Sampled tracking for linear delayed plants using piecewise functioning controller

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Abstract: - This article presents a control method to realize tracking of linear plants that deliver a delayed and sampled state. To do so, we make use of a class of piecewise functioning controller (PFC): bi-sampled controller. The use of this type of controller allows sampled tracking with a delay twice as much as that on the state's output. Thus, we propose an optimal mathematical approach of the dynamics of our controller. We then give an adaptation of this mathematical approach, using only the delayed and sampled state. Computer simulation results are given so as to enhance the theoretical aspect of our method.

Key-Words: - sampled tracking, bi-sampled controller, delayed sampled state

1 Introduction

Discrete control of continuous plants has been studied in various works dealing with different criteria corresponding to control context. Many interesting papers have been published in this area. Among them, Kabamba [4] has proposed the use of generalized sampled-data hold function (GSHF) in the control of linear time-invariant systems. The main idea of GSHF is to periodically sample the output of the system, in order to generate a control by means of a periodical matrix. Urikura and Nagata [5] have proposed a discrete control with reduction of intersample ripples. Yamamoto [6] used a concept of piecewise defined function for which the state is observed at sampling times and during the sampling periods.

More recently, Koncar and Vasseur [7-10] have defined a class of piecewise continuous systems (PCS) using two input spaces and two time spaces. Each input corresponds to one specific time space. The first refers to the discrete time space $S=\{t_k, k=0,1,2,\cdots\}$ called switching space. The second refers to the continuous time space $t\in\Im-S$ with $\Im=\{t\in[0,\infty]\}$. Between two switching instants, the plant is controlled from a first input space U^r. At each switching instant, the plant is controlled from a second input space V^σ. The PCS controller based on this type of systems assumes that the system is continuous between two switching instants. In reference to the classification of Tittus and Egardt [1], this class of control systems has hybrid properties and extends the concept of compound control proposed by Laurent [2] and Vasseur [3]. Moreover, according to Branicky's taxonomy of hybrid systems [11], these control units are characterized by autonomous switching and controlled impulses.

Shortly later, Koncar and Vasseur [12] have studied the case where the plant is defined as sampled data system. Thus, a control based on a PFC bi-sampled controller has been developed in order to realize the tracking of a state trajectory.

But in their first works they considered that the linear time-invariant plant's state is entirely available. In this article, we deal with the case

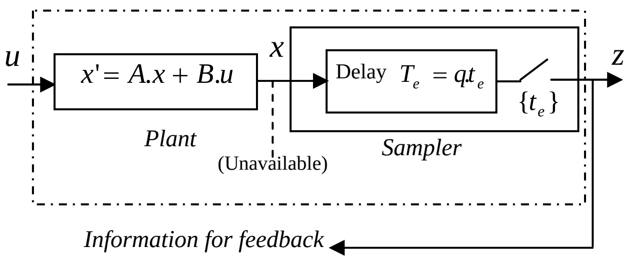

Fig.1: *System for disposal*

illustrated in Fig.1, where the plant's state is not available directly. We assume that the only available feedback signal information is from the plant delivered by a digital sensor (sampling period t_e) introducing a delay T_e corresponding to the time needed to process the information.

This structure of systems is frequently encountered in the industry when we make use of digital technology for measure (e.g. camera) and/or control. If we take the state-space approach, the system's behavior can be represented as follows:

$$x' = A.x + B.u \tag{1}$$

$$z(t) = x^*(t - T_e) = x^*(t - q.t_e) \tag{2}$$

Matrices $A \in \Re^{n\times n}$ and $B \in \Re^{n\times r}$ are commonly used state matrices with appropriate dimensions and * represent sampling at period t_e.

The paper is organized as follows. In the section 2, we will present piecewise functioning controller (PFC): bi-sampled controllers. In section 3, we give results of tracking with the optimal control based on the delayed and sampled state. Finally section 4 is dedicated to the conclusion of the present method, the solution of tracking based on the delayed and sampled output and an overview of our future research.

2 PFC bi-sampled controller

The principle of PFC control is to build an associated PFC controller whose output constitutes the input of the plant for disposal.

