij}]$ we define the real matrix $|N|$ as the matrix formed by the components $|n_{ij}|$.

Now, in order to establish our main stability result, we need some technical key role result which is provided by the following well-known lemma.

Lemma 3.1 *[11] Let M be a real matrix and N be a complex matrix such that $|N| \leq M$ ($M - |N|$ is a nonnegative matrix), then $\rho(N) \leq \rho(M)$.*

Lemma 3.2 *[2] Assume that the matrices A_{11}, A_{12}, A_{21} and A_{22} are constant nonnegative matrices (or equivalently that System (9) is positive). Then, the following statements are equivalent:*

(i) 1-D system described by (9) is asymptotically stable.

(ii) $\rho(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) < 1.$

(iii) there exists a positive diagonal matrix $P \in R^{(n_1+n_2)\times(n_1+n_2)}$ satisfying

$$A^T PA - P < 0$$

where $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

(iv) There exists a vector $d \in R^{n_1+n_2}$ such that

$$\begin{bmatrix} A_{11} - I_{n_1} & A_{12} \\ A_{21} & A_{22} - I_{n_2} \end{bmatrix} d < 0, \quad d > 0. \tag{10}$$

In what follows we present equivalent conditions with regard to the asymptotic stability of 2-D positive system described by the Roesser model (8).

Theorem 3.1 *Assume that the system (8) is positive or equivalently that the matrices A_{11}, A_{12}, A_{21} and A_{22} are nonnegative. Then, the following statements are equivalent:*

(i) $\begin{cases} \mathbf{det}(\begin{bmatrix} I_{n_1} - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_{n_2} - z_2 A_{22} \end{bmatrix}) \neq 0, \\ \forall (z_1, z_2) \in \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}. \end{cases}$

(ii) $\rho(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) < 1.$

Proof: $(i) \Rightarrow (ii)$ by setting $z_1 = z_2 = z$ in condition (i), then we have obviously

$$det(I - z \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) \neq 0, |z| \leq 1, \tag{11}$$

which, in turn, is equivalent to the condition (ii).
$(ii) \Rightarrow (i)$ Let z_1 and z_2 be any arbitrary complex numbers such that $|z_1| \leq 1, |z_2| \leq 1$. So we can easily see that

$$\begin{bmatrix} |z_1 A_{11}| & |z_1 A_{12}| \\ |z_2 A_{21}| & |z_2 A_{22}| \end{bmatrix} \leq \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

then, by using the spectral property given in Lemma 3.1, we obtain

$$\rho(\begin{bmatrix} z_1 A_{11} & z_1 A_{12} \\ z_2 A_{21} & z_2 A_{22} \end{bmatrix}) \leq \rho(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) < 1.$$

Since z_1 and z_2 are arbitrary complex number with modulus less or equal to one; the above inequality, in turn, implies condition (i) and the proof is complete. ■

Now, we are in position to state the main result of this section.

Corollary 3.1 *The following statements are equivalent:*

(i) 2-D system described by Roesser model (8) is positive and asymptotically stable.

(ii) 1-D system described by (9) is positive and asymptotically stable.

(iii) there exists a positive diagonal matrix $P \in R^{(n_1+n_2)\times(n_1+n_2)}$ satisfying

$$A^T PA - P < 0$$

where $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

(iv) The matrices $A_{11}, A_{12}, A_{21}, A_{22}$ are nonnegative and there exists $d \in R^{n_1+n_2}$ such that:

$$\begin{bmatrix} A_{11} - I_{n_1} & A_{12} \\ A_{21} & A_{22} - I_{n_2} \end{bmatrix} d < 0, \quad d > 0.$$

