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On robustness of suboptimal min-max model predictive control *

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Abstract: - With the hard computation of an exact solution to non-convex optimization problem in a limited time, we propose a suboptimal min-max model predictive control (MPC) scheme for nonlinear discrete-time systems subjected to constraints and disturbances. The idea of input-to-state stability (ISS) is introduced and a Lyapunov-like sufficient condition for ISS is presented. Based on this, we show that the suboptimal predictive controller obtained here holds back the disturbance robustly in the present of constraints on states and inputs.

Key-Words: -Nonlinear predictive control; Suboptimal control; Input-to-state stability; Constraints

1 Introduction

Model predictive control (MPC) or receding horizon control (RHC), as a class of on-line optimization control technique, has received much attentions in both academic and industrial societies due to its ability to handle multivariable, constrained systems [1]. Meanwhile, a "standard" theoretical framework containing a terminal cost and/or a terminal inequality constraint in the optimization problem has been developed to design stabilizing nonlinear model predictive controllers (NMPC) [2][3]. However, the stability may be lost when the dynamics is subject to disturbances and constraints, because of the failure of solving the optimization problem, i.e., infeasibility [4]. One way to avoid the failure is to redefine the optimization problem by such approaches as soft-constraint, minimal-time etc. [5]; the other way is to exploit min-max MPC schemes that are minimized by inputs and maximized by disturbances [2][6]~[9]. The present work concentrates on the latter method, i.e., min-max MPC scheme that is formulated by the standard framework.

It is well known that an exact global solution of the non-convex nonlinear programming of NMPC, leaded generally by the nonlinear model equality constraint, cannot usually be obtained or is highly computationally expensive[1][10]. Scokaert *et al.* [10] presented, however, a suboptimal NMPC algorithm with guaranteed nominal stability (closed-loop stability when the model is perfect and with no disturbances) even if a globally optimal solution is not available. Therefore, it is meaning to study the suboptimal NMPC problem with robustness analysis when the model is subject to disturbances and a globally (or locally) optimal solution is not available. On the other hand, input-to-state stability (ISS) has became one of the most important notions to investigate the robustness of nonlinear systems and has been recently introduced in the study of NMPC resulting systems under disturbances [8][9] [11]~[14]. By employing nominal models (which usually results in hard conservativeness), references [12]~[14] derived the ISS stability for linear MPC, and [11] for NMPC. Moreover, Limon *et al.* [8] presented an input-to-state practically stable min-max NMPC under bounded uncertainties. Nevertheless, this result did not directly consider disturbance attenuation specifications in objective functions, where the disturbance term should have a different sign from that of the state term as a game opposer. Magni *et al.* [8] added the disturbance term to the cost and obtained regional ISS of min-max MPC for nonlinear perturbed systems if the overall optimization problem is solved exactly and its feasibility under horizon length of a single step holds at all time. Thus, this design is very rigid and may be severe conservative since it is likely to have a very small feasible region or a poor performance.

In order to increase the applicability of NMPC, the suboptimal versions based on min-max open-loop and closed-loop NMPC formulations are

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proposed for nonlinear discrete-time systems subjected to disturbances and constraints. Similar to the strategy in [10], we exploit the initial feasible solution that is updated iteratively in the process of the on-line optimization. The purpose of this paper is to obtain some tractable sufficient conditions under which the robust stability of the suboptimal NMPC is guaranteed. We obtain our main goals by means of an ISS-Lyapunov approach of nonlinear systems, which does not require that the optimized cost function and the NMPC resulting system are continuous. Therefore, the suboptimal strategy achieves the robust stability of NMPC in the sense of disturbance rejection and satisfies the constraint on state and input as well.

The paper is organized as follows: Section 2 states the problem to be studied, including the system description and the formulation of min-max NMPC. In Section 3, we derive a preliminary result on ISS, which is slightly different from the one presented in [15]. In Section 4, we present our main results on the robustness in suboptimal min-max NMPC case. Finally, we conclude the paper in Section 5.

