# Frequencies of propagation of electromagnetic waves in a hexagonal waveguide

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*Abstract*—In this work, cut-off frequencies of propagation of electromagnetic waves in a hexagonal waveguide are calculated using two-dimensional (2-D) finite element method. The numerical approach is a standard one and involves six finite elements. A new type of hexagonal waveguide structure for the simple homogeneous dielectric case has been considered. The starting point is Maxwell's equations in conjunction to the exponential dependence of the fields on the Z- coordinates. For the homogeneous case, it results in the Helmholtz equations. Finally, finite element method has been used to derive approximate values of the possible propagation constant for each frequency.

*Keywords*—Finite-element-method, Variational principle, Eigenvector, Matrix Equation, frequencies of propagation, hexagonal waveguide.

# I. INTRODUCTION

THE finite element method (FEM) has been widely used over the decades in the analysis of waveguide components. It is because the propagation characteristics of arbitrarily shaped waveguides of is based on a spatial discretization of cross-section [1], [2], [3], [6]. This approximation allows handling of waveguide cross section geometries which are very similar to the real structures employed in practice. As a consequence, FEM constitutes a promising tool to characterize such problems [7].

Modern phased array radars imply the requirements for polarization agility of wideband array elements. Surface hexagonally poled lithium niobate for two dimensional non-linear interactions in optical waveguide structures has been reported [8], [9]. One possible choice for a radiating element with this property is the hexagonal waveguide. In this paper, a numerically efficient finite-element formulation is proposed to solve waveguides problems. Propagation modes obtained by this formulations may be used to analyse problems involving linear systems of arbitrary complex tensor permittivity and permeability. The solution of these eigenvalue problems results in the approximate fields for all components of different eigenmodes in the waveguide which can further be used to obtain the corresponding eigenvalues [10], [11], [12]. A possible comparison of the proposed methodology with the available theoretical results has also been presented herein this paper to claim the accuracy and reliability of the solution method.



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Fig. 1. hexagonal cross section of the waveguide

### II. THE FINITE ELEMENT FORMULATION

The basic idea of taking hexagonal cross-section and dividing the cross-section in to finite number of elements has been taken from elsewhere [2], [6]. In this paper, the equilateral hexagonal cross section of the waveguide is divided into a number of finite elements. An element is considered to be first-order triangular in shape. An schematics of a triangular finite element in the hexagonal waveguide is shown in figure 1. Consider a triangle having vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ . We draw a vector  $\vec{u}$  joining  $(x_1, y_1), (x_2, y_2)$  and another vector  $\vec{v}$  joining  $(x_1, y_1)$  and  $(x_3, y_3)$ .

Let

$$d_1 = |u| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
(1)

and

$$d_2 = |v| = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$$
(2)

The unit vector along the two directions u and v are

$$\hat{u} = \frac{u}{|u|} = \frac{(x_2 - x_1, y_2 - y_1)}{d_1} \tag{3}$$

and

$$\hat{v} = \frac{v}{|v|} = \frac{(x_3 - x_1, y_3 - y_1)}{d_2} \tag{4}$$

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any point(x, y) inside this triangle can be represented as

$$(x,y) = (x_1, y_1) + u.\hat{u} + v.\hat{v}$$
  
=  $(x_1, y_1) + \frac{u(x_2 - x_1, y_2 - y_1)}{d_1}$   
+  $\frac{v(x_3 - x_1, y_3 - y_1)}{d_2}$ 

so

$$(x - x_1) = \frac{u(x_2 - x_1)}{d_1} + \frac{v(x_3 - x_1)}{d_2}$$
(5)

and

$$(y - y_1) = \frac{u(y_2 - y_1)}{d_1} + \frac{v(y_3 - y_1)}{d_2}$$
(6)

Equations (5) and (6) are two linear equations for the variable u and v and solving them gives us u, v as linear functions of x, y. The area measure is given by

$$ds(u,v) = |\vec{u} \times \vec{v}| du.dv$$

where

$$|\vec{u} \times \vec{v}| = \sin(\alpha)$$

here angle  $\alpha$  between the vectors u and v defined as

$$cos(\alpha) = \frac{u.v}{d_1.d_2} = \frac{(x_2 - x_1)(x_3 - x_1) + (y_2 - y_1)(y_3 - y_1)}{d_1.d_2}$$
(7)