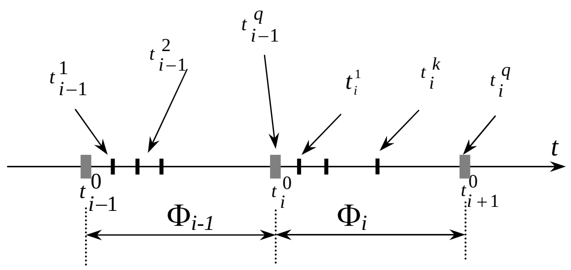

Fig.2: *Two sampled-data time scale*

The most important concept in this type of controller is the two sampled time scales, as illustrated in Fig.2. The discrete instants $\{\ i.T_e + k.t_e\ \}$, noted t_i^k with $T_e = q.t_e$, are characterized by:

- i gives the time scale relative to switching instants, $i \in S = \{i.T_e, i = 0,1,2\cdots\}$. The discrete state of PFC: bi-sampled controller is switched to forced values at instants $\{i.T_e\}$, noted t_i^0; and two successive switching instants t_i^0 and t_{i+1}^0 delimit a piece, noted Φ_i.
- k is the time scale relative to system evolution between two switching instants. And within the two switching instants, the sampling period of the PFS bi-sampled controller is t_e.

The equations describing the behavior of a PFC bi-sampled controller in the Φ_i piece are:

$$\lambda_i^{k+1} = \alpha_i.\lambda_i^k + \beta_i^c.a_i^k\ ,\ k = 0,\dots, q-1 \tag{3}$$

$$\lambda_i^0 = \beta_i^d.\psi_i^0 \tag{4}$$

$$u_i^k = \gamma_i.\lambda_i^{k+1} \tag{5}$$

The state and the output of the PFC bi-sampled controller at instant the t_i^k are respectively denoted by $\lambda_i^k \in R^{\hat{n}}$ and $u_i^k \in R^r$.

The two input spaces for the controller are defined by: $a_i^k \in R^{\hat{r}}$ which is the control between two successive switching instants, and $\psi_i^0 \in R^{\hat{\sigma}}$ which is a control imposed at switching instants for generating the initial state λ_i^0, as illustrated in Fig.3a. Fig.3b gives symbolic representation of the

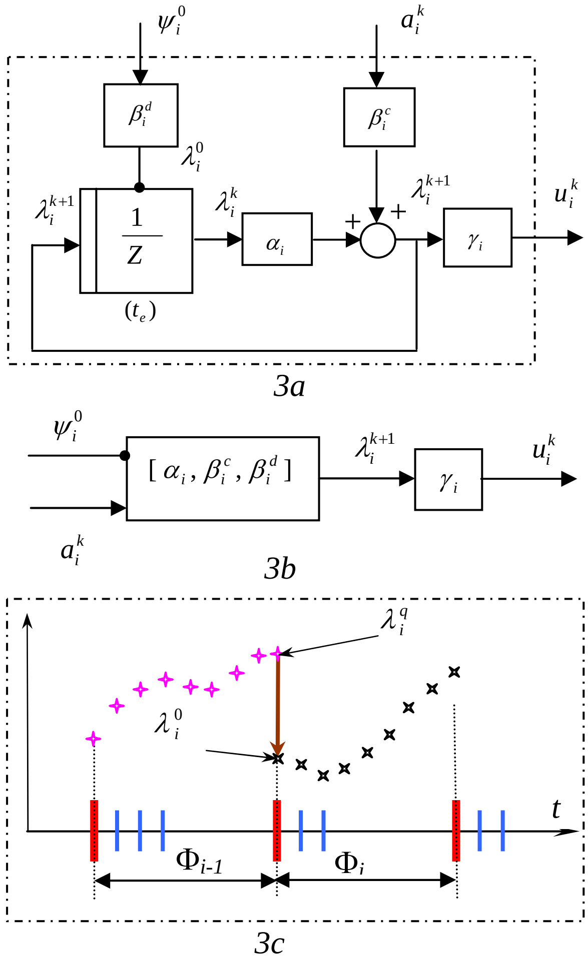

Fig.3: *PFS: bi-sampled controller*
a Detailed representation
b Symbolic representation
c Discontinuity at commutation instants

controller. Matrices $\alpha_i \in R^{\hat{n}\times\hat{n}}$, $\beta_i^c \in R^{\hat{n}\times\hat{r}}$ and $\gamma_i \in R^{r\times\hat{n}}$ are state representation matrices with appropriate dimensions. The additionnal matrix $\beta_i^d \in R^{\hat{n}\times\hat{\delta}}$ defines the relation between λ_i^0 and the input control ψ_i^0. Generally, $\lambda_i^0 \neq \lambda_{i-1}^{q_{i-1}}$ implies discontinuity at switching instants, as illustrated in Fig.3c.