Notation: The symbol "**:=**" represents that the left-hand term is defined as the right-hand side. $\mathbf{Z}_+$ denotes the set of all nonnegative integers and $\mathbf{R}_+$ the set of all nonnegative real numbers. For any vector x, x' is its transpose and $|x|$ its Euclidean norm. The l^n_∞-norm of a signal u is defined as $\|u\|=sup\{|u(t)|: u(t)\in\mathbf{R}^n, t\in\mathbf{Z}_+\}<\infty$. We use $\alpha_1\circ\alpha_2$ to denote the composition of two mappings α_1 and α_2, $\alpha^{[k]}=\alpha\circ\alpha^{[k]}$ and α^{-1} its inverse mapping. A continuous function α: $\mathbf{R}_+\to\mathbf{R}_+$ is a ***K***-function if it is strictly increasing and $\alpha(0)=0$, $\alpha(t)>0$ for all $t>0$; it is a ***K***$_\infty$-function if it is a K-function and $\alpha(t)\to\infty$ as $t\to\infty$. A continuous function β: $\mathbf{R}_+\times\mathbf{R}_+\to\mathbf{R}_+$ is a ***KL***-function if $\beta(s, t)$ is a ***K***-function in s for each fixed $t\geq 0$; it is strictly decreasing in t for every fixed $s>0$ and $\beta(s, t)\to 0$ as $t\to\infty$.

2 Problem Formulation

Consider nonlinear, uncertain discrete-time systems of the general form

$$x(t+1)=f(x(t),u(t),w(t)) \tag{1}$$

where $x(t)\in\mathbf{R}^n$ is the state, $u(t)\in\mathbf{R}^m$ is the control variable and $w(t)\in\mathbf{R}^q$ is the disturbance, for each time instant $t\in\mathbf{Z}_+$. The trajectory of system (1) with initial state $x(0)=x_0$ and inputs u and w is denoted as $x(\cdot;x_0,u,w)$, which is supposed to be uniquely on $\boldsymbol{Z}_+$. The state and control variables are subject to the following constraints:

$$u(t)\in U,\ x(t)\in X \tag{2}$$

Here sets X and U are compact subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$, and contain the origin as an interior point, respectively. We assume that the state of the system is available for state feedback and the origin is an equilibrium point of the system.

The main objective of this paper is to obtain a suboptimal controller for the system subjected to the following disturbance $w(t)$, based on a class of "standard" min-max MPC scheme [2].

$$w(t)\in W(x,u) \tag{3}$$

where W is a compact set for each pair (x, u) and contains the origin. The obtained controller is required to regulate the state of the system to a ball containing the origin whose radius is a function of the supremum norm of the disturbance while satisfying the constraints (2).

In what follows, we review the "standard" min-max MPC scheme [2].

Problem 1 (**FHODG**). Given the positive integer N (time horizon), the stage cost l_1: $X\times U\to\mathbf{R}_+$ and l_2: $W\to\mathbf{R}_+$, the terminal cost E: $X\to\mathbf{R}_+$ and the terminal region $\Omega\subseteq X$ containing the origin, then the *Finite Horizon Optimal Differential Game* (FHODG) problem is defined as

$$\min_u \max_w \{E(x(t+N))+\sum_{k=t}^{t+N-1} l_1(x(k),u(k))-l_2(w(k))\} \tag{4}$$

$$\begin{aligned} s.t.\quad & x(k+1)=f(x(k),u(k),w(k)) \\ & u(k)\in U,\quad w(k)\in W \\ & x(k)\in X,\quad \forall k\in[t+1,t+N] \\ & x(t+N)\in\Omega,\quad x(t)=x \end{aligned} \tag{5}$$

where,

$$u:=\{u_t,\cdots,u_{t+N-1}\} \tag{6a}$$

or

$$u:=\{\pi_0(x(t)),\pi_1(\cdot),\cdots,\pi_{N-1}(\cdot)\} \tag{6b}$$

with control laws π_i: $\mathbf{R}^n\to\mathbf{R}^m$, $i=0,1,\ldots,N-1$. If we choose the control action of (6a) as the decision variables of Problem 1, the scheme is called min-max open-loop MPC scheme; or else, the scheme selecting control law (6b) is said to be min-max closed-loop (also called feedback) MPC scheme. For simplicity of discussion, we write in this work them uniformly as $u=\{u(t), \ldots, u(t+N-1)\}$.