Proof: Recall that the equivalence $(ii) \Leftrightarrow (iii)$ results from Lemma 3.2 and then the proof will be complete if we only show $(i) \Leftrightarrow (iii)$.
$(i) \Rightarrow (iii)$ First, using Proposition 2.1 we have that $A_{11}, A_{12}, A_{21}, A_{22}$ are nonnegative. Next, since by Lemma 2.1 the asymptotic stability of the 2D system (8) is equivalent to

$$\begin{cases} \mathbf{det}(\begin{bmatrix} I_{n_1} - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_{n_2} - z_2 A_{22} \end{bmatrix}) \neq 0, \\ \forall (z_1, z_2) \in \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}, \end{cases}$$

which by Theorem 3.1 is also equivalent to $\rho(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) < 1$,. Finally by using Lemma 3.2 this implies (iii).
Reciprocally, to show that $(iii) \Rightarrow (i)$ it suffices to follow the same line of arguments by utilizing Proposition 2.1 combined with (in this order) Lemma 3.2, Theorem 3.1 and Lemma 2.1. ■

4 Controller synthesis

This section studies the stabilization problem of linear 2-D systems described by Roesser model for which the control law to be investigated has the state-feedback form

$$u(i,j) = K \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + v(t).$$

This control law is designed in order to ensure the positivity and the asymptotic stability of the resulting closed-loop system given in (5).
The following Lemmas will be useful in the subsequent development [12], [10].

Lemma 4.1 *Given the open-loop system (1) and the controller given by (4), the closed-loop system (5) is positive if and only if* $\bar{A} \in R_+^{(n_1+n_2)\times(n_1+n_2)}$, $B \in R_+^{(n_1+n_2)\times m}$, $\bar{C} \in R_+^{l\times(n_1+n_2)}$ *and* $D \in R_+^{l\times m}$.

Lemma 4.2 *Given the open-loop system (1) and the controller given by (4), assume that the closed-loop system in (5) is positive. Then it is asymptotically stable if and only if there exists a positive diagonal matrix* $P \in R^{(n_1+n_2)\times(n_1+n_2)}$ *satisfying*

$$\bar{A}^T P \bar{A} - P < 0 \tag{12}$$

In what follows we provide the main result of this section.

Theorem 4.1 *The closed-loop Roesser system (5) is positive and asymptotically stable for any initial boundary conditions, if and only if there exist a positive diagonal matrix* $Q \triangleq diag\{q_1,\ q_2, ...q(n_1+n_2)\}$ *and matrix* $\bar{K} \triangleq [\bar{k_{ij}}] \in R^{m\times(n_1+n_2)}$ *such that*

$$\begin{bmatrix} -Q & AQ + B\bar{K} \\ * & -Q \end{bmatrix} < 0 \tag{13}$$

$$a_{ij}q_j + \sum_{z=1}^{m} b_{iz}\bar{k}_{zj} \geq 0,\ \ 1 \leq i,\ j \leq n_1 + n_2 \tag{14}$$

$$c_{ij}q_j + \sum_{z=1}^{m} d_{iz}\bar{k}_{zj} \geq 0,\ \ 1 \leq i,\ j \leq n_1 + n_2 \tag{15}$$

with $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ *and* $D = [d_{ij}]$.
Under the above conditions, the matrix gain of desired controller (4) is given by

$$K = \bar{K}Q^{-1}. \tag{16}$$

Proof: $(Sufficiency)$ First from (16), we have $k_{zj} = \bar{k}_{zj}q_j^{-1}$. By noticing $q_j > 0$, (14) and (15) trivially ensure that $\bar{A} \in R_+^{(n_1+n_2)\times(n_1+n_2)}$ and $\bar{C} \in R_+^{l\times(n_1+n_2)}$. Then, by the positivity of $B \in R_+^{(n_1+n_2)\times m}$ and $D \in R_+^{l\times m}$, from Lemma 4.1 we know that the closed-loop system is positive.
Second, from (16), we have

$$\bar{K} = KQ. \tag{17}$$

By substituting (17) into (13), we obtain

$$\begin{bmatrix} -Q & AQ + BKQ \\ * & -Q \end{bmatrix} < 0 \tag{18}$$

By applying to (18) the congruence transformation defined by $diag\{Q^{-1}, Q^{-1}\}$ and keeping in mind (6), one obtains

