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Optimal Control and Filtering of Weakly Coupled Linear Discrete Stochastic Systems by The Eigenvector Approach

Naser Prljaca [1], Zoran Gajic [2]
1) Faculty of Electrical Engineering, University of Tuzla, Franjevacka 2, Tuzla 75000, Bosnia and Herzegovina, e-mail: naser.prljaca@untz.ba , www.untz.ba
2) Department of Electrical and Computer Engineering, Rutgers University, 94 Brett Road, Piscataway 08854-8058, NJ, USA e-mail: gajic@ece.rutgers.edu, www.ece.rutgers.edu

Abstract:-In this paper the regulator and filter algebraic Riccati equations, corresponding to the steady state optimal control and filtering of weakly coupled linear discrete stochastic systems are solved in terms of reduced-order sub problems by using the eigenvector approach. The eigenvector method outperforms iterative methods (fixed point iterations, Newton method) of solutions to reduced-order sub problems in case of higher level of coupling between subsystems. In such cases the iterative methods could fail to produce solutions of the corresponding algebraic Riccati equations.

Key Words:- Optimal control, Linear discrete stochastic systems, Weakly coupled systems, Order reduction, Eigenvector method

1 Introduction

The work in this paper is influenced by the work done in the theory of weakly coupled systems. The theory of weakly coupled control systems has attracted a lot of attention in the control literature [1], [2], [3], [4]. In [3], a transformation was introduced for decomposition of the weakly coupled algebraic Riccati equation, which is based on the closed-loop decomposition technique. The algebraic equations comprising the transformation have the form of general non symmetric nonsquare Riccati equations. These equations can be efficiently solved by iterative methods (fixed point iterations, Newton method) for a small value of coupling between subsystems [2]. For a larger value of coupling between subsystems, iterative methods might diverge and the desired transformation could not be found. In [5], the transformation was used in order to decompose corresponding algebraic Riccati equations of the optimal regulator and Kalman filter of weakly coupled linear discrete-time stochastic systems. The eigenvector approach to the solution of optimal control of continues-time singularly perturbed and weakly coupled systems was introduced in [10], [11]. This work extends applicability of the eigenvector method to the problem of optimal control and filtering of weakly coupled linear discrete-time stochastic systems.

2 Decomposition of the linear-quadratic control problem

Consider a linear time-invariant discrete-time system

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

with the quadratic performance criterion

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left[x(k)^T Qx(k) + u(k)^T Ru(k)\right] \tag{2}$$

The weakly coupled structure of (1) and (2) implies the following partitions

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 & \varepsilon A_2 \\ \varepsilon A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & \varepsilon B_2 \\ \varepsilon B_3 & B_4 \end{bmatrix} \tag{3}$$

$$Q = \begin{bmatrix} Q_1 & \varepsilon Q_2 \\ \varepsilon Q_2^T & Q_3 \end{bmatrix}, \quad R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

$$S = BR^{-1}B^T = \begin{bmatrix} S_1 & \varepsilon S_2 \\ \varepsilon S_2^T & S_3 \end{bmatrix}$$

where x_1, x_2 are vectors of subsystem state variables of appropriate dimensions (n_1, n_2), u_1, u_2 are vectors of control inputs (m_1, m_2), and ε is a small coupling parameter. A, B are system constant matrices, Q and R are constant weighting matrices. In addition, it is assumed that A_1 and A_4 are nonsingular. The well known solution to the above optimal control problem is given by

$$u(k) = -R^{-1}B^T\lambda(k+1) = -(R + B^T P_r B)^{-1} B^T P_r Ax(k)$$
$$u(k) = -Fx(k) \tag{4}$$

where $\lambda(k)$ is a vector of costate variables and P_r is the positive-semidefinite stabilizing solution of the discrete Riccati equation given by

$$P_r = Q + A^T P_r A - A^T P_r B(R + B^T P_r B)^{-1} B^T P_r A \tag{5}$$

The solution to this equation exists under the standard stabilizability-detectibility assumption imposed on the triple (A,B,Q).
The Hamiltonian form of the optimal control problem is given by [9]

$$\begin{bmatrix} x(k+1) \\ \lambda(k+1) \end{bmatrix} = H \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \tag{6}$$

where

$$H = \begin{bmatrix} A + BR^{-1}B^T A^{-T} Q & -BR^{-1}B^T A^{-T} \\ -A^{-T}Q & A^{-T} \end{bmatrix} \tag{7}$$