The integral of a function can be evaluated as

$$I(\phi) = \frac{1}{2} \int_0^{d_1} \int_0^{d_2} \phi[x_1 + \frac{u(x_2 - x_1)}{d_1} + \frac{v(x_3 - x_1)}{d_2}]$$
$$y_1 + \frac{u(y_2 - y_1)}{d_1} + \frac{v(y_3 - y_1)}{d_2}]sin\alpha.dudv$$
(8)

if  $\phi = 1$  then we get

$$I(\phi) = \frac{d_1 \cdot d_2 \sin\alpha}{2} \tag{9}$$

which is the correct formula for the area of the triangle. Suppose we write

$$V(x,y) = ax + by + c$$

for

$$x, y \in \Delta$$

with  $\Delta$  as the area bounded by the triangle. a, b, c are chosen so that V at the vertices are given, i.e.,

$$V(x_1, y_1) = V_1$$
  
 $V(x_2, y_2) = V_2$   
 $V(x_3, y_3) = V_3$ 

Thus,

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$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Thus we find that

$$a = \frac{V_1(y_2 - y_3) + V_2(y_3 - y_1) + V_3(y_1 - y_2)}{\Delta}$$
(10)

$$b = \frac{V_1(x_2 - x_3) + V_2(x_3 - x_1) + V_3(x_1 - x_2)}{\Delta}$$
(11)

$$e = \frac{V_1(x_2.y_3 - x_3.y_2) + V_2(x_3.y_1 - x_1y_3) + V_3(x_1.y_2 - x_2.y_1)}{\Delta}$$
(12)

where

(

$$\Delta = x_2 \cdot y_3 - x_3 \cdot y_2 + x_3 \cdot y_1 - x_1 \cdot y_3 + x_1 \cdot y_2 - x_2 \cdot y_1 \quad (13)$$

thus for

$$x, y \in \Delta$$

we have

$$V(x,y) = ax + by + c$$
  
=  $V_1\phi_1(x,y) + V_2\phi_2(x,y) + V_3\phi_3(x,y)$ 

$$\phi_1(x,y) = \frac{(y_2 - y_3)x + (x_2 - x_3)y + (x_2 \cdot y_3 - x_3 \cdot y_2)}{\Delta}$$
(14)

$$\phi_2(x,y) = \frac{(y_3 - y_1)x + (x_3 - x_1)y + (x_3.y_1 - x_1.y_3)}{\Delta}$$
(15)

$$\phi_3(x,y) = \frac{(y_1 - y_2)x + (x_1 - x_2)y + (x_1.y_2 - x_2.y_1)}{\Delta}$$
(16)

The following two integrals occur when one uses the finite element method

first

second

$$I_1 = \int_{\Delta} V_{(x,y)}^2 dx.dy$$

$$I_2 = \int_{\Delta} |\nabla V|^2 dx. dy$$

 TABLE I

 Nodal Coordinates of the Finite Element Mesh of Figure 2

S.No.	Element no.	Coordinates
1	Element 1	(0.0,0.0), (1.0,1.73), (2.0,0.0)
2	Element 2	(0.0,0.0), (1.0,-1.73), (2.0,0.0)
3	Element 3	(0.0,0.0), (-1.0,-1.73), (1.0,-1.73)
4	Element 4	(0.0,0.0), (-2.0,0.0), (-1.0,-1.73)
5	Element 5	(0.0,0.0), (-1.0,1.73), (-2.0,0.0)
6	Element 6	(0.0.0.0), (-1.0.1.73), (1.0.1.73)



Fig. 2. A finite element mesh (6 elements and 7 nodes). The numbers shown in circles represent the elements.