$$\begin{bmatrix} -Q & Q^{-1}\bar{A} \\ * & -Q \end{bmatrix} < 0$$

By defining $P \triangleq Q^{-1}$, we readily obtain (12) via schur compliment equivalence [1]. Then, from Lemma , we know that the closed-loop system is asymptotically stable.
$(Necessity)$ Suppose there exists a controller of form given in (4) such that the closed-loop system given in (5) is asymptotically stable and positive. Then, from Lemmas and , we know that $\bar{A} \in R_+^{(n_1+n_2)\times(n_1+n_2)}$, $\bar{C} \in R_+^{l\times(n_1+n_2)}$, and there exists a positive diagonal matrix $P \triangleq diag\{p_1, p_2, ..., p_{(n_1+n_2)}\} \in R^{(n_1+n_2)\times(n_1+n_2)}$ satisfying (12). First, by Schur compliment, (12) is equivalent to

$$\begin{bmatrix} -P & P\bar{A} \\ * & -P \end{bmatrix} < 0 \tag{19}$$

By applying to (19) the congruence transformation defined by $diag\{P_{-1}, P_{-1}\}$ and keeping in mind (6), one obtains

$$\begin{bmatrix} -P^{-1} & AP^{-1} + BKP^{-1} \\ * & -P^{-1} \end{bmatrix} < 0$$

By defining

$$Q \triangleq P^{-1}, \;\; \bar{K} \triangleq KQ \qquad (20)$$

we readily obtain (13).
Second, $\bar{A} \in R_+^{(n_1+n_2)\times(n_1+n_2)}$ and $\bar{C} \in R_+^{l\times(n_1+n_2)}$ trivially imply (14) and (15), respectively, by noticing (20). ■

The importance of the above result is relevant, because it provides not only checkable necessary and sufficient condition but also a simple approach to address numerically the computation of the problem. Then, these conditions can be solved as a standard Linear Matrix Inequalities problem. In addition, based on the same formulation we can take into account the positiveness of the state feedback control law by just adding an additional constraint on the variables $\bar{k}_{ij}$. This is shown in the following result.

Theorem 4.2 *The following statements are equivalent:*
(i) There exists a positive state-feedback law $u(i,j) = K \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + v(t)$ *such that the closed-loop (5) is positive and asymptotically stable.*
(ii) There exists a matrix $K \in R^{m\times(n_1+n_2)}$ *such that* $K \geq 0$ *and* $\bar{A}$ *is nonnegative stable matrix.*
(iii) there exist a positive diagonal matrix $Q \triangleq diag\{q_1,\ q_2, ...q_{(n_1+n_2)}\}$ *and matrix* $\bar{K} \triangleq [\bar{k_{ij}}] \in R_+^{m\times(n_1+n_2)}$ *such that*

$$\begin{bmatrix} -Q & AQ + B\bar{K} \\ * & -Q \end{bmatrix} < 0 \qquad (21)$$

$$a_{ij}q_j + \sum_{z=1}^{m} b_{iz}\bar{k}_{zj}, \geq 0 \quad 1 \leq i,\ j \leq n_1 + n_2 \quad (22)$$

$$c_{ij}q_j + \sum_{z=1}^{m} d_{iz}\bar{k}_{zj}, \geq 0 \quad 1 \leq i,\ j \leq n_1 + n_2 \quad (23)$$

$$\bar{k}_{ij} \geq 0, \qquad i = 1,..,m, \;\; j = 1,...,n_1 + n_2 \quad (24)$$

with $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ *and* $D = [d_{ij}]$. *Moreover, the gain matrix K is given by:*

$$K = \bar{K}Q^{-1}.$$

Proof: By a simple convexity argument the proof is straightforward. ■

Now, some significant remarks are provided.

Remarks 4.1 *Note that if a negative state-feedback control law is to be considered it suffices to impose* $\bar{k}_{ij} \leq 0$ *instead of* $\bar{k}_{ij} \geq 0$ *in the previous LMI formulation.*

Remarks 4.2 *As the matrices* B *and* D *are invariant state-feedback law (4), their positivity is necessary for closed-loop system (5) to be positive. However, no such condition is imposed on* A *and* C *for open-loop system (1), which means that the open-loop system (1)is not necessarily positive. Therefore the controller is designed not only to stabilize the system, but also to render the closed-loop system positive*