Hamiltonian form represents the closed-loop solution to the optimal control problem, where $\lambda(k) = P_r x(k)$.
Partitioning the state vector x and the corresponding costate vector λ and interchanging second and third rows, the Hamiltonin form can be written as [3]

$$\begin{bmatrix} x_1(k+1) \\ \lambda_1(k+1) \\ x_2(k+1) \\ \lambda_2(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_{1r} & \bar{S}_{1r} & \varepsilon\bar{A}_{2r} & \varepsilon\bar{S}_{2r} \\ \bar{Q}_{1r} & \bar{A}_{11r}^T & \varepsilon\bar{Q}_{2r} & \varepsilon\bar{A}_{21r}^T \\ \varepsilon\bar{A}_{3r} & \varepsilon\bar{S}_{3r} & \bar{A}_{4r} & \bar{S}_{4r} \\ \varepsilon\bar{Q}_{3r} & \varepsilon\bar{A}_{12r}^T & \bar{Q}_{4r} & \bar{A}_{22r}^T \end{bmatrix} \begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix} \tag{8}$$

or

$$\begin{bmatrix} U(k+1) \\ V(k+1) \end{bmatrix} = \begin{bmatrix} T_{1r} & \varepsilon T_{2r} \\ \varepsilon T_{3r} & T_{4r} \end{bmatrix} \begin{bmatrix} U(k) \\ V(k) \end{bmatrix} \tag{9}$$

with obvious meanings of vectors $U(k)$, $V(k)$ and matrices T_{1r}, T_{2r}, T_{3r}, T_{4r}.
The system (9) can be block diagonalized by the means of the following nonsingular similarity transformation [3]

$$\begin{bmatrix} \eta(k+1) \\ \zeta(k+1 \end{bmatrix} = \begin{bmatrix} I & -\varepsilon H \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \varepsilon L & I \end{bmatrix} \begin{bmatrix} T_{1r} & \varepsilon T_{2r} \\ \varepsilon T_{3r} & T_{4r} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\varepsilon L & I \end{bmatrix} \begin{bmatrix} I & \varepsilon H \\ 0 & I \end{bmatrix} \begin{bmatrix} \eta(k) \\ \xi(k) \end{bmatrix} \tag{10}$$

or

$$\begin{bmatrix} \eta(k+1) \\ \xi(k+1) \end{bmatrix} = \begin{bmatrix} I - \varepsilon^2 H_r L_r & -\varepsilon H_r \\ \varepsilon L_r & I \end{bmatrix} \begin{bmatrix} T_{1r} & \varepsilon T_{2r} \\ \varepsilon T_{3r} & T_{4r} \end{bmatrix} \begin{bmatrix} I & \varepsilon H_r \\ -\varepsilon L_r & I - \varepsilon^2 L_r H_r \end{bmatrix} \begin{bmatrix} \eta(k) \\ \xi(k) \end{bmatrix} \tag{11}$$

and the relationship between old and new coordinates is then given by

$$\begin{bmatrix} U(k) \\ V(k) \end{bmatrix} = \begin{bmatrix} I - \varepsilon^2 H_r L_r & -\varepsilon H_r \\ \varepsilon L_r & I \end{bmatrix} \begin{bmatrix} \eta(k) \\ \xi(k) \end{bmatrix} = T_r \begin{bmatrix} \eta(k) \\ \xi(k) \end{bmatrix} \tag{12}$$

The transformation leads to two completely decoupled subsystems

$$\eta(k+1) = (T_{1r} - \varepsilon^2 T_{2r} L_r)\eta(k)$$
$$\xi(k+1) = (T_{4r} + \varepsilon^2 L_r T_{2r})\xi(k) \tag{13}$$

where

$$\eta(k) = \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix}, \quad \xi(k) = \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} \tag{14}$$

and L_r and H_r satisfying

$$L_r T_{1r} - T_{4r} L_r + T_{3r} - \varepsilon^2 L_r T_{2r} L_r = 0$$
$$H_r(T_{4r} + \varepsilon^2 L_r T_{2r}) - (T_{1r} - \varepsilon^2 T_{2r} L_r)H_r - T_{2r} = 0 \tag{15}$$

