# The Theory and Application of an Adaptive Moving Least Squares for Non-uniform Samples

Xianping Huang, Qing Tian, Jianfei Mao\*, Li Jiang, Ronghua Liang College of Computer Science and Technology Zhejiang University of Technology Hangzhou, 310023, P. R. China Corresponding author: Jianfei Mao mjf@zjut.edu.cn http://www.zjut.edu.cn

*Abstract:* - Moving least squares (MLS) has wide applications in scattering points approximation fitting and interpolation. In this paper, we improve a novel MLS approach, adaptive MLS, for non-uniform sample points fitting. The size of radius for MLS can be adaptively adjusted according to the consistency of the sampled data points. Experiments demonstrate that our method can produce higher quality approximation fitting results than the MLS.

Key-Words: - MLS; Sample Points; Non-uniform; Points Set; Approximation Fitting; Interpolation

# **1** Introduction

A curve or surface can be easily drawn if we know its explicit representation. However, in most engineering application, an explicit formation can not be provide for such curve or surface. So, we often need to have a dataset of points that sampled in a specific scope [1][2][3], with which the curve or the surface can be represented by interpolation or fitting. The basic idea of interpolation is to do an approximate estimation to unknown points with the given discrete points, then connect these discrete points to obtain the entire curve (curved surface). Although the whole smoothness of interpolation is good enough, in the boundary of the points set, the potential error of interpolation is greater. Thus fitting is more frequently used, such as the reconstruction of curves. Classical fitting methods include Radial Basis Function (RBF) method[4], Least-squares method[5], and the Moving Least Squares with superior property developed from Least-squares method(Moving Least Squares, hereinafter referred to as MLS)[5-11]. In recent years, MLS has attracted great attention, and now is becoming a hot focus in related researching fields. A major factor that affects the quality of fitting in MLS is the selection of the radius of influence domain. In order to obtain high-quality fitting, many approaches were presented. In literature [2][12][13] heuristic algorithm was used, for instance, weight space[14] ball which contains k-nearest neighbors and Voronoi triangularization[15] can be used to search the influence domain of being fitted points. But the limitation of the above method is that only when the discrete points are distributed regularly or the density of the sampled points is basically constant, can it be effective. When it comes to non-uniform distribution of sampled points, it would be quite difficult to get the fitting details by MLS, even hardly can it be realized when it comes to larger convex (concave) hull, and still, the fitting error is still relatively huge.

We present an adaptive moving least squares in this paper, in which the radius of influence domain is adjusted dynamically according to the density of the sampled points. This new method could do superior fitting than classical MLS for either dense or sparse sampled points. Moreover, when the basis function takes the second basis or higher, the smoothness of fitting will be better.

## 2 The Moving Least Square Algorithm

According to fitting region, fitting function f(x) is figured as:

$$f(x) = \sum_{i=1}^{m} a_{i}(x) p_{i}(x) = p(x)^{T} a(x)$$
(1)

In above function,  $a(x) = [a_1(x), a_2(x), ..., a_m(x)]^T$  is the coefficient of the fitting function f(x). In the process of calculation, a(x) dynamically changes along with the being fitted points. p(x) = [p1(x), p2(x), ..., pm(x)] is the base function which is a base-order complete polynomial.

In order to let the fitting function f(x) close in upon the true value u(x) better, at each point x of the fitting area, similar to the least-squares principle, the value of J in formula (2) should be as small as possible.

$$J = \sum_{i=1}^{n} w((x - \chi_i) / R) [p(\chi_i)^T a(x) - u(x)]^2$$
(2)