Now

$$I_1 = \int_{\Delta} V_{(x,y)}^2 dx. dy = \int_{\Delta} (ax + by + c)^2 dx. dy$$
 (17)

By substituting the value of (x,y) in terms of  $(x_i, y_i)$  in equation(17), we get

$$I_{1} = \int_{\Delta} V_{(x,y)}^{2} dx.dy$$

$$= \frac{\sin\alpha}{2} \int_{0}^{d_{1}} \int_{0}^{d_{2}} [a(x_{1} + \frac{u(x_{2} - x_{1})}{d_{1}} + \frac{v(x_{3} - x_{1})}{d_{2}}) + b(y_{1} + \frac{u(y_{2} - y_{1})}{d_{1}} + \frac{v(y_{3} - y_{1})}{d_{2}}) + c]^{2} du dv$$
(18)

The use of method of variable separation for u and v results in the following

$$I_{1} = \int_{\Delta} V_{(x,y)}^{2} dx. dy = \frac{\sin\alpha}{2} \int_{0}^{d_{1}} \int_{0}^{d_{2}} [u] \\ (\frac{a(x_{2} - x_{1}) + b(y_{2} - y_{1})}{d_{1}}) + v \\ (\frac{a(x_{3} - x_{1}) + b(y_{3} - y_{1})}{d_{2}}) + c']^{2}$$
(19)

where

$$c' = ax_1 + by_1 + c$$

Equation (19) can be written as

$$I_1 = \int_{\Delta} V_{(x,y)}^2 dx dy = T_1 + T_2 + T_3 + T_4 + T_5 + T_6$$
(20)

Here

$$T_1 = \frac{\sin \alpha}{2} \int_0^{d_1} \int_0^{d_2} \left[ \frac{a(x_2 - x_1) + b(y_2 - y_1)}{d_1} \right]^2 u^2 du dv$$
(21)

$$T_{2} = \frac{\sin \alpha}{2} \int_{0}^{d_{1}} \int_{0}^{d_{2}} 2 \left[ \frac{a(x_{2} - x_{1}) + b(y_{2} - y_{1})}{d_{1}} \right] \\ \left[ \frac{a(x_{3} - x_{1}) + b(y_{3} - y_{1})}{d_{2}} \right] uv du dv$$
(22)

$$T_3 = \frac{\sin \alpha}{2} \int_0^{d_1} \int_0^{d_2} \left[ \frac{a(x_3 - x_1) + b(y_3 - y_1)}{d_2} \right]^2 v^2 du dv$$
(23)

$$T_4 = \frac{\sin \alpha}{2} \int_0^{d_1} \int_0^{d_2} \left[ \frac{2C'(a(x_2 - x_1) + b(y_2 - y_1))}{d_1} \right] u du dv$$
(24)

$$T_5 = \frac{\sin \alpha}{2} \int_0^{d_1} \int_0^{d_2} \left[ \frac{2C'(a(x_3 - x_1) + b(y_3 - y_1))}{d_2} \right] v du dv$$
(25)

$$T_{6} = \frac{\sin \alpha}{2} \int_{0}^{d_{1}} \int_{0}^{d_{2}} C'^{2} du dv$$
(26)

The above integration for first element is given as follows

$$I_{1} = \int_{\Delta} V_{(x,y)}^{2} dx.dy$$
  
= .576756 $v_{1}^{2}$  + 2.307 $v_{1}v_{2}$  + 2.018646 $v_{2}^{2}$   
+.5000774565 $v_{1}(-1.73)v_{2}$  - .5000774565  
 $v_{1}(-1.73)v_{3}$  - 2.018645998 $v_{2}v_{3}$   
+.5767559995 $v_{3}^{2}$  - 1.7303 $v_{1}v_{3}$   
(27)

After calculating the above integrals for each element in the above stated manner, we will find the sum of these integrals over the elements in which we have divided the cross-section. Here we have divided the cross-section into 6 elements (Fig. 2). Summation of these integrals will result in a matrix B of size  $7 \times 7$ .

# III. Calculation of integral $|\nabla V|^2$

$$\int_{0}^{d_{1}} \int_{0}^{d_{2}} |\nabla V|^{2} du dv = \int_{0}^{d_{1}} \int_{0}^{d_{2}} \left[ \left( \frac{\partial V}{\partial x} \right)^{2} + \left( \frac{\partial V}{\partial y} \right)^{2} \right] dx dy$$
(28)

Here,

$$dxdy = Jdudv \tag{29}$$

The Jacobian J is given by

$$J = \begin{pmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{pmatrix} = \begin{pmatrix} \frac{x_2 - x_1}{d_1} & \frac{x_3 - x_1}{d_2} \\ \frac{y_2 - y_1}{d_1} & \frac{y_3 - y_1}{d_2} \end{pmatrix}.$$
 (30)