The first equation has a form of the asymmetric nonsquare Riccati equation, while the second is a Sylvester type linear equation. The solution of the above equations will be discussed later in the paper.
The rearrangement of variables in (8) is done by the means of a similarity transformation E of the form

$$\begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix} = \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & I_{n2} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = E \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} \tag{16}$$

The relationship between original and new coordinates is given by

$$\begin{bmatrix} \eta_1 \\ \xi_1 \\ \eta_2 \\ \xi_2 \end{bmatrix} = E^T T_r E \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = \Pi_r \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = \begin{bmatrix} \Pi_{1r} & \Pi_{2r} \\ \Pi_{3r} & \Pi_{4r} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} \tag{17}$$

Since $\lambda = P_r x$, where P_r satisfies the discrete algebraic Riccati equations (5), it follows

$$\begin{bmatrix} \eta_1(k) \\ \zeta_1(k) \end{bmatrix} = (\Pi_{1r} + \Pi_{2r} P_r) \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$
$$\begin{bmatrix} \eta_2(k) \\ \zeta_2(k) \end{bmatrix} = (\Pi_{3r} + \Pi_{4r} P_r) \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{18}$$

The decupled subsystems (13) also represent the closed-loop solution of the optimal control problem in the new coordinates. Based on this fact the equations (13) can be written as

$$\begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1) \end{bmatrix} = \begin{bmatrix} a_{1r} & a_{2r} \\ a_{3r} & a_{4r} \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix}$$

$$\begin{bmatrix} \xi_1(k+1) \\ \xi_2(k+1) \end{bmatrix} = \begin{bmatrix} b_{1r} & b_{2r} \\ b_{3r} & b_{4r} \end{bmatrix} \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} \tag{19}$$

where

$$\eta_2(k) = P_{ra}\eta_1(k), \quad \xi_2(k) = P_{rb}\xi_1(k)$$

or

$$\begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} = \begin{bmatrix} P_{ra} & 0 \\ 0 & P_{rb} \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} \tag{20}$$

and P_{ra} and P_{rb} satisfy nonsymetric Riccati equations of the form

$$\begin{aligned} P_{ra}a_{1r} - a_{4r}P_{ra} - a_{3r} + P_{ra}a_{2r}P_{ra} &= 0 \\ P_{rb}b_{1r} - b_{4r}P_{rb} - b_{3r} + P_{rb}b_{2r}P_{rb} &= 0 \end{aligned} \tag{21}$$

leading to

$$\begin{aligned} \eta_1(k+1) &= (a_{1r} + a_{2r}P_{ra})\eta_1(k) \\ \xi_1(k+1) &= (b_{1r} + b_{2r}P_{rb})\xi_1(k) \end{aligned} \tag{22}$$

It follows from (18) and (20)

$$\begin{bmatrix} P_{ra} & 0 \\ 0 & P_{rb} \end{bmatrix} = (\Pi_3 + \Pi_4 P_r)(\Pi_1 + \Pi_2 P_r)^{-1} \tag{23}$$

This equation can be solved for P_r giving

$$P_r = \left(\begin{bmatrix} P_{ra} & 0 \\ 0 & P_{rb} \end{bmatrix} \Pi_2 - \Pi_4 \right)^{-1} \left(\Pi_3 - \begin{bmatrix} P_{ra} & 0 \\ 0 & P_{rb} \end{bmatrix} \Pi_1 \right) \tag{24}$$

which gives the solution of the global discrete Riccati equation (5) in terms of reduced order continues time nonsymmetric Riccati equations (21) and decupling transformation matrix (12). The matrix inversion in (24) is guaranteed for sufficiently small ε [3]. In order to realize the above presented decomposition procedure, it is necessary to solve continues-time nonsquare and nonsymmetric Riccati equations (15) and (21). The solution of equations (15) and (21) will be discussed in the section 5 of the paper.