Formula (2) also can be written in matrix form as

$$J = (pa - u)^T w(x)(pa - u)$$
<sup>(3)</sup>

And

$$p = \begin{bmatrix} p_{1}(x_{1}) & p_{2}(x_{1}) & \cdots & p_{m}(x_{1}) \\ p_{1}(x_{2}) & p_{2}(x_{2}) & \cdots & p_{m}(x_{2}) \\ \vdots & \vdots & \ddots & \vdots \\ p_{1}(x_{n}) & p_{2}(x_{n}) & \cdots & p_{m}(x_{n}) \end{bmatrix}$$
(4)  
$$w(x) = \begin{bmatrix} w(x - x_{1}) & 0 & \cdots & 0 \\ 0 & w(x - x_{2}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(x - x_{n}) \end{bmatrix}$$
(5)

where n represents the number of sampled points in the fitting region. R is the radius of influence, and w(x) is the weight function. According to mathematical knowledge, J takes the derivative on a(x) to zero, then **Eqs.**(3) and (4) are gotten as

$$\frac{\partial J}{\partial a} = A(x)a(x) - B(x)u = 0 \tag{6}$$

Formula (6) also can be written in matrix form as

$$\begin{bmatrix} (p_{1}, p_{1}) & (p_{1}, p_{2}) & \dots & (p_{1}, p_{m}) \\ (p_{2}, p_{1}) & (p_{2}, p_{2}) & \dots & (p_{2}, p_{m}) \\ \vdots & \vdots & \ddots & \vdots \\ (p_{m}, p_{1}) & (p_{m}, p_{2}) & \dots & (p_{m}, p_{m}) \end{bmatrix} \begin{bmatrix} a_{1}(x) \\ a_{2}(x) \\ \vdots \\ a_{m}(x) \end{bmatrix} = \begin{bmatrix} (p_{1}, u_{i}) \\ (p_{2}, u_{i}) \\ \vdots \\ (p_{m}, u_{i}) \end{bmatrix}$$
(7)  
Where

$$A = \begin{bmatrix} (p_{1}, p_{1}) & (p_{1}, p_{2}) & \dots & (p_{1}, p_{m}) \\ (p_{2}, p_{1}) & (p_{2}, p_{2}) & \dots & (p_{2}, p_{m}) \\ \vdots & \vdots & \ddots & \vdots \\ (p_{m}, p_{1}) & (p_{m}, p_{2}) & \dots & (p_{m}, p_{m}) \end{bmatrix}$$
(8)  
$$B(x)u = \begin{bmatrix} (p_{1}, u_{i}) \\ (p_{2}, u_{i}) \\ \vdots \end{bmatrix}$$
(9)

$$a(x) = A^{-1}(x)B(x)u$$
 (10)

where

١

$$u = \begin{vmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{vmatrix}$$

 $(p_{\ldots}, u_{i})$ 

Via **Eqs.** (1) and (10), using an appropriate sub-base function according to the actual needs, you can obtain the approximate value of being fitted point.

All above is the basic principle of the moving least squares(MLS). Taking the fitting accuracy, smoothness, calculational amount and so on into account, in practical application, 1 or 2 times is usually chosen for the basis function. Weight function usually is spline function as (11) and (12), Gaussian function as (13) or exponential function as (14) in which the symbol  $\beta$  is the figure parameter.

$$w(x) = \begin{cases} \frac{2}{3} - 4x^2 + 4x^3 & x \le 0.5 \\ \frac{4}{3} - 4x + 4x^2 - \frac{4}{3}x^3 & 0.5 \le x \le 1 \\ 0 & x > 1 \end{cases}$$
(11)

$$w(x) = \begin{cases} 1 - 6x^2 + 8x^3 - 3x^4 & x \le 1 \\ 0 & x > 1 \end{cases}$$
(12)

$$w(x) = \begin{cases} \frac{e^{-x^2\beta^2} - e^{-\beta^2}}{1 - e^{-\beta^2}} & x \le 1\\ 0 & x > 1 \end{cases}$$
(13)

x > 1

$$w(x) = \begin{cases} e^{-(x/\beta)^2} & x \le 1 \\ 0 & x > 1 \end{cases}$$
(14)

But, according to a latest literature [16] in MLS, the weight function should be defined as Eq.(15)which has been improved by us through experiment

$$w(x) = \begin{cases} (1-x)^4 (1+2x) & (0 \le x \le 1) \\ 0 & (x > 1 \text{ or } x < 0) \end{cases}$$
(15)

The result of fitting will be better when the weight function w(x) defined as Eq.(15) rather than spline function, and in order to cut down the fitting time, in this paper, we take Eq.(15) as the weight function rather than Gaussian function acquiescently.