The idea is to make use of the PFC bi-sampled controller to achieve sampled tracking of a state trajectory $c(t)$ by the plant's inaccessible state $x(t)$ at each switching instant $i.T_e$ and with a delay equal to $T_e = q.t_e$. Hence, we define a control strategy to ensure $x_{i+1}^0 = c_i^0 \quad \forall\, i = 0,1,2,\cdots$ by using the feedback signal $z(t)$, which is defined in equation (2). Due to its sampled nature, this signal $z(t)$ will be denoted by z_i^k as from now on.

Thus, the PFS bi-sampled controller should allow us to achieve:

$$z_{i+2}^0 = c_i^0,\ \forall\, i = 0,1,2,\cdots.$$

3 Tracking by delayed and sampled state

In [12], a PFC bi-sampled controller using state feedback is defined. In this section, we assume that the state of the plant is only available in a delayed and sampled format. We thus adapt the PFC bi-sampled controller for this case.

3.1 Control strategy

First, we consider the plant's state equation, and then we use it to define the PFC bi-sampled controller based on the principle of Pontryagin in order to minimize the error between the delayed sampled state and the desired tracking trajectory.

3.1.1 Plant's definition

In reference to the two time scales, we suppose that the time in the Φ_i piece is noted $t = \varphi + i.T_e$, where $\varphi \in [0, T_e]$. We can write the equations as follows:

$$x(t) = x(i.T_e + \varphi) = x_i\,(\varphi) \tag{6}$$

$$u(t) = u_i\,(\varphi) \tag{7}$$

$$z_i\,(\varphi) = z(i.T_e + \varphi) = x_{i-1}(\varphi) \tag{8}$$

So that we can write the state equation of the plant by the feedback signal $z(t)$:

$$z_{i+1}^{'}(\varphi) = A.z_{i+1}(\varphi) + B.u_i\,(\varphi) \tag{9}$$

And letting $u_i\,(\varphi) = v_{i+1}(\varphi)$, we have:

$$z_{i+1}^{'}(\varphi) = A.z_{i+1}(\varphi) + B.v_{i+1}(\varphi) \tag{10}$$

So in the Φ_{i+1} piece, the discrete plant's equation is:

$$z_{i+1}^{k+1} = f.z_{i+1}^k + h.v_{i+1}^k,\quad k = 0,\dots,\ q-1 \tag{11}$$

With: $f = e^{A.t_e}$ and $h = \int_0^{t_e} e^{A.(t_e-\tau)}.B.d\tau$.

3.1.2 Controller definition

In this part we apply the optimal control based on the principle of Pontryagin to define the PFC bi-sampled controller.

The cost criterion including the additional constraint in the Φ_{i+1} piece is defined as follows:

$$r = \frac{1}{2} * \sum_{k=0}^{q-1}\left[\left(c_{i-1}^k - z_{i+1}^k\right)^T.E.\left(c_{i-1}^k - z_{i+1}^k\right) + v_{i+1}^{k\ T}.G.v_{i+1}^k\right] \tag{12}$$

In this expression the matrices E and G are symmetric and positive and they have appropriate dimensions. The minimization of the cost criterion r ensures the reduction of intersample ripples and the moderation of control magnitude. Following the optimal control theory the corresponding Hamiltonian is denoted as in [12]:

$$H = -\frac{1}{2} * \sum_{k=0}^{q-1}\left[\left(c_{i-1}^k - z_{i+1}^k\right)^T.E.\left(c_{i-1}^k - z_{i+1}^k\right) + v_{i+1}^{k\ T}.G.v_{i+1}^k\right] + \sum_{k=0}^{q-1}\lambda_{i+1}^{k+1T}.\left[f.z_{i+1}^k + h.v_{i+1}^k\right]$$