3 Decomposition of the optimal filtering problem

Let the linear discrete-time invariant stochastic weakly coupled system be given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_1 & \varepsilon A_2 \\ \varepsilon A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} G_1 & \varepsilon G_2 \\ \varepsilon G_3 & G_4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{25}$$

with corresponding measurements

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C_1 & \varepsilon C_2 \\ \varepsilon C_3 & C_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix} \tag{26}$$

where x_i are state vectors, w_i and v_i are independent zero-mean white Gaussian processes with intensities W and V, and y_i are system measurements. A_i, G_i, C_i are constant system matrices ($i = 1, 2$). The well known optimal Kalman filter is is given by

$$\hat{x}(k+1) = A\hat{x}(k) + K(y(k) - C\hat{x}(k)) \tag{27}$$

or in the closed-loop form as

$$\hat{x}(k+1) = (A - KC)\hat{x}(k) + Ky(k) \quad (28)$$

where

$$A = \begin{bmatrix} A_1 & \varepsilon A_2 \\ \varepsilon A_3 & A_4 \end{bmatrix}, C = \begin{bmatrix} C_1 & \varepsilon C_2 \\ \varepsilon C_3 & C_4 \end{bmatrix}, K = \begin{bmatrix} K_1 & \varepsilon K_2 \\ \varepsilon K_3 & K_4 \end{bmatrix} \quad (29)$$

The Kalman gain is given by

$$K = AP_f C^T (V + CP_f C^T)^{-1}, \quad V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \quad (30)$$

where P_f is the positive-semidefinite stabilizing solution of the discrete-time algebraic Riccati equation given by

$$P_f = AP_f A^T - AP_f C^T (V + CP_f C^T)^{-1} CP_f A^T + GWG^T \quad (31)$$

with

$$G = \begin{bmatrix} G_1 & \varepsilon G_2 \\ \varepsilon G_3 & G_4 \end{bmatrix}, \quad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \quad (32)$$

Using the decomposition procedure given in the previous section and the duality property between the optimal regulator and optimal filter, will result in the decomposition of the global filter to the completely decupled reduced order subsystem filters both driven by system measurements.

By duality between the optimal filter and regulator, the filter Riccati equation (31) can be solved by using the same decomposition method presented in the previous section with

$$A \to A^T, Q \to GWG^T, B \to C^T, BR^{-1}B^T \to C^T V^{-1} C \quad (33)$$

which leads to the Hamiltonian state-costate filter closed-loop form. Partitioning the state vector x and the corresponding costate vector λ and interchanging second and third rows, the Hamiltonian form can be written as

$$\begin{bmatrix} x_1(k+1) \\ \lambda_1(k+1) \\ x_2(k+1) \\ \lambda_2(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_{1f} & \bar{S}_{1f} & \varepsilon \bar{A}_{2f} & \varepsilon \bar{S}_{2f} \\ \bar{Q}_{1f} & \bar{A}_{11f}^T & \varepsilon \bar{Q}_{2f} & \varepsilon \bar{A}_{21f}^T \\ \varepsilon \bar{A}_{3f} & \varepsilon \bar{S}_{3f} & \bar{A}_{4f} & \bar{S}_{4f} \\ \varepsilon \bar{Q}_{3f} & \varepsilon \bar{A}_{12f}^T & \bar{Q}_{4f} & \bar{A}_{22f}^T \end{bmatrix} \begin{bmatrix} x_1(k) \\ \lambda_1(k) \\ x_2(k) \\ \lambda_2(k) \end{bmatrix} \quad (34)$$

$$\begin{bmatrix} U(k+1) \\ V(k+1) \end{bmatrix} = \begin{bmatrix} T_{1f} & \varepsilon T_{2f} \\ \varepsilon T_{3f} & T_{4f} \end{bmatrix} \begin{bmatrix} U(k) \\ V(k) \end{bmatrix} \quad (35)$$

As it was shown in the previous section , this system can be diagonalized by the means of the similarity transformation given by

$$\begin{bmatrix} U(k) \\ V(k) \end{bmatrix} = \begin{bmatrix} I - \varepsilon^2 H_f L_f & -\varepsilon H_f \\ \varepsilon L_f & I \end{bmatrix} \begin{bmatrix} \eta(k) \\ \xi(k) \end{bmatrix} = T_f \begin{bmatrix} \eta(k) \\ \xi(k) \end{bmatrix} \quad (36)$$

and L_f and H_f satisfying

$$\begin{aligned} & L_f T_{1f} - T_{4f} L_f + T_{3f} - \varepsilon^2 L_f T_{2f} L_f = 0 \\ & H_f (T_{4f} + \varepsilon^2 L_f T_{2f}) - (T_{1f} - \varepsilon^2 T_{2f} L_f) H_f - T_{2f} = 0 \end{aligned} \quad (37)$$