To solve Problem 1 at each time is to yield a control sequence, the first of which is employed as MPC law. However, the global optimality of the control is usually not guaranteed since the Problem is a non-convex nonlinear programming problem. In fact, the suboptimality in overall optimization problem is more interesting in terms of computational burden. In suboptimal case, the control is not unique (usually defined by inequalities) and may be discontinuous, too [10]. Moreover, the nonnegative condition of the objective function in (4) will be lost due to the suboptimality in the maximization subproblem. Hence, the robustness of suboptimal MPC should be considered more carefully. In the next section, we will present the preliminary results that are capable to deal with the cases resulted from the suboptimality and to simplify the subsequent analysis.

3 ISS properties

Consider the following nonlinear systems

$$x(t+1)=\chi(x(t),u(t)) \quad (7)$$

with $\chi(0,0)=0$. The response of system (7) with initial state $x(0)=x_0$ and input u is denoted as $x(t;x_0,u)$ for all $t\in\mathbf{Z}_+$.

Definition 1 [15]. The system (7) is input-to-state stable (ISS) if there exist some functions $\alpha\in K$ and $\beta\in KL$ such that, for each input $u\in\mathbf{R}^m$ and each state $\xi\in\mathbf{R}^n$

$$|x(t;\xi,u)|\le\beta(|\xi|,t)+\alpha(\|u\|) \quad (8)$$

for each $t\in\mathbf{Z}_+$.

Inequality (8) guarantees that for any bounded input u, the state of system (7) x is bounded. Moreover, as time increases, the state will ultimately approach a ball around the origin whose radius is determined by a function of class K_∞ of the upper limit of the input [16]. Therefore, ISS implies that the system (7) with null-input or decaying input is asymptotically stable to the origin.

Lemma 1. The system (7) is ISS if there is a function V: $\mathbf{R}^n\to\mathbf{R}_+$ such that, for all $u\in\mathbf{R}^m$ and all $x\in\mathbf{R}^n$

$$\alpha_1(|x|)\le V(x)\le\alpha_2(|x|)+\sigma_1(\|u\|) \quad (9a)$$

$$V(\chi(x,u))-V(x)\le\sigma_2(|u|)-\alpha_3(|x|) \quad (9b)$$

where $\alpha_i\in K_\infty$, i =1,2,3 and $\sigma_i\in K$, i=1,2. Then function $V(\cdot)$ is called an *ISS-Lyapunov* function of system (7).

Proof. The process of proof here is similar to the one in [8] for input-to-state practically stability.

Let $\alpha_4(s)=\alpha_2(s)+\sigma_1(s)$ and $\alpha_4(s)$ be a K_∞-function. Based on property (p6) (See **Appendix**), we derive that, from the inequality (9a)

$$|x|+\|u\|\ge\alpha_4^{-1}\circ V(x) \quad (10)$$

Set the K_∞-function $\alpha_6(s)=min(\alpha_5(0.5s)+\alpha_3(0.5s))$ where $\alpha_5(s)$ is a given K_∞-function. Apply property (p7) with $s=|x|+\|u\|$ and the inequality (10) to yield

$$\alpha_3(|x|)+\alpha_5(\|u\|)\ge\alpha_6\circ\alpha_4^{-1}\circ V(x) \quad (11)$$

where $\alpha_6\circ\alpha_4^{-1}(s)$ is a K_∞-function. Substituting (11) into (9b) yields

$$V(\chi(x,u))-V(x)\le\sigma_3(\|u\|)-\alpha_6\circ\alpha_4^{-1}\circ V(x) \quad (12)$$

where $\sigma_3(s)=\alpha_5(s)+\sigma_2(s)$. In terms of appendix, there exists a K_∞-function $\alpha_7(s)$ such that $\alpha_7(s)\le\alpha_6\circ\alpha_4^{-1}(s)$ for all $s\ge0$ and both functions $\psi_1(s)=s-\alpha_7(s)$ and $\psi_2(s)=s-0.5\alpha_7(s)$ are K-functions. Then inequality (12) yields

$$V(\chi(x,u))\le\sigma_3(\|u\|)+\psi_1\circ V(x) \quad (13)$$

Define function $\alpha_8(s)=\alpha_7^{-1}(2s)$. Then, we have that $\alpha_8(s)>2s$ and

$$\psi_2\circ\alpha_8(s)=\alpha_8(s)-0.5\alpha_7\circ\psi_2(s)=\alpha_8(s)-s \quad (14)$$