In this expression λ_{i+1}^{k+1} is the n dimension Lagrange multiplication vector. The principle of Pontryagin leads to:

$$\lambda_{i+1}^{k+1} = (f^T)^{-1}\lambda_{i+1}^k - (f^T)^{-1}.E.(c_{i-1}^k - z_{i+1}^k) \tag{13}$$

$$v_{i+1}^k = G^{-1}h^T.\lambda_{i+1}^{k+1} \tag{14}$$

In reference to section 2, in the Φ_{i+1} piece the equations (13) and (14) can be interpreted respectively as the state equation and the output equation of the PFC bi-sampled controller. We can identify the controller's state matrix $\alpha_{i+1} = (f^T)^{-1}$ and $\beta_{i+1}^c = -(f^T)^{-1}.E$, the controller's output matrix $r_{i+1} = G^{-1}.h^T$, and the controller's input $a_{i+1}^k = c_{i-1}^k - z_{i+1}^k$, which defines a unit gain feedback between two switching instants. On the other side, the value of λ_{i+1}^0 has to be determined in order to satisfy the tracking condition: $z((i+2).T_e) = c(i.T_e), \forall\, i = 0,1,2,\cdots$, which is the same as the notation $z_{i+1}^q = c_i^0$, $\forall\, i = 0,1,2,\cdots$. To calculate λ_{i+1}^0, we use the equations (11), (13) and (14) to compose the new state equation:

$$\begin{bmatrix} z_{i+1}^{k+1} \\ \lambda_{i+1}^{k+1} \end{bmatrix} = \mathrm{H}.\begin{bmatrix} z_{i+1}^{k} \\ \lambda_{i+1}^{k} \end{bmatrix} + \mathrm{K}.c_{i-1}^k \qquad (15)$$

with: $\mathrm{H} = \begin{bmatrix} f + hG^{-1}h^T(f^T)^{-1}E & hG^{-1}h^T(f^T)^{-1} \\ (f^T)^{-1}E & (f^T)^{-1} \end{bmatrix}$, and

$$\mathrm{K} = \begin{bmatrix} -h.G^{-1}.h^T.(f^T)^{-1}.E \\ -(f^T)^{-1}.E \end{bmatrix} \qquad (16)$$

The resolution of equation (15) is:

$$\begin{bmatrix} z_{i+1}^{q} \\ \lambda_{i+1}^{q} \end{bmatrix} = \mathrm{H}^q.\begin{bmatrix} z_{i+1}^{0} \\ \lambda_{i+1}^{0} \end{bmatrix} + \left[\mathrm{H}^{q-1}.\mathrm{K}\cdots\mathrm{H}^{q-2}.\mathrm{K}\cdots\mathrm{H}^0.\mathrm{K}\right]\begin{bmatrix} c_{i-1}^0 \\ c_{i-1}^1 \\ . \\ . \\ c_{i-1}^{q-1} \end{bmatrix} \qquad (17)$$

Note: $\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \mathrm{H}^q$ and

$$\begin{bmatrix} \mathrm{I}_h \\ \mathrm{I}_b \end{bmatrix} = \left[\mathrm{H}^{q-1}.\mathrm{K}\cdots\mathrm{H}^{q-2}.\mathrm{K}\cdots\mathrm{H}^0.\mathrm{K}\right]\begin{bmatrix} c_{i-1}^0 \\ c_{i-1}^1 \\ . \\ . \\ c_{i-1}^{q-1} \end{bmatrix} \qquad (18)$$

This leads to the following writing for z_{i+1}^q: $z_{i+1}^q = \Theta_{11}.z_{i+1}^0 + \Theta_{12}.\lambda_{i+1}^0 + \mathrm{I}_h$. From the tracking condition, we can replace z_{i+1}^q by c_i^0. Then at each switching instant, we have:

$$\lambda_{i+1}^0 = \beta_{i+1}^d.\psi_{i=1}^0 = \Theta_{12}^{-1}[c_i^0 - \Theta_{11}.z_{i+1}^0 - I_h] \qquad (19)$$