## **3 Adaptive MLS Algorithm**

## 3.1 The Shortcoming of Moving Least **Squares in Non-uniform Points Set Fitting**

When the sampled points which are being fitted by traditional MLS are non-uniform, the radius R of the influence domain of point to be fitted (i.e., compact support domain) usually depends on the density of the sparse sampled area. As shown in Fig.1, symbol V true represents the true value of point V, Vdr represents the fitted value of point V as

the radius of influence domain adjusts dynamically, Vsr represents the fitted values of V points by traditional MLS method. dR represents the radius of influence domain obtained while fitting dynamically, and sR represents the radius of influence domain in traditional MLS at point V, in order to make sure matrix A(x) reversible in Eq.(4) and the fitting smooth, R will make the point to be fitted contains too many sampled points in it's influence domain, which will result in the fitting value far more less than the real. But actually, V's proper radius of influence is less than R. If we let the radius of influence domain be dR, the fitted value Vdr would be quite closed to the real value. Vdr is really more accurate than Vsr. Thus, the whole fitting effect would be improved very much if we could adjust the radius of influence domain according to the density of the sampled points dynamically.





(c) Fitting diagram

**Fig.** 1 The relationship between the radius of influence and the fitting

#### 3.2 An Adaptive MLS

The local shape in some one of the non-uniform points is relative to the neighbors merely, so before fitting, we need to find the effective sampled points for current point marked V fitting, and this can be done by seeking the k-nearest neighbor points of V. In this paper, we present an adaptive MLS to do non-uniform points set fitting betterly. The matrix  $A^{-1}(x)$  must exist before using the adaptive MLS to do fitting, This could be fulfilled by searching V's k-nearest neighbors which contains k sampled points not all in the same line. The process of the adaptive MLS fitting is as follows:

(1)Assuming the domain to be fitted be in x[x<sub>min</sub>, x<sub>max</sub>], y[y<sub>min</sub>, y<sub>max</sub>]. Divide the fitting area with squares of which side length is L, and 
$$L = \alpha \sqrt{\frac{k^{\alpha}}{n}} (x_{max} + 1 - x_{min}) (y_{max} + 1 - y_{min}) \qquad \alpha$$

denotes regulating factor of L. According to literature[17], We let it equal 1.1. k represents the k-nearest neighbors, and the n is the number of sampled points.

(2)To current point V(x', y') coming to be fitted, calculate the square S[[x'/L]][[y'/L]] where it lies according to the square side length L got from Eq.(1) (|.| expresses the flooring operation).

(3)To the sampled points in square S[|x'/L|][|y'/L|], sort the points from small to large according to the euclidean distance to the V. If the current square S [|x'/L|] [|y'/L|] contains less than k sampled points or the euclidean distance of the  $j^{th}(j \le k)$  nearest sampled point to V  $D_{sk}$  is larger than D<sub>min</sub> (D<sub>min</sub> is the shortest distance of the current being fitted point to the four sides of S), go to step 4; else consider the distance of the k<sup>th</sup> nearest sampled point to V as the radius of influence domain and these sampled points are linearly independent. Else find out the k+i-nearest(i=

1,2,...), until in which there are k sampled points linearly independent in the current V's k+i-nearest points or having considered all the sampled points in the current S, less than k linearly independental sampled points are found, then go to step 4. All these could be shown as **Fig**.2.

(4)If there are divided squares besides the square  $S^{\lfloor x'/L \rfloor} [\lfloor y'/L \rfloor]$  in it's four directions, then take these sampled points into current squares account together and let current  $D_{min}$  and L treble, then go to step (3); else make the point who has the furthest distance within the sampled points of current S to V as the size of radius of influence domain. Just here, the whole algorithm is over.