The transformation leads to two decoupled sub systems

$$\begin{aligned} \eta(k+1) &= \begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1) \end{bmatrix} = (T_{1f} - \varepsilon^2 T_{2f} L_f) = \begin{bmatrix} a_{1f} & a_{2f} \\ a_{3f} & a_{4f} \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} \\ \xi(k+1) &= \begin{bmatrix} \xi_1(k+1) \\ \xi_2(k+1) \end{bmatrix} = (T_{4f} + \varepsilon^2 L_f T_{2f}) = \begin{bmatrix} b_{1f} & b_{2f} \\ b_{3f} & b_{4f} \end{bmatrix} \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} \end{aligned} \quad (38)$$

where

$$\eta_2(k) = P_{fa} \eta_1(k), \quad \xi_2(k) = P_{fb} \xi_1(k)$$

or

$$\begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} = \begin{bmatrix} P_{fa} & 0 \\ 0 & P_{fb} \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} \quad (39)$$

and P_{fa} and P_{fb} satisfy nonsymetric Riccati equations of the form

$$\begin{aligned} & P_{fa} a_{1f} - a_{4f} P_{fa} - a_{3f} + P_{fa} a_{2f} P_{fa} = 0 \\ & P_{fb} b_{1f} - b_{4f} P_{fb} - b_{3f} + P_{fb} b_{2f} P_{fb} = 0 \end{aligned} \quad (40)$$

leading to

$$\begin{aligned} & \eta_1(k+1) = (a_{1f} + a_{2f} P_{fa}) \eta_1(k) \\ & \xi_1(k+1) = (b_{1f} + b_{2f} P_{fb}) \xi_1(k) \end{aligned} \quad (41)$$

The overall transformation between the new and original coordinates is given by

$$\begin{bmatrix} \eta_1 \\ \xi_1 \\ \eta_2 \\ \xi_2 \end{bmatrix} = E^T T_f E \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = \Pi_f \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = \begin{bmatrix} \Pi_{1f} & \Pi_{2f} \\ \Pi_{3f} & \Pi_{4f} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} \tag{42}$$

Since $\lambda = P_f\, x$, where P_f satisfies the discrete algebraic Riccati equations (31), it follows

$$\begin{bmatrix} \eta_1(k) \\ \zeta_1(k) \end{bmatrix} = (\Pi_{1f} + \Pi_{2f} P_f) \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \Omega \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$\begin{bmatrix} \eta_2(k) \\ \zeta_2(k) \end{bmatrix} = (\Pi_{3f} P_f + \Pi_{4f}) \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{43}$$

It follows from (43) and (39)

$$\begin{bmatrix} P_{fa} & 0 \\ 0 & P_{fb} \end{bmatrix} = (\Pi_{3f} + \Pi_{4f} P_f)(\Pi_{1f} + \Pi_{2f} P_f)^{-1} \tag{44}$$

This equation can be solved for P_f giving

$$P_f = \left(\begin{bmatrix} P_{fa} & 0 \\ 0 & P_{fb} \end{bmatrix} \Pi_{2f} - \Pi_{4f} \right)^{-1} \left(\Pi_{3f} - \begin{bmatrix} P_{fa} & 0 \\ 0 & P_{fb} \end{bmatrix} \Pi_{1f} \right) \tag{45}$$

which gives the solution of the filter global discrete Riccati equation (31).
Applying the transformation Ω (43) to the Kalman filter equation (28) leads to

$$\begin{bmatrix} \hat{\eta}_1(k+1) \\ \hat{\xi}_1(k+1) \end{bmatrix} = \Omega^{-T} (A - KC) \Omega^T \begin{bmatrix} \hat{\eta}_1(k) \\ \hat{\xi}_1(k) \end{bmatrix} + \Omega^{-T} K y(k) \tag{46}$$

or

$$\begin{aligned} \hat{\eta}_1(k+1) &= (a_{1f} + a_{2f} P_{fa})^T \hat{\eta}_1(k) + K_1 y(k) \\ \hat{\xi}_1(k+1) &= (b_{1f} + a_{2f} P_{fb})^T \hat{\eta}_1(k) + K_2 y(k) \end{aligned} \tag{47}$$

which completely decomposes the global Kalman filter into two reduced order subfilters, that can be implemented independently. Again, as it was the case in the previous section, in order to realize the above presented decomposition procedure it is necessary to solve continues-time nonsquare and nonsymetric Riccati equations (37) and (40).