But in the equation (19), at instant t_i^k, the information z_{i+1}^0 is not available, we only have z_i^k. In this situation, we can define a function based on z_i^k and $u_i^k = v_{i+1}^k$ to evaluate z_{i+1}^k:

$$z_{i+1}^k = \eta(z_i^k, v_{i+\&}^k) = f^q.z_i^k + \left[f^{q-1}.h\cdots f^{q-2}.h\cdots f^0.h\right]\begin{bmatrix} v_{i+1}^{k-q} \\ v_{i+1}^{k-q+1} \\ . \\ . \\ v_{i+1}^{k-1} \end{bmatrix} \qquad (20)$$

Therefore, in the Φ_{i+1} piece, the PFC bi-sampled controller is completely defined with: $\alpha_{i+1} = (f^T)^{-1}$, $\beta_{i+1}^c = -(f^T)^{-1}.E$, $r_{i+1} = G^{-1}.h^T$, $\beta_{i+1}^d = \Theta_{12}^{-1}$, $a_{i+1}^k = c_{i-1}^k - z_{i+1}^k$ and $\psi_{i+1}^0 = c_i^0 - \Theta_{11}.z_{i+1}^0 - I_h$.

Our controller exists with the conditions that are defined in [12]. To summarize, the matrix Θ_{12} must be non-singular and *q* greater than or equal to the order of plant.

3.1.3 Structure of command

The structure of command using the PFC bi-sampled controller is represented in Fig.4. In the block the parts $\Sigma(\cdot)$ and $\eta(z_i^k, v_{i+1}^k)$ are defined respectively by equations (18) and (20). The controller has two

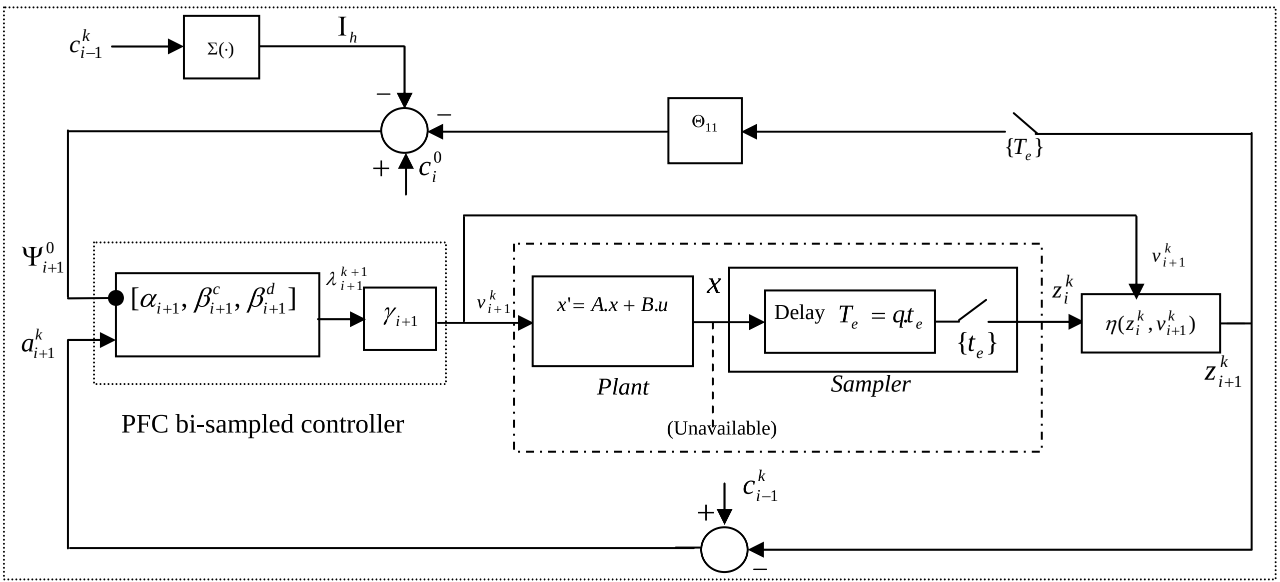

Fig.4: *Structure of command*

inputs ψ_{i+1}^{0} and a_{i+1}^{k} which correspond to the two time scales T_e and t_e. It is important to note that in this present method, we have considered only the case where $C = I_n \in R^{n \times n}$.

