The Evaluation Of Nearly Singular Integrals In The Direct Regularized Boundary Element Method

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Abstract: - The numerical analysis of boundary layer effect is one of the major concerned problems in boundary element method (BEM). The accuracy of this problem depends on the precision of the evaluation of the nearly singular integrals. In the boundary element analysis with direct formulation, the hyper-singular integral will arise from the potential derivative boundary integral equations (BIEs). Thus the nearly strong singular and hyper-singular integrals need to be calculated when the interior points are very close to the boundary. For nearly hyper-singular integrals, it is thought, generally, more difficult to calculate. In this paper, a general nonlinear transformation is adopted and applied to calculating the potential and its derivative at the interior points very close to the boundary. Numerical examples demonstrate that the present algorithm is efficient and can overcome the boundary layer effect successfully even when the interior points are very close to the boundary.

Key-Words: - BEM; potential problems; nearly singular integrals; boundary layer effect; transformation; Numerical method

1 Introduction

Research on numerical methods of differential equations is a hot topic. Many efficient methods for finding numerical solutions of differential equations have been presented so far such as in [1-4].

The BEM is power and efficient computational methods if integrals are evaluated accurately, and the main advantages of the BEM resulting from the reduction of the dimension of the boundary value problem are well-known. However, it is popular as well that the standard BEM formulations include singular and nearly singular integrals, and thus the integrations should be performed very carefully. Other than the nearly singular integral, many direct and indirect algorithms for singular integral have been developed and used successfully [5-15]. Therefore, the key point in achieving the required accuracy and efficiency of the BEM is not the singular integral but the nearly singular integral. Although that difficulty can be overcome by using very fine meshes, the process requires too much preprocessing and CPU time.

Nearly singular integrals are not singular in the sense of mathematics. However, from the point of view of numerical integrations, these integrals can not be calculated accurately by using the conventional numerical quadrature since the integrand oscillates very fiercely within the integration interval. The accurate evaluation of nearly singular integrals plays an important role in many engineering problems. In general, these include evaluating the solution near the boundary in potential problems and calculating displacements and stresses near the boundary in elasticity problems, for example, contact problems, displacement around crack tips, sensitivity problems and thin-body problems.

In the past decades, tremendous effort was devoted to derive convenient integral forms or sophisticated computational techniques for calculating the nearly singular integrals. These proposed methods can be divided on the whole into two categories: “indirect algorithms” and “direct algorithms”. The indirect algorithms, which benefit from the regularization ideas and techniques for the singular integrals, are mainly adopted to calculate indirectly or avoid calculating the nearly singular integrals by establishing new regularized BIE [11-18]. The direct algorithms are employed to calculate the nearly singular integrals directly. They usually include interval subdivision method [19-21], special Gaussian quadrature method [22], exact integration method [23-31], and various nonlinear transformation methods [32-40]. In a recent study, the above methods have been reviewed in detail by Zhang and Sun [41].

Although great progresses have been achieved for each of the above methods, it should be pointed out that the geometry of the boundary element is often depicted by using linear shape functions when nearly singular integrals need to be calculated [42, 43]. In fact, to the authors' best knowledge, no work is found in the literature which can be used to calculate the nearly singular integrals under high-order geometry effectively. However, most engineering processes occur mostly in complex geometrical domains, and obviously, higher order geometry elements are expected to be more accurate to solve such practical problems. When the geometry of the boundary element is approximated by using high-order elements—usually of second order, the Jacobian $J(\xi)$ is not a constant but a non-rational function which can be expressed as $\sqrt{a+b\xi+c\xi^2}$, where a, b and c are constants, ξ is the dimensionless coordinate; The distance r between the field points and the source point is a non-rational function of the type $\sqrt{p_4(\xi)}$, where $p_4(\xi)$ is the fourth order polynomial. Thus, the forms of the integrands in boundary integrals become more complex, and it is, unfortunately, more difficult to implement when nearly singular integrals need to be calculated.