Let $x(0)=\xi$. Since $\alpha_8(s)>2s$, we obtain that, from (13)

$$V(x(1))\le\alpha_8\circ\sigma_3(\|u\|)+\psi_2\circ V(\xi)$$

Using induction, we assume that $V(x(k))\le\alpha_8\circ\sigma_3(\|u\|)+\psi_2^{[k]}\circ V(x_0)$. Apply properties (p8)~(p9) and (14) to obtain

$$\begin{aligned}V(x(k+1))&\le\sigma_3(\|u\|)+\psi_1\circ V(x(k))\\&\le\sigma_3(\|u\|)+\psi_1(\alpha_8\circ\sigma_3(\|u\|)+\psi_2^{[k]}\circ V(\xi))\\&\le\sigma_3(\|u\|)+\psi_2\circ\alpha_8\circ\sigma_3(\|u\|)+\psi_2^{[k+1]}\circ V(\xi)\\&=\alpha_8\circ\sigma_3(\|u\|)+\psi_2^{[k+1]}\circ V(\xi)\end{aligned} \quad (15)$$

Substituting (15) into (9a) and using property (p5) yield

$$\begin{aligned}\alpha_1(|x(k)|)&\le\alpha_8\circ\sigma_3(\|u\|)+\psi_2^{[k]}\circ V(\xi)\\&\le\alpha_8\circ\sigma_3(\|u\|)+\psi_2^{[k]}(\alpha_2(|\xi|)+\sigma_1(\|u\|))\\&\le\alpha_8\circ\sigma_3(\|u\|)+\psi_2^{[k]}\circ2\alpha_2(|\xi|)+2\sigma_1(\|u\|)\\&:=\sigma_4(\|u\|)+\psi_2^{[k]}\circ2\alpha_2(|\xi|)\end{aligned}$$

where $\sigma_4(s)=\alpha_8\circ\sigma_3(s)+2\sigma_1(s)$. Furthermore, we obtain the size of the system state

$$\begin{aligned}|x(k)|&\le\alpha_1^{-1}(\sigma_4(\|u\|)+\psi_2^{[k]}\circ2\alpha_2(|\xi|))\\&\le\alpha_1^{-1}\circ2\sigma_4(\|u\|)+\alpha_1^{-1}\circ4\psi_2^{[k]}\circ\alpha_2(|\xi|)\\&:=\alpha(\|u\|)+\beta(|\xi|,k)\end{aligned}$$

With the properties of the K-functions, it is apparent that $\alpha(\cdot)$ is a K-function and $\beta(\cdot,\cdot)$ is a KL-function. Hence, the system (7) is input-to-state stable. ■

Remark 1. From (9a), we know that the ISS-Lyapunov function may not be bounded by a K_∞-function of the state. In addition, the function may not be continuous and the result obtained here is independent of the continuity of system (7). Therefore, the result is suitable for discontinuous systems such as the system (1) controlled by discontinuous feedback control laws resulted from suboptimal MPC controllers.

4 ISS of suboptimal MPC

Let π**:** $\Omega\to U$, with $\pi(0)=0$, is a local control of system (1). For a given feasible (not necessarily optimal) pair of solution ($\hat{u}$, $\hat{w}$), the value function of FHODG is defined as

$$J(x,N) = E(x(t+N)) + \sum_{k=t}^{t+N-1} l_1(x(k),\hat{u}(k)) - l_2(\hat{w}(k))\} \quad (16)$$

Now we present a suboptimal, fixed-horizon version corresponding to Problem 1.

Problem 2 (**suboptimal FHODG**):

(1) Fix a suitable time horizon N and pick $\mu \in (0,1]$.