5 Synthesis with uncertain plant

An important issue in the control design is *robust stability*, that is, ensuring stability under uncertainty or against possible perturbations. In this section, we consider robust stabilization of Roesser systems for which the dynamics are not exactly known and subject to uncertainties which are captured in a polytopic domain. Consider the following uncertain system:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A_\alpha \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + B_\alpha u(i,j), \qquad (25)$$

where the matrices $A_\alpha \in R^{(n_1+n_2)\times(n_1+n_2)}$ and $B_\alpha \in R^{(n_1+n_2)\times m}$ are supposed to be not exactly known but it is assumed that they belong to the following convex set:

$$\begin{bmatrix} A_\alpha & B_\alpha \end{bmatrix} \in \mathbf{D},$$

$$D := \Big\{ \sum_{p=1}^{r} \alpha_p \begin{bmatrix} A^p & B^p \end{bmatrix}, \sum_{p=1}^{r} \alpha_p = 1,\ \alpha_p \geq 0 \Big\},$$

where $[A^1 \;\; B^1], \ldots, [A^r \;\; B^r]$ are known matrices. Our robust synthesis design consists in finding a single constant gain matrix K for which the following closed-loop system is positive and asymptotically stable for every $[A_\alpha \ \ B_\alpha] \in \mathbf{D}$:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = (\bar{A_\alpha}) \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}. \qquad (26)$$

where $\bar{A_\alpha} = (A_\alpha + B_\alpha K)$ This kind of uncertainties in the model (5.1) can be directly handled by the proposed approach as is stated in the following result.

Theorem 5.1 *The closed-loop Roesser system is positive and asymptotically stable for any initial boundary conditions, if and only if there exist a positive diagonal matrix* $Q \triangleq diag\{q_1,\ q_2, ...q_{n_1+n_2}\}$ *and matrix* $\bar{K} \triangleq [\bar{k_{ij}}] \in R^{m\times(n_1+n_2)}$ *such that*

$$\begin{bmatrix} -Q & A^pQ + B^p\bar{K} \\ * & -Q \end{bmatrix} < 0 \qquad (27)$$

$$a^p_{ij}q_j + \sum_{z=1}^{m} b^p_{iz}\bar{k}_{zj} \geq 0,$$

$$1 \leq i,\ j \leq n_1 + n_2,\ p = 0, ..., r \qquad (28)$$

$$c^p_{ij}q_j + \sum_{z=1}^{m} d^p_{iz}\bar{k}_{zj} \geq 0$$

$$1 \leq i,\ j \leq n_1 + n_2,\ p = 0, ..., r \qquad (29)$$

with $A^p = [a^p_{ij}]$ *and* $B^p = [b^p_{ij}]$.
Under the above conditions, the matrix gain of desired controller (4) is given by

$$K = \bar{K}Q^{-1}. \qquad (30)$$

Proof: By a simple convexity argument the proof is straightforward. ■

6 Numerical Examples

Example 6.1 (Stability synthesis)
As an illustration of our stability synthesis design, we treat the following Roesser system (1) described by the matrices:

$$A = \left[\begin{array}{cc|c} -1.2 & 0.1 & 0 \\ 0.2 & 0.5 & 0.3 \\ \hline 1 & 1 & 0.2 \end{array}\right],\ B = \left[\begin{array}{c} 1 \\ 1 \\ \hline 0 \end{array}\right].,$$

$$C = \left[\begin{array}{cc|c} -0.1 & 0.3 & 0.2 \end{array}\right],\ D = 0.1.$$

Of course, as the matrices A and C are not nonnegative, the open-loop system (1) is not positive. This fact is also illustrated by the open-loop response of x_1^h *depicted in Figure1 (starting from initial nonnegative boundary). According to the result given in [4], the system in open-loop is also unstable, i.e the spectral radius* $\rho(A_{11} + A_{12}(zI_{n_2} - A_{22})^{-1}A_{21})) > 1$*, where* $z = e^{j\omega}$*, for all* $\omega \in [-\pi, \pi]$ *(see Figure1 and Figure2). Our purpose is to design a state-feedback controller on the form (4) such that the closed-loop system is asymptotically stable and positive. By applying Theorem 4.1, we obtain the following matrix variables:*