(a) The number of the sampled points is less than k in S



(b) The distance from k- nearest neighbors of V to V D<sub>sk</sub>>D<sub>min</sub>



(c) The distance from S is less than k neighbors of V

to V  $D_{sk} \leq D_{min}$ 

Fig. 2 The distribution map about the distance from sorted sampled point in S to V





influence domain obtained through above steps of the adaptive MLS, the fitting could be carried out finally as shown in **Fig**.3.

### **4 Experiment Evaluation and Results**

#### 4.1 Curve Fitting

Select two representative formulas (16) and (17), with 12 non-uniform sampled points as [-2.9 -2 -1.2 -0.8 -0.7 -0.45 0 0.5 1 1.8 2.1 2.7] in the interval  $x \in [-3,3]$ , to make curve fitting respectively on the first base function p(x) = [1 x] and the second base function  $p(x) = [1 x x^2]$  ( the results are similar when the base of the p(x) takes third or larger times):

$$F_{x1} = 8e^{-\frac{(9x-5)^2}{16}} + 7e^{-\frac{(9x-11)^2}{196}} + 6e^{-\frac{(9x+5)^2}{16}} - 3e^{-\frac{(9x+1)^2}{4}}$$
(16)

$$F_{x2} = e^{(-1+\cos(x)^2)} - x\sin(x) + x^2$$
(17)

The fitting results are shown as **Figs**.4 and 5. We can see from the **Figs**. that the fitting when the adaptive MLS takes the second base function is closer to the real results more. In **Fig.** 4 and 5, the graphs drawn with solid lines indicate the corresponding real graph, and the ones with broken lines are the graphs by the MLS or the adaptive MLS.





(b) MLS second base

(a) MLS first base fitting (R=2)





(c) Adaptive MLS first base fitting (d) Adaptive MLS second base fitting

Fig. 4 The comparative experiment results of function  $F_{x1}$  fitting



#### 4.2 Surface Fitting

Select two representative formulas (18) and (19), in the interval  $x \in [-3,3]$   $y \in [-3,3]$  shown in **Fig.** 6 about 36 non-uniform sampled points, to do the surface fitting on the first base function  $p(x) = [1 x y^2 x y^2]$ respectively (the results are similar when the base of the p(x) takes third or larger times):



$$F_{ml} = 8e^{\frac{(9x-5)^{2}+(9y+5)^{2}}{16}} + 7e^{\frac{(9x-11)^{2}}{19} + \frac{9y-11}{20}} + 5e^{\frac{(9x+5)^{2}+(9y-3)^{2}}{16}} - 2e^{\frac{(9x+11)^{2}+(9y+5)^{2}}{4}}_{[18]}(18)$$

$$F_{m2} = \sin\frac{1}{x^{2} + y^{8} + 1} - \sin\frac{1}{x^{8} + y^{2} + 1}$$
(19)

The fitting results are shown as Figs. 7-8









(b) MLS first base fitting

(c) MLS second base fitting

(R=2.5)

(R=3.5)





(d) Adaptive MLS first (e) Adaptive MLS second base fitting base fitting Fig. 7 The comparative experiment results of function  $F_{m1}$  fitting



(a) The original graph



(b) MLS first base fitting

(R=2.5)





#### 4. 3 Results Analysis

Based on the curve fitting and surface fitting experiment obtained by traditional MLS and the adaptive MLS presented in this paper respectively, we do error analysis about the fitting results. We take B(i) and Q<sub>method</sub> as the fitting error difference function and fitting performance function

$$B(j) = F_{MIS_j} - F_j |-|F_{RSRMIS_j} - F_j| (j=1,...,N)$$
(20)

$$Q_{method} = \frac{\sum_{j=1}^{N} (2^{t} - 1)}{N} * 100\%$$
(21)

When the method is the adaptive MLS in formula(21), if formula(21) satisfies that B(j)>0, t is valued 1, else t is valued 0; when the method is MLS, if formula(21) satisfies that B(j) < 0, t is valued 1, else t is valued 0. N is the number of points have been fitted.