(2) At time t=0, initial state x_0, find a pair of solution, $\hat{u}_0=\{\hat{u}(0|0),..., \hat{u}(N\text{-}1|0)\}$ and $\hat{w}_0=\{\hat{w}(0|0),..., \hat{w}(N\text{-}1|0)\}$, satisfying (5); Set $u(0)=\hat{u}(0|0)$.

(3) At time t+1, initial state x_{t+1}, calculate the value $\hat{J}(x_{t+1}, N)$ with $u_{t+1}=\{\hat{u}(t+1|t),..., \hat{u}(t+N\text{-}1|t), \pi(x(t+N; x_t, \hat{u}_t, \hat{w}_t))\}$ and $\hat{w}_{t+1}=\{\hat{w}(t+1|t),..., \hat{w}(t+N\text{-}1|t), w(t+N)\}$ with any given $w(t+N)\in W$.

(4) For disturbance $\hat{w}_{t+1}$ in (3), pick a control $\hat{u}_{t+1}=\{\hat{u}(t+1|t+1),..., \hat{u}(t+N|t+1)\}$ satisfying (5) and (20); Set $u(t+1)= \hat{u}(t+1|t+1)$.

$$J(x_{t+1},N) \le \mu\hat{J}(x_{t+1},N) \quad (17)$$

(5) Set t: =t+1 and go back to (3).

If a suboptimal control $\hat{u}$ (i.e. satisfying (5) and (17)) is found, according to the receding horizon mechanism, the first element of $\hat{u}$ is chosen as the MPC law, i.e.

$$u^{MPC}(t) = \hat{u}(t \mid t) \quad (18)$$

and the resulting system is formulated as

$$x(t+1) = f(x(t), u^{MPC}(t), w(t)) \quad (19)$$

for all $t\in \mathbf{Z}_+$.

Remark 2. Similar to the strategy presented in [10], Problem 2 does not need to solve the overall optimization (including both minimization and maximization parts) problem exactly and the solution of control obtained at previous time serves as an initial guess for the current nonlinear programming, which considerably reduce the online computational burden. However, the nonnegative condition of value function (16) in overall optimality case, which is ensured by using the properties of differential game theory [9], may be lost due to the suboptimality of the maximization subproblem. Thus, the value function is not suitable for a candidate Lyapunov function. Notice, that Larger μ values make the problem easier to achieve, which, however, generally results in more conservativeness and worse performance.

Let $\Theta(N)$ denote the set of initial states $x(0)\in X$ in which there exists a feasible solution of the FHODG. In order to guarantee the robust stability of the suboptimal controllers with set $\Theta(N)$, some definitions and assumptions are introduced below:

Definition 2 [17]. Set S is robustly invariant for the system (1)-(3) if there exists an input $u\in U$ such that $f(x,u,w)\in S$ for all $x\in S$ and all $w\in W$.

Assumption 1. There are some functions $\theta_1\in K_\infty$ and $\theta_2\in K$ such that $l_1(x, u) \ge \theta_1(|x|)$ and $l_2(w) \le \theta_2(|w|)$ for all $u\in U$, $x\in X$ and $w\in W$.

Assumption 2. Set Ω is a feasible robust invariant set and the local control π**:** $\Omega\rightarrow U$ satisfies the following:

$$E(f(x,\pi(x),w)) - E(x) + l_1(x,\pi(x)) - l_2(w) \le 0 \quad (20)$$

for all $x\in\Omega$ and $w\in W$.

Assumption 3. There exist some K_∞-functions θ_3 and θ_4 such that, for all $x\in\Omega$ and $w\in W$

$$\theta_3(|x|) \le E(x) \le \theta_4(|x|) \quad (21)$$

Assumption 4. The set of initial state, $\Theta(N)$ is a robust invariant set of resulting system (19).

Note that the above assumptions on the design parameters (l_1, l_2, E, Ω, π) in are typical and not trivial at all. For instance, references [18]~[20] gave some approaches to compute these parameters for a class of nonlinear systems and quadratic cost functions, based on H_∞ control theory. However, Assumption 4 does not a-priori holds to the case of suboptimal controller (18). One method to guarantee it is to restrict properly the state constraints and the terminal region in Problem 2. For details, one can see reference [11].