$$Q = diag\{0.8866, 13.4709, 34.1739\}$$

$$\bar{K} = [1.2097 \quad -1.1171 \ \ 0.4538]$$

Then, according to (16), the feedback gain matrix K of the controller (4) is given by:

$$K = [1.3643 \quad -0.0829 \ \ 0.0133] \qquad (31)$$

so the matrices of closed-loop system (5) are given by

$$\bar{A} = \left[\begin{array}{cc|c} 0.1643 & 0.0171 & 0.0133 \\ 1.5643 & 0.4171 & 0.3133 \\ \hline 1.0000 & 1.0000 & 0.2000 \end{array}\right],\ \bar{B} = \left[\begin{array}{c} 1 \\ 1 \\ \hline 0 \end{array}\right].,$$

$$\bar{C} = \left[\begin{array}{cc|c} 0.0364 & 0.2917 & 0.2013 \end{array}\right].,\ \bar{D} = 0.1.$$

Hence, it suffices to look at the entries of the matrices $\bar{A}, \bar{B}, \bar{C}, \bar{D}$*, to conclude that the closed-loop system is positive (according to proposition 2.1). In addition, according to Corollary 3.1, the closed-loop system is asymptotically stable (also, it can be checked that the matrix* $\bar{A}$ *has all eigenvalues inside the unit circle, namely* $\lambda_1 = 0.9357$*,* $\lambda_2 = 0.0961$ *and* $\lambda_3 = -0.2504$*. The closed-loop responses of the closed-loop system for* x_1^h *is shown in Figure 3 (The other state responses are similar, and hence, omitted). Thus, the simulation results shows that the closed-loop is positive and asymptotically stable.*

Example 6.2 (Positive feedback)
Let us consider the following non-positive Roesser system (1) described by:

$$A = \left[\begin{array}{c|c} 0.2 & 0.7 \\ \hline 0.9 & -0.8 \end{array}\right], B = \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right].$$

Note that the system in open-loop is unstable, since the spectral radius $\rho(A_{11} + A_{12}(zI_{n_2} - A_{22})^{-1}A_{21})) > 1$ *where* $z = e^{j\omega}$, *for all* $\omega \in [-\pi, \pi]$ *(see Figure 4). Here, our task is to utilize a positive state-feedback controller of the following form:*

$$u(i, j) = K \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} \quad (32)$$

such that the closed-loop system

$$\begin{bmatrix} x^h(i + 1, j) \\ x^v(i, j + 1) \end{bmatrix} = \bar{A} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} \quad (33)$$

where $\bar{A} = A + BK$
in order to stabilize the system and enforce the state to be positive. By applying Theorem 4.2 we obtain the following matrix variables:

$$Q = diag\{4.1364, 5.9289\}$$

$$\bar{K} = [0.4338 \ \ 5.1540]$$

Then, according to (16), the feedback gain matrix K *of the controller (4) is given by:*

$$K = [0.1049 \ \ 0.8693] \quad (34)$$

so the matrix of closed-loop system (31) is given by

$$\bar{A} = \left[\begin{array}{c|c} 0.2000 & 0.7000 \\ \hline 1.0049 & 0.0693 \end{array}\right].$$

It can be seen that the closed-loop response of the closed-loop system for x_1^h *is convergent see Figure 5 and asymptotically stable, (The other state responses are similar, and hence, omitted). It can be also checked that matrix* $\bar{A}$ *has all eigenvalues inside the unit circle, namely* $\lambda_1 = 0.9759$, *and* $\lambda_2 = -0.7066$ *.From this it follows that the system is asymptotically stable according to condition 3 of Theorem 3-1, by simple looking at the entries of dynamic matrices* $\bar{A}$ *and* $\bar{B}$, *we can conclude that the governed system is positive according to Proposition 3.1.*

Example 6.3 *(***Uncertain plan)**
In this example, we consider an uncertain Roesser system (25) subject to a parametric perturbation as follows:

$$A_\alpha = \left[\begin{array}{cc|c} -1.2 & 0.1 & 0 \\ 0.2 & 0.5 & 0.3 \\ \hline 1 & 1 & 0.2 - 0.01\alpha \end{array}\right],$$

$$B_\alpha = \left[\begin{array}{c} 1 \\ 1 - 0.01\alpha \\ \hline 0 \end{array}\right],$$

where $0 \leq \alpha \leq 1$.