According to the criterion given by formula(20) and (21), do error difference analysis to the result **Figs.** 4 and 5 of functions  $F_{x1}$  and  $F_{x2}$  obtained by MLS and the adaptive MLS fitting and result Figs. 7 and 8 of functions  $F_{m1}$  and  $F_{m2}$ , we can come to see the percent of error done by the adaptive MLS fitting in all the fitting area to  $F_{x1}$ ,  $F_{x2}$ ,  $F_{m1}$  and  $F_{m2}$ is less than all that of MLS, which is shown in the following table1.

Table 1. The whole fitting region Adaptive MLS fitting error is less than that of MLS

fitted function	F <sub>x1</sub>	$F_{x2}$	$F_{m1}$	$F_{m2}$
times of base function				
first base function	83%	80%	72%	70%

From the error comparing analysis between MLS and the adaptive MLS in curve and surface fitting, we can easily reach the conclusion that the fitting error is less when using the adaptive MLS this paper presented compared with that obtained by traditional MLS based on the same sampled points. After the detail comparisons of Figs. 4 and 5 and Figs. 7 and 8, we also find the effect of adaptive MLS is much better than that of traditional MLS in detail fitting. Especially, the fitting of typical complex function  $F_{x1}$  in **Fig**. 4, the result graph obtained by adaptive MLS fitting almost completely overlap with the real graph in condition that 12 sampled points spread non-uniformly. And the result is even better when base function p(x) is linear, while MLS only could give a generally trend. Function F<sub>m1</sub> has 3 polar

peaks as shown in **Fig.** 7(a). And they all haven been fitted nearly by the adaptive MLS, while the largest polar peak gotten by traditional MLS. In **Fig.** 8, function  $F_{m2}$  has two not very clear polar peaks. They were both fitted obviously by adaptive MLS, while neither of them gotten by traditional MLS. In addition, in dealing with more complex functions, more realistic function trends could be fitted better by adaptive MLS compared with the traditional MLS even the sampled points very sparse. Above all, the fitting performance of the adaptive MLS is better than that of traditional one.

# **5 Deformation Based on the Adaptive**

## MLS

Besides fitting, the Adaptive MLS also could be used in 2D or 3D graphics deformation with excellent performance. Now we assume  $p_i$  be the set of control points and  $q_i$  the deformed new positions of the control points, and then we need to find an appropriate affine function  $g_v(p_i)$  with which the new  $q_i$  could be calculated, and the deformation at the other points in the graphics could be calculated according to the control points a(s)formula(22)<sup>[19]</sup>

$$\min \sum_{i} \omega_{i}((v - p_{i})/R) |g_{v}(p_{i}) - q_{i}|^{2}$$
(22)

Where,  $\omega_i$  is the weight function, R the radius of influence, and the nearer distance from v to  $p_i$ , the less value of weight function. What's more, the R could be adjusted dynamically according to the density of neighboring control points.

In order to obtain the affine function  $g_{y}(p_{i})$ , we

must minimize formula(22), which could be fulfilled via mathematical theories. And with the presented adaptive MLS we could deform the graphics with some non-uniform control points to new shape as

#### shown in Fig. 9



(a) Original graphics



(b) Non-uniform control points



(c) Deformation 1



(d) Deformation 2 Fig. 9 The deformation based on the adaptive MLS

**Fig.** 9 The deformation based on the adaptive MLS As shown in **Fig.** 9, we could see that the deformation performance of the adaptive MLS is excellent, especially the **Fig.** 9(c) and **Fig.** 9(d) which are vivid.

## **6** Conclusion

In this paper, we presented an adaptive MLS, in which k-nearest neighbor theory was adopted to select appropriate sampled points to fit current unknown point. By this way, the radius of influence domain can be adjusted dynamically according to the density of the surrounding sampled points, and the curve or surface can be reconstructed via the presented adaptive MLS. Through experiment we can draw a conclusion that the fitting performance of the adaptive MLS is superior compared with traditional MLS algorithm. Besides fitting, the presented adaptive MLS also could be used in graphics deformation with excellent performance. Furtherly, the adaptive MLS could be applied in massive 3D discrete sampled data recovery, or fitting sampled points in visualization etc.