Before the statement of our main results, an important lemma similar to [2] is given below:

Lemma 2 [2]. Under Assumptions **1~3** and for a given initial state x, the value function (16) has the following monotonic property on the horizon length T

$$J(x,T+1) \le J(x,T), \quad \forall T \ge T_{\min} \quad (22)$$

where T_{min} is the shortest time horizon that ensures the existence of a feasible solution of FHOFG at start time. ■

Remark 3. In general, the length of T_{min} is dependent on the initial state of FHODG at initial time t=0. Longer length of T_{min} results in a larger set of initial states, $\Theta(T)$ and raises the feasibility of the optimization problem. Obviously, for initial state $x\in\Omega$ this monotonic property is available even if T_{min} =1.

Now we present the result on robust stability for this suboptimal MPC controller.

Theorem 1. Consider system (1)-(3) and suppose Assumption **1~4** hold. Then, the suboptimal resulting system (19) is ISS with the set $\Theta(N)$ provided that the feasibility of FHODG is satisfied at initial time t=0.

Proof. Because of the feasibility of FHODG at initial time and the robustness of $\Theta(N)$, the controller is well defined at all future times [2]. Given an initial state $x(t)=x\in\Theta(N)$. By Remark 2, we make $V(x)=J(x, N)+Nl_2(\|w\|)$ as a candidate ISS-Lyapunov function of system (19). Then, we have that

$$V(x) \geq l_1(x,u) \geq \theta_1(|x|) \quad (23)$$

Define $B_c=\{x\in X : |x| \leq c\}$ such that $B_c\subseteq\Omega$. Note that the origin in Ω implies the existence of B_c. We now consider two cases of initial states in order to obtain an upper bound of $V(x)$.

(i) $x\in\Omega$: By Lemma 2 with $T_{min}=0$ and, from Remark 3, we obtain that

$$J(x,N) \leq J(x,N-1) \leq \cdots \leq J(x,0) = E(x)$$

which yields

$$J(x,N) \leq \theta_4(|x|) \quad (24)$$

(ii) $x\notin\Omega$: In this case, $x\notin B_c$ and hence, $x<|c|$. With the feasibility of FHODG with a finite horizon length N, all states $x\in\Theta(N)$ can be steered into Ω in N steps by some feasible controls. Combining the properties of disturbances, we have that the value function $J(x, N)$ exists a sufficiently larger number $\Pi<\infty$ such that $J(x, N) \leq\Pi$ for all $x\in\Theta(N)$. Let $\lambda=max(1, \Pi/\theta_4(c))$ and defined a K_∞-function $\theta_5(s)=\lambda\theta_4(s)$. It is apparent that $\theta_5(s)\geq\theta_4(s)$ for all $s\in\mathbf{R}_+$. Then

$$J(x,N) \leq \Pi\,\theta_4(|x|)/\theta_4(d) = \theta_5(|x|) \quad (25)$$

Thus, integrating (24) and (25) to obtain

$$V(x) \leq \theta_5(|x|) + Nl_2(\|w\|) \leq \theta_5(|x|) + N\theta_2(\|w\|) \quad (26)$$

Let $\hat{u}_t=\{\hat{u}(t),\ldots, \hat{u}(t+T-1)\}$ be a suboptimal control at time t. Solving Problem 2, we find a suboptimal control $\hat{u}_{t+1}$ at time $t+1$. Applying this control to system (1) at time $t+1$, we have that

$$\begin{aligned} & J(x^+,N) - J(x,N) \leq \hat{J}(x^+,N) - J(x,N) \\ & \leq E(x(t+T+1)) - E(x(t+T)) + \\ & l_1(x(t+T),\pi(x(t+T))) - l_2(w(t+T)) - \\ & l_1(x,\hat{u}(t)) + l_2(w(t)) \end{aligned} \quad (27)$$

where $x^+=f(x, \hat{u}(t), w(t))$ denotes the successor of the current state x. Substituting (20) into (27), we derive that

$$J(x^+,N) - J(x,N) \leq -l_1(x,\hat{u}(t)) + l_2(w(t)) \quad (28)$$

Moreover, apply Assumption **1** to obtain

$$J(x^+,N) - J(x,N) \leq \theta_2(|w|) - \theta_1(|x|)$$

which is equivalent to

$$V(x^+) - V(x) \leq \theta_2(|w|) - \theta_1(|x|), \forall x \in \Theta(N) \quad (29)$$