In this paper, a general nonlinear transformation is adopted and applied to calculating the potential and its derivative at the interior points very close to the boundary in 2D potential problems. The proposed transformation is constructed based on the idea of diminishing the difference of the orders of magnitude or the scale of change of operational factors. After the BIEs are discretized on the boundary, the nearly weakly singular, strongly singular and hyper-singular integrals can be calculated accurately by using the present method. The nonlinear transformation is available for linear and quadratic elements. Both temperatures and its derivative at the interior points very close to the boundary are accurately computed. The algorithm derived in this paper substantially simplifies the programming and provided a general computational method for solving thin coating problems.

2 Non-singular boundary integral equations (BIEs)

It is well known that the domain variables would be computed by using integral equations only after all the boundary quantities have been obtained, and the accuracy of boundary quantities directly affects the validity of the interior quantities. However, when calculating the boundary quantities, we have to deal with the singular boundary integrals, and a good choice is to use the regularized BIEs. In this paper, we always assume that Ω is a bounded domain in R^2, Ω^c is its open complement; $\Gamma = \partial\Omega$ denotes their common boundary; $\boldsymbol{t}(\boldsymbol{x})$ and $\boldsymbol{n}(\boldsymbol{x})$ are the unit tangent and outward normal vectors of Γ to domain Ω at point $\boldsymbol{x}$, respectively. For two dimensional potential problems, the equivalent non-singular BIEs with direct variables are given in [12].

$$\int_\Gamma u^*(\boldsymbol{x},\boldsymbol{y})q(\boldsymbol{x})d\Gamma = \int_\Gamma [u(\boldsymbol{x})-u(\boldsymbol{y})]q^*(\boldsymbol{x},\boldsymbol{y})d\Gamma,\ y \in \Gamma \tag{1}$$

where we can take the fundamental solution $u^*(\boldsymbol{x},\boldsymbol{y})$ for Eq. (1) as

$$u^*(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{2\pi}\ln r \tag{2}$$

in which $r=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}$; y_1 and y_2 are the coordinates of source point $\boldsymbol{y}$; x_1 and x_2 are the coordinates of field point $\boldsymbol{x}$.

After Eq. (1) is discretized to numerically evaluate the boundary unknown variables, the potential at interior points can be obtained by using the following integral equation

$$u(\boldsymbol{y}) = \int_\Gamma u^*(\boldsymbol{x},\boldsymbol{y})q(\boldsymbol{x})d\Gamma - \int_\Gamma q^*(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{x})d\Gamma \tag{3}$$

In order to determine the flux at interior point $\boldsymbol{y}$, taking derivative of Eq. (3) with respect to the coordinates of the source point $\boldsymbol{y}$ there is

$$\nabla_y u(\boldsymbol{y}) = \int_\Gamma \nabla_y u^*(\boldsymbol{x},\boldsymbol{y}) q(\boldsymbol{x}) d\Gamma - \int_\Gamma u(\boldsymbol{x}) \nabla_y q^*(\boldsymbol{x},\boldsymbol{y}) d\Gamma \tag{4}$$

The Gaussian quadrature is directly used to calculate the integrals in discretized equations (3) and (4) in the conventional boundary element method. However, when the field point $\boldsymbol{y}$ is very close to the integral element Γ_e, the distance r between the field point $\boldsymbol{y}$ and the source point $\boldsymbol{x}$ tends to zero. This causes the integrals in the discretized equations (3) and (4) nearly singular. Therefore, the physical quantities at interior points cannot be calculated accurately by using the conventional Gaussian quadrature.