We are looking for a robust state-feedback control which stabilizes and enforces the positivity of all the plants between the two extreme plants ($\alpha = 0$ *and* $\alpha = 1$*). By applying Theorem 5.1, we obtain the following matrix variables:*

$$Q = diag\{17.2102, 259.0649, 654.3670\}$$

$$\bar{K} = [23.1671 \ \ -21.0130 \ \ 9.9411]$$

Then, according to (16), the feedback gain matrix K *of the controller (4) is given by:*

$$K = [1.3461 \ \ -0.0811 \ \ 0.0152] \quad (35)$$

Hence, with this gain all the closed-loop systems between the two extreme plants ($\alpha = 0$ *and* $\alpha = 1$*) are positive and asymptotically stable. The state responses of the closed-loop system for* x_2^h *for the two extreme plants (*$\alpha = 0$ *and* $\alpha = 1$*), starting from initial positive boundaries, are depicted in Figure 6 and Figure 7 (The other state responses are similar, and hence, omitted).*

7 Conclusion

The control problem for stability and positivity is treated in this paper for 2D linear discrete-time systems described by the Roesser model. Necessary and sufficient conditions are derived in terms of LMIs for the existence of desired controllers guaranteeing the closed-loop system to be asymptotically stable and positive. Also, it has been shown how our method can take into account the positivity of the control and also the uncertainties in the model. Numerical examples are provided to illustrate the proposed results.

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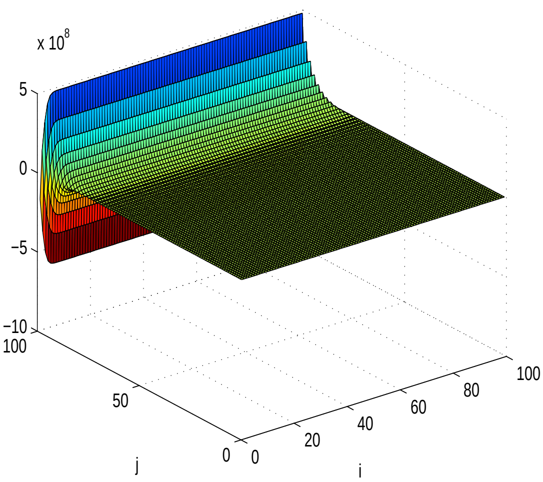