## Acknowledgment

This work was partly supported by Major Program

of Zhejiang Province Natural Foundation under grant No.Z1090630, project from Science and Technology Department of Zhejiang Province under grant No.2009C33043 and the Nature Science Foundation of Zhejiang Province of China under grant No.Y1090335.

#### References:

- Hoppe H., Derose T., Duchamp T., et al. Surface Reconstruction from Unorganized Points. Computer Graphics, 1992, 26(2):71-78.
- [2] Alexa M., Behr J., Cohen-or D., et al. Point Set Surfaces. In Proceedings of IEEE Visualization'01, 2001: 21-28.
- [3] Ohtake Y., Belyaey A., Alexa M., et al. Multi-level Partition of Unity Implicits[J]. ACM Trans. Graph, 2003, 22(3):463-470.
- [4] Carr J. C., Beaton R. K., Cherrie J. B., et al. Reconstruction and Representation of 3D Objects with Radial Basis Functions. ACM SIGGRAPH, 2001: 67-76.
- [5] Nealen A., An As-Short-As-Possible Introduction to the Least Squares, Weighted Least Squares and Moving Least Squares Methods for Scattered Data Approximation and Interpolation. Tech. Rep., TU Darmstadt, 2004.
- [6] Li J. F., Wang R. F., Robust Denoising of Point-sampled Surfaces. WSEAS Trans. on Com. 2009,8(1):153-162.
- Yaghmaei M., Naei M. H., Nonlinear Bending of Functionally Graded Thick Circular Plates by Using Element Free Galerkin Method. Proceedings of the 3<sup>rd</sup> IASME/WSEAS Int. Con. on Continuum Mechanics, 2007:137-145.
- [8] Kuragano T., Yamaguchi A., Nurbs Curve Shape Modification and Fairness Evaluation for Computer Aided Aesthetic Design. WSEAS Transactions on Computers, 2008,7(4):174-183.
- [9] Kuragano T., Kasono K., Curve Generation and Modification Based on Radius of Curvature Smoothing. Proceedings of the 10<sup>th</sup> WSEAS Int. Conf. on Mathematical and Computational

methods in science and engineering, 2009:80-87.

- [10] Levin D., The Approximation Power of Moving Least-squares. Mathematics Computing, 1998, 224(67):1517-1523.
- [11] Lipman Y., Cohen-or D., Levin D., Data-dependent MLS for Faithful Surface Approximation. Eurographics Symposium on Geometry Processing, 2007.
- [12] Pauly M., Gross M., Kobbelt L. P., Efficient Simplification of Point-sampled Surfaces. In VIS '02: Proceedings of the conference on Visualization '02 (Washington, DC, USA), IEEE Computer Society, 2002: 163–170.
- [13] Pauly M., Keiser R., Kobbelt L. P., et al. Shape Modeling with Point-sampled Geometry. ACM Trans. Graph, 2003, 22(3):641–650.
- [14] Wei W., Zhang L. Y., Zhou L. S., A Spatial Sphere Algorithm for Searching k-Nearest Neighbors of Massive Scattered Points. Acta aeronautica et astronautica sinica, 2006, 5(27):944-948.

- [15] Floater M. S., Reimers M., Meshless Parameterization and Surface Reconstruction, Comput. Aided Geom. 2001, 18(2):77–92.
- [16] Cheng Z. Q., Wang Y. Z., Li B., et al. A Survey of Methods for Moving Least Squares Surfaces, IEEE/EG Symposium on Volume and Point-Based Graphics, 2008.
- [17] Xiong B. S., He M. Y., Yu H. J., Algorithm for Finding K-Nearest Neighbors of Scattered Points in Three Dimensions. Journal of Computer Aided Design & Computer Graphics. 2004, 16(7):909-917.
- [18] Lipman Y., Cohen-or D., Levin D., Error Bounds and Optimal Neighborhoods for MLS Approximation. Eurographics Symposium on Geometry Processing, 2006.
- [19] Schaefer S., MacPhail T., Warren J., Image Deformation Using Moving Least Squares. ACM Transactions on Graphics, 2006, 25(3):533-540.