By (23), (26) and (29), it is showed that function $V(x)$ is an ISS-Lyapunov function of system (19). Therefore, the MPC resulting system is ISS with robust invariant set $\Theta(N)$. This completes the proof of Theorem 1. ■

Remark 4. From a practical point of views, the results obtained in this paper may be more applicable than that guaranteed by the overall optimality of the FHODG problem (e.g. [8][9]). In addition, these results are well established by min-max MPC scheme based either on open-loop or on closed-loop optimization. However, the controller obtained by the open-loop optimization is generally more conservative than that based on the closed-loop optimization though the computation of an open-loop optimization is more trivial.

5 Conclusion

In this paper, we proposed a suboptimal min-max MPC scheme for nonlinear discrete-time systems subjected to disturbances and constraints on states and inputs, and obtained the robustness of the controller. The concept of ISS has played an important role to derive the sufficient conditions on the robustness analysis. The assumption that the system to be controlled and the controller are continuous is not necessary. Based on these results, we showed that the suboptimal controller obtained in this paper is robustly against the disturbance if the FHODG problem is feasible initially. Therefore, the computational requirements for application of this class of MPC controllers will be more reasonable.

Appendix. Properties of comparison functions

In the sequence, a collection of some well-known properties of comparison functions used in this paper is presented (see [8], [15] and [21]).

Properties. Let $\varepsilon_1:[0,a_1]\rightarrow\mathbf{R}_+$ and $\varepsilon_2:[0,a_2]\rightarrow\mathbf{R}_+$ be both K-functions; $\varepsilon_3(\cdot)$ and $\varepsilon_4(\cdot)$ be K_∞-functions and let $\beta(\cdot,\cdot)$ be a KL-function. Then

(p1) $\varepsilon_1^{-1}(\cdot)$ is a K-function defined in $[0, \varepsilon_1(a_1)]$;

(p2) $\varepsilon_1\circ\varepsilon_2(\cdot)$ is a K-function defined in $[0, b]$ with $b=min(a_2, \varepsilon_2^{-1}(a_1))$;

(p3) $\varepsilon_1\circ\beta(\cdot,\cdot)$ is a KL-function;

(p4) $max(\varepsilon_1(s), \varepsilon_2(s))$ and $min(\varepsilon_1(s), \varepsilon_2(s))$ are both K-functions defined in $[0, b]$ with $b=min(a_2, a_1)$;

(p5) $\varepsilon_1(s_1+s_2) \leq \varepsilon_1(2s_1)+\varepsilon_1(2s_2)$ for all $s_1, s_2\in[0, 0.5a_1]$;

(p6) $\varepsilon_1(s_1)+\varepsilon_2(s_2) \leq \varepsilon_1(s_1+s_2)+\varepsilon_2(s_1+s_2)$ for all $s_1+s_2 \leq min(a_2, a_1)$;

(p7) $\varepsilon_1(s_1)+\varepsilon_2(s_2) \geq min(\varepsilon_1(0.5(s_1+s_2)), \varepsilon_2(0.5(s_1+s_2)))$ for all $s_1\in[0,a_1]$ and $s_2\in[0,a_2]$ satisfying $s_1+s_2\leq 2min(a_2, a_1)$;

(p8) There exists a K_∞-function $\varepsilon_5(s)$ such that $\varepsilon_5(s) \le \varepsilon_3(s)$ for all $s \ge 0$ and $\varepsilon_6(s)=s- \varepsilon_5(s)$ is a K-function;
(p9) Let K-function $\varepsilon_7(s)=s-0.5\varepsilon_5(s)$, then $\varepsilon_6(s_1+s_2) \le \varepsilon_7(s_1)+ \varepsilon_7(s_2)$;
(p10) Let $\varepsilon_8(s)$ be a K-function such that $\varepsilon_8(s) < s$ for all $s \ge 0$, then the function $\beta_1(s, k)=\varepsilon_8^{[k]}(s)$ is a KL-function.

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