Figure 1. Open-loop response of $x_1^h(i,j)$

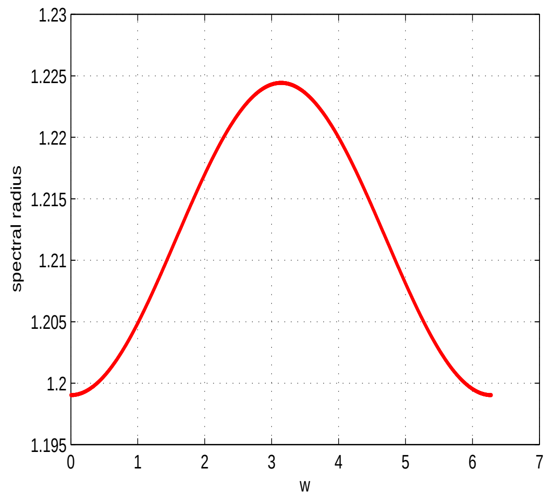

Figure 2. The spectral radius in open-loop system

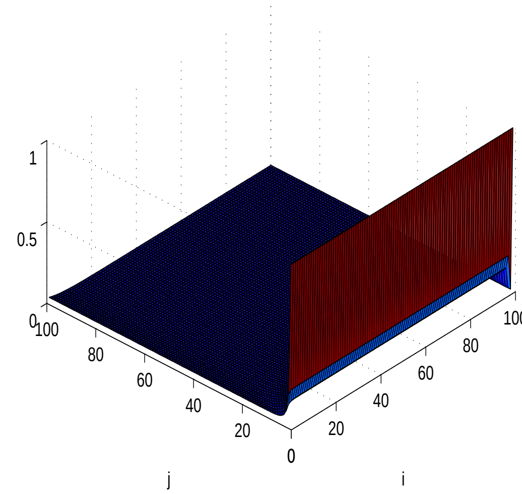

Figure 3. Closed-loop response of $x_1^h(i,j)$

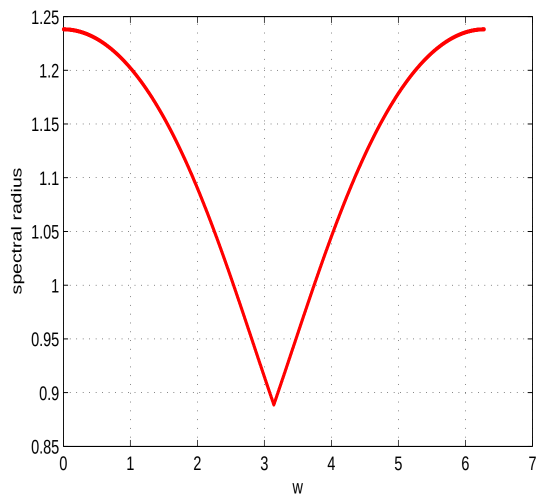

Figure 4. The spectral radius in open-loop system

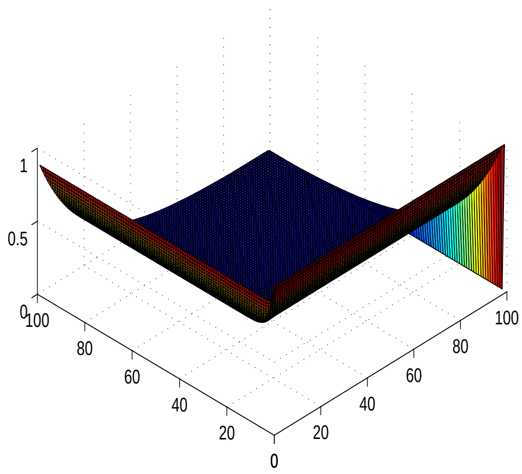

Figure 5. Closed-loop response of $x_1^h(i,j)$

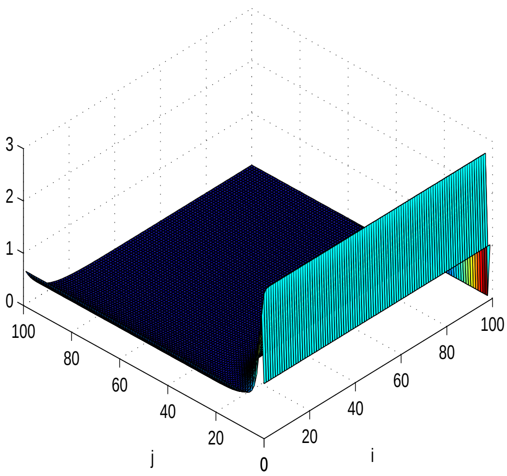

Figure 6. Closed-loop response of $x_2^h(i,j) for \alpha = 0$

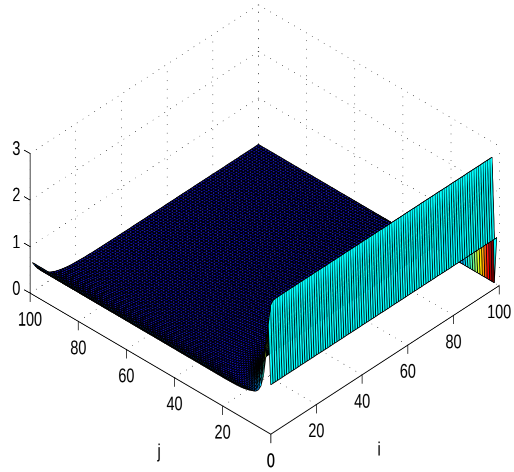

Figure 7. Closed-loop response of $x_2^h(i,j) for \alpha = 1$