3.2 Illustrating example

In view of validating of our method we have simulated the block diagram of Fig.4 by means of Matlab®/Simulink®.

We use the same plant as defined in [12]:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

The t_e-sampled plant's state equation leads to

$$f = e^{A.t_e} \text{ and } h = \int_0^{t_e} e^{A.(t_e - \tau)}.B.d\tau .$$

In this example we consider that $T_e = 1\,s$.

Furthermore, we suppose that the desired state trajectory is:

$$c(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}.$$

We choose $E = I_2$ and $G = 100$ for our controller. The results correspond to different values of the couple $\{t_e ; q\}$ that satisfy $T_e = q.t_e = 1s$.

We choose three couples: $\{2 ; 0.5\}$, $\{4 ; 0.25\}$ and $\{20 ; 0.05\}$ for simulations and the results are illustrated in Fig.5. Note that for comparison sake, we define:

$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} c_1(t - 2.T_e) \\ c_2(t - 2.T_e) \end{bmatrix}$$

It is easy to realize that the delayed sampled state z_i^k follows perfectly the trajectory $w(t) = c(t - 2.Te)$ without oscillations between two switching instants except a short transition at the beginning. With the increase of *q*, the input control *u(t)* is smoother and of lower magnitude. Discontinuities at switching instants are less pronounced. And we can notice its piecewise functioning nature between with switching instants.

We also found that the results of tracking can be ameliorated when we either increase the value of *G* or decrease the value of *E*.

4 Conclusions and future researches

In this paper, we present a method, which is appropriate for control of linear plants in cases where the only available feedback comes from a sensor delivering the plant's state in a delayed (of $T_e = q.t_e$) and sampled (at t_e) format.

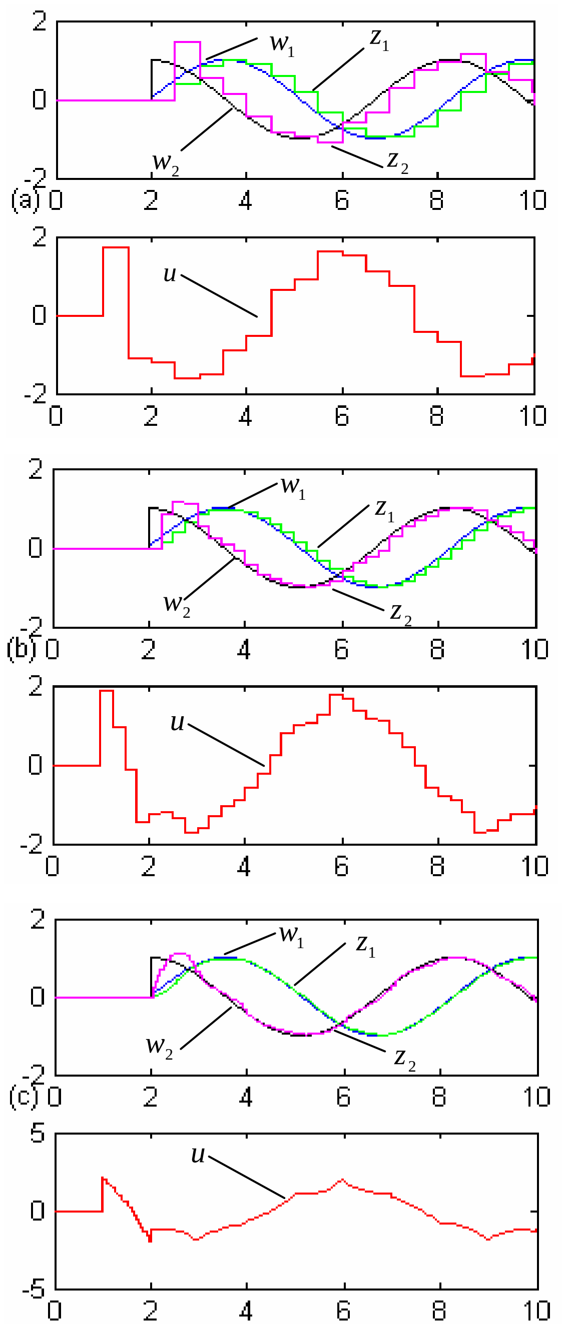

Fig.5: *Tracking by delayed and sampled state*
(a) $q = 2, t_e = 0.5\,s, T_e = 1\,s.$
(b) $q = 4, t_e = 0.25\,s, T_e = 1\,s.$
(c) $q = 20, t_e = 0.05\,s, T_e = 1\,s.$