4 LQG control problem

The well known linear quadratic Gaussian control problem is defined as follows. Given the linear discrete-time stochastic system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_1 & \varepsilon A_2 \\ \varepsilon A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 & \varepsilon B_2 \\ \varepsilon B_3 & B_4 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} + \begin{bmatrix} G_1 & \varepsilon G_2 \\ \varepsilon G_3 & G_4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C_1 & \varepsilon C_2 \\ \varepsilon C_3 & C_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} v_1(k) \\ v_2(k) \end{bmatrix} \tag{48}$$

with performance criterion

$$J = \frac{1}{2} \mathrm{E} \left\{ \sum_{k=0}^{\infty} x(k)^T Q x(k) + u(k)^T R u(k) \right\} \tag{49}$$

Find the control low which minimizes the criterion. The optimal control law is given by [8]

$$u(k) = -F\hat{x}(k) \tag{50}$$

where F is found according to the section 2.

with the optimal filter

$$\hat{x}(k+1) = (A - KC)\hat{x}(k) + Ky(k) + Bu(k) \tag{51}$$

which is decomposed into reduced order filters according to the section 3 as

$$\begin{aligned} \hat{\eta}_1(k+1) &= (a_{1f} + a_{2f} P_{fa})^T \hat{\eta}_1(k) + K_1 y(k) + \Phi_1 u(k) \\ \hat{\xi}_1(k+1) &= (b_{1f} + a_{2f} P_{fb})^T \hat{\eta}_1(k) + K_2 y(k) + \Phi_2 u(k) \end{aligned} \tag{52}$$

where

$$\begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} = \Omega^T \begin{bmatrix} \hat{\eta}_1(k) \\ \hat{\xi}_1(k) \end{bmatrix}, \quad \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \Omega^{-T} B \tag{53}$$

5 The eigenvector solution to nonsymmetric algebraic Riccati equation

The eigenvector method for solving the algebraic symmetric and square, nonsymmetric and nonsquare Riccati equations has received considerable attention in the literature [8], [9].

Without loss of generality, let us consider the algebraic square and nonsymmetric Riccati equation (ARE) given by

$$AX + XB + C + XDX = 0 \tag{54}$$

where matrices *A, B, C, D* are of appropriate dimensions (*n x n*) and *X* is the sought solution of dimension (*n x n*).

Let the matrix *R* be associated with the ARE

$$R = \begin{bmatrix} B & D \\ -C & -A \end{bmatrix} \tag{55}$$

The matrix *R* can be diagonalized by the matrix *M* consisting of eigenvectors of the matrix *R* as follows. Calculate all *2n* eigenvalus of *R*, $\lambda_i = a_i + jb_i$ and all corresponding eigenvectors $v_i = x_i + jy_i$. Arrange in the ($2n \times 2n$) matrix *M* all real eigenvectors (x_i) and for each complex-conjugate pair use consecutively the real and imaginary parts of one eigenvector only (x_i, y_i). There are many ways to form matrix *M*.

Then, it follows

$$M^{-1}RM = \Lambda, \quad RM = M\Lambda = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \tag{56}$$

where M_1 contains the first *n* columns and M_2 contains the remaining *n* columns of *M*. Λ_1 and Λ_2 are diagonal or block diagonal matrices.

The equation (59) may be rewritten as

$$RM_1 = M_1\Lambda_1, \quad RM_2 = M_2\Lambda_2 \tag{57}$$

By partitioning M_1 as

$$M_1 = \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \tag{58}$$

we get from (57)

$$BM_{11} + DM_{21} = M_{11}\Lambda_1, \quad -CM_{11} - AM_{21} = M_{21}\Lambda_1 \tag{59}$$