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Now

$$dx.dy = \frac{(x_2 - x_1)(x_3 - x_1)}{d_1.d_2} du.dv$$
(31)

Finally

$$\int_{0}^{d_{1}} \int_{0}^{d_{2}} |\nabla V|^{2} du dv = \int_{0}^{d_{1}} \int_{0}^{d_{2}} \left[ \left( \frac{\partial V}{\partial x} \right)^{2} + \left( \frac{\partial V}{\partial y} \right)^{2} \right]$$
$$\frac{(x_{2} - x_{1})(x_{3} - x_{1})}{d_{1} d_{2}} du dv \quad (32)$$

Here

$$\left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right] = a^2 + b^2 \tag{33}$$

After substituting all the values in above equation and integrate, we get

$$\int_{0}^{d_{1}} \int_{0}^{d_{2}} |\nabla V|^{2} du dv = .577098v_{1}^{2} + .5760718862v_{1}v_{2} + .577098v_{2}^{2} - .578124(v_{1}v_{3} + v_{2}v_{3} - v_{3}^{2})$$
(34)

Here  $v_1, v_2, v_3 \cdots v_n$  are the nodal potential. Solution of integration of  $|\nabla V|^2 du dv$  for all the 6 element, computed in same manner will result in a matrix A of size  $7 \times 7$ .

# IV. FINDING EIGEN VALUES OF THE MATRIX

Now the process of finding eigen values of the matrix is as follows.

$$(V^T A V - k^2 V^T B V)$$

when minimized over V gives the quadratic form defined by

$$\int_{\Delta} |\nabla V|^2 dx dy - k^2 \int_{\Delta} V^2 dx dy \tag{35}$$

here

$$\delta \int_{\Delta} (\nabla \overrightarrow{V}, \nabla \overrightarrow{V}) dx dy = 2 \int_{\Delta} (\overrightarrow{\nabla}, \delta \overrightarrow{\nabla} V) dx dy$$
$$= -2 \int \delta V \nabla^2 V dx dy \quad (36)$$

and

$$\delta \int_{\Delta} V^2 dx dy = \int 2V \delta V dx dy \tag{37}$$

Now from equation (33)

$$-2\int \delta V \nabla^2 V dx dy - 2k^2 \int \delta V V dx dy = 0 \qquad (38)$$

or

$$\int \delta V (\nabla^2 + k^2) V dx dy = 0 \tag{39}$$

$$(\nabla^2 + k^2)V dx dy = 0 \tag{40}$$

equation 38 when evaluated approximately using the finite element method gives

$$AV - k^2 BV = 0 \tag{41}$$

$$(A - k^2 B)V = 0 \tag{42}$$

$$det(A - k^2 B) = 0 \tag{43}$$

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Here V is the vector of vertex nodal field values. Solution of this matrix will give the eigen values. These eigen values are the propagation frequencies of the waveguide. Using above method we can calculate the propagation frequencies of a waveguide of any type of cross-section.

# V. SIMULATION RESULTS

In order to validate the procedure, the computed result is compared with those obtained from the theoretical analysis. Table 2 compares the Eigenvalues of the fundamental frequencies of propagation of the hexagonal waveguide with the theoretical and computed. MATLAB software has been used here for the simulations [13].

TABLE II EIGENVALUES OF THE FUNDAMENTAL FREQUENCIES OF PROPAGATION OF THE HEXAGONAL WAVEGUIDE

S. No.	eigen value(This work)	eigen value(Theoretical)
1	.451	.460
2	.657	.665
3	.7709	.765
4	$1.22666 \times 10^{-8} + 1.439i$	1.220
5	$1489 \times 10^{-6} + .0006328i$	149
6	4513	452
7	657	658

# VI. CONCLUSION

In this paper an advantageous finite-element- method for the hexagonal cross-sectional waveguide problem has been developed by which complex propagation characteristics may be obtained for arbitrarily shaped waveguide . The extension to higher order elements is straightforward. By suitable modifications of the method it is possible to treat other types of waveguides as well, e.g. dielectric waveguides with impedance walls and open unbounded dielectric waveguides properly treating the region of infinity.

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