The control unit is based on an optimal PFC bi-sampled controller whose switching instants matches the regular period T_e equal to *q* times of the sample rate t_e of the sensor, thus designed for sampled feedback. Moreover, the adaptation of this kind of controller ensures sampled tracking at every switching instant $i.T_e, \forall i = 0,1,2,...$, with a delay of T_e concerning the state and $2.T_e$ concerning the sensor's output such that: $z_i^0 = x_{i-1}^0 = c_{i-2}^0$, $\forall\, i = 0,1,2,\cdots$.

Note that in every case, the PFC bi-sampled controller shows better efficiency for small values of T_e, implying faster switching of the controller. The method, tested in computer simulation, is reliable, and robust against slight time variations of the parameters (A and B) of the plant.

We have also the solution to adapt the PFS bi-sampled controller to realize the tracking trajectory $z_i^0 = C.c_{i-2}^0$, $\forall\, i = 0,1,2,\cdots$ where the output matrix is $C \in R^{m \times n} \neq I_n$. In this case, the feedback is based on the delayed and sampled output. This part will be published later.

As a perspective of study, we consider further tests of the method in practical system and its establishment on nonlinear processes.

Acknowledgements

This work is supported by the European Union under Grants 15010/02Y0064/03-04 CAR/Presage N° 4605 Obj. 2-2004:2 - 4.1 – N° 160/4605.

References:

[1] M. Tittus and B. Egart, Control Design for Integrator Hybrid Systems, *IEEE Transaction on Automatic Control*, Vol.43, No.4, 1998, pp. 491-500.

[2] F. Laurent, Sur la commande d'un filtre linéaire par des impulsions multimodulées. *C.R. Acad. Sc, Paris*, t. 270, pp. 288-289.

[3] C. Vasseur, Contribution à l'étude des systèmes échantillonnés commandés par impulsions multimodulées, Ph.D. Thesis, University Lille 1, Villeneuve d'Ascq, France, 1972.

[4] P. T. Kabamba, Control of Linear Systems Using Generalized Sampled-Data Hold Functions, *IEEE Transaction on Automatic Control*, Vol.AC-32, NO.9, 1987, pp. 772-783.

[5] S. Urikura and A. Nagata, Ripple-Free Deadbeat Control for Sampled-Data Systems, *IEEE Transaction on Automatic Control*, Vol.AC-32, NO.6, 1987, pp. 474 482.

[6] Y. Yamamoto, A Function Space Approach to Sampled Data Control Systems and Tracking Problems, *IEEE Transaction on Automatic Control*, Vol.39, NO.4, 1994, pp. 703-713.

[7] V. Koncar and C. Vasseur Tracking by Compound Control, *Studies in Informatics and Control*, N° SIC Vol. 4, 2000.

[8] V. Koncar and C. Vasseur, Suivi de Trajectoire par Commande Composite, *Session invitée, CIMASI 2000, Casablanca, Maroc*, 2000.

[9] V. Koncar and C. Vasseur, Systèmes à fonctionnement par morceaux et poursuite échantillonnée, *APII-JESA*, Vol.35, No.5, 2001, pp. 665-689.

[10] V. Koncar and C. Vasseur, Control of linear systems using piecewise continuous systems, *IEE Control Theory & Applications*, Vol.150, n° 6, 2003, pp. 565-576.

[11] M. S. Branicky, V. S. Borkar and S. K. Mitter, A unified framework for hybrid control, *In Proceedings IEEE Conf.Decision Contr.*, Lake Buena Vista, 1994, pp. 4248-4234.

[12] V. Koncar and C. Vasseur, Piecewise functioning systems: bi-sampled controllers, *Studies in Informatics and Control*, Vol.11, No.2, 2002.

[13] P. Borne, C. Dauphin-Tanguy, J.P. Richard, F. Rotella, I. Zambetakis, *Commande et Optimisation des Processus*, Editions Technip, 1990.