Rearranging the last two equations and using the substitution

$$X = M_{21}M_{11}^{-1} \tag{60}$$

leads to

$$AX + XB + C + XDX = 0 \tag{61}$$

which proves that *X* is a solution to (54). Since the matrix *M* can be formed in many ways. It follows that all solutions to (54) have the form

$$X_k = M_{k21}M_{k22}^{-1} \tag{62}$$

Let the spectrum of *R* be $S = \{\lambda_1, \ldots \lambda_{2n}\}$ or $S = S_1 \cup S_2$, where $S_1 = \{\lambda_1, \ldots, \lambda_n\}$ and $S_2 = \{\lambda_{n+1}, \ldots, \lambda_{2n}\}$. If the corresponding eigenvalues of eigenvectors used to form M_1 are $S_1 = \{\lambda_1, \ldots, \lambda_n\}$ and to form M_2 are $S_2 = \{\lambda_{n+1}, \ldots, \lambda_{2n}\}$, then eigenvalues of ($B+DX$) are S_1 and eigenvalues of $-(A+XD)$ are S_2 [9]. This is easily justified by transforming the matrix *R* as follows

$$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} B & D \\ -C & -A \end{bmatrix} \begin{bmatrix} I & 0 \\ X & 0 \end{bmatrix} = \begin{bmatrix} B+DX & D \\ 0 & -(A+XD) \end{bmatrix} \tag{63}$$

Further, the matrix *R* can be put in the block diagonal form by using another transformation matrix

$$\begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} \begin{bmatrix} B+DX & D \\ 0 & -(A+XD) \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} B+DX & 0 \\ 0 & -(A+XD) \end{bmatrix} \tag{64}$$

where Y satisfies the Sylvester equation

$$(B+DX)Y + Y(A+XD) + D = 0 \tag{65}$$

6 Numerical example

Consider the system with problem matrices given by ($\varepsilon = 1$)

$$A = \begin{bmatrix} 0.8674 & -0.3024 & 0.4092 & 0.2066 \\ -0.9509 & -0.2256 & 0.3904 & 0.0966 \\ 0.9218 & 0.5582 & -0.3639 & -0.3696 \\ -0.3360 & -0.1248 & 0.1511 & 0.3564 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0190 & 0.0030 \\ 0.1800 & 0.0578 \\ 0.0152 & 0.0190 \\ -0.1641 & 0.1810 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$Q = 0.1I_4, \quad R = I_2, \quad W = I_2, \quad V = I_2$$

The obtained solutions for LQG problem according to the presented methodology (note that iterative methods in this case do not converge) are summarized as follows

$$P_r = \begin{bmatrix} 391.0855 & -50.8930 & 96.3664 & 49.4185 \\ -50.8930 & 8.3975 & -14.4404 & -7.3477 \\ 96.3664 & -14.4404 & 26.0110 & 13.2175 \\ 49.4185 & -7.3477 & 13.2175 & 6.8589 \end{bmatrix}$$

$$P_f = \begin{bmatrix} 1.7500 & -0.9361 & 0.7142 & -0.4013 \\ -0.9361 & 0.6377 & -0.4928 & 0.2313 \\ 0.7142 & -0.4928 & 0.4316 & -0.2111 \\ -0.4013 & 0.2313 & -0.2111 & 0.1718 \end{bmatrix}$$

$$\hat{\eta}(k+1) = \begin{bmatrix} -0.1339 & -1.0531 \\ -0.6604 & 0.1458 \end{bmatrix} \hat{\eta}(k) + \begin{bmatrix} 0.2742 & 0.3128 \\ -0.4997 & -0.1248 \end{bmatrix} y(k) + \begin{bmatrix} 0.0799 & -0.0984 \\ 0.1798 & 0.0495 \end{bmatrix} u(k)$$

$$\hat{\xi}(k+1) = \begin{bmatrix} 0.1849 & -0.1127 \\ 0.0289 & 0.2919 \end{bmatrix} \hat{\xi}(k) + \begin{bmatrix} 0.0066 & -0.0114 \\ -0.0053 & 0.0185 \end{bmatrix} y(k) + \begin{bmatrix} 0.1479 & -0.0104 \\ -0.1987 & 0.1899 \end{bmatrix} u(k)$$

$$u(k) = -\begin{bmatrix} -5.7198 & 5.5950 \\ 6.2312 & -6.6806 \end{bmatrix} \hat{\eta}(k) - \begin{bmatrix} -11.0231 & -5.7617 \\ 12.5405 & 6.5756 \end{bmatrix} \hat{\xi}(k)$$

7 Conclusion

In this paper the algebraic Riccati equation decomposition and eigenvector method have been used in order to solve the optimal control and filtering of the discrete-time linear weakly coupled stochastic system. This approach can be used in case of higher level of coupling between the subsystems. Beside providing reduction and parallelism in on-line computation of control and filtering tasks, it gives new insights into the optimal control and filtering of weakly coupled systems.

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