The above mentioned nearly singular integrals can be expressed as the following generalized integrals:

$$\begin{cases} I_1 = \int_\Gamma \psi(\boldsymbol{x}) \ln r^2 d\Gamma \\ I_2 = \int_\Gamma \psi(\boldsymbol{x}) \dfrac{1}{r^{2\alpha}} d\Gamma \end{cases} \tag{5}$$

where $\alpha > 0$, $\psi(\boldsymbol{x})$ is a well-behaved function including the Jacobian, the shape functions and ones which arise from taking the derivative of the integral kernels. Under such a circumstance, either a very fine mesh with massive integration points or a special integration technique needs to be adopted. In the last two decades, numerous research works have been published on this subject in the BEM literature. Most of the work has been focused on the numerical approaches, such as subdivisions of the element of integration, adaptive integration schemes, exact integration methods and so on. However, most of these earlier methods are either inefficient or can not provide accurate results when the interior points are very close to the boundary. In this paper, a very efficient transformation method is employed to calculate the nearly singular integrals in the discretized equations (3) and (4). Consequently, the accurate results of the physical quantities at interior points very close to the boundary are obtained.

3 Nearly singular integrals under linear elements

The quintessence of the BEM is to discretize the boundary into a finite number of segments, not necessarily equal, which are called boundary elements. Two approximations are made over each of these elements. One is about the geometry of the boundary, while the other has to do with the variation of the unknown boundary quantity over the element. In this section, the geometry segment is modeled by a continuous linear element.

Assuming $\boldsymbol{x}^1 = (x_1^1, x_2^1), \boldsymbol{x}^2 = (x_1^2, x_2^2)$ are the two extreme points of the linear element Γ_j, then the element Γ_j can be expressed as

$$x_k(\xi) = N_1(\xi) x_k^1 + N_2(\xi) x_k^2 \ , \xi \in [-1,1], k = 1,2 \tag{6}$$

where $N_1(\xi) = (1-\xi)/2, N_2(\xi) = (1+\xi)/2$.

Letting $s_i = x_i^2 - x_i^1, w_i = y_i - (x_i^2 + x_i^1)/2$, one has

$$r_{,i} = \frac{r_i}{r} = \frac{y_i - x_i}{r} = \frac{s_i \xi/2 + w_i}{r} \tag{7}$$

$$r^2 = |\boldsymbol{x} - \boldsymbol{y}|^2 = r_i r_i = A\xi^2 + B\xi + E = L^2[(\xi - \eta)^2 + d^2] \tag{8}$$

Where

$A = s_i s_i / 4, B = s_i w_i, E = w_i w_i, \eta = -B/2A,$

$L = \sqrt{A}, d = \sqrt{4AE - B^2}/2A$.

With the aid of the Eq. (8), the nearly singular integrals in Eq. (5) can be rewritten as

$$\begin{cases} I_1 = \left\{ \int_{-1}^{\eta} + \int_{\eta}^{1} \right\} g(\xi) \ln[(\xi-\eta)^2 + d^2] d\xi + \ln L^2 \int_{-1}^{1} g(\xi) d\xi \\ I_2 = \left\{ \int_{-1}^{\eta} + \int_{\eta}^{1} \right\} \dfrac{g(\xi)}{L^{2\alpha}[(\xi-\eta)^2 + d^2]^\alpha} d\xi \end{cases} \tag{9}$$

where $g(\cdot)$ is a regular function that consists of shape function and Jacobian.

4 Nearly singular integrals under curvilinear elements

The linear element is not an ideal one as it can not approximate with sufficient accuracy for the geometry of curvilinear boundaries. For this reason, it is recommended to use higher order elements, namely, elements that approximate geometry and boundary quantities by higher order interpolation polynomials—usually of second order. In this paper, the geometry segment is modeled by a continuous parabolic element, which has three knots, two of which are placed at the extreme ends and the third somewhere in-between, usually at the mid-point. Therefore the boundary geometry is approximated by a continuous piecewise parabolic curve. On the other hand, the distribution of the boundary quantity on each of these elements is depicted by a discontinuous quadratic element, three nodes of which are located away from the endpoints.

Assume $\boldsymbol{x}^1=(x_1^1,x_2^1)$ and $\boldsymbol{x}^2=(x_1^2,x_2^2)$ are the two extreme points of the segment Γ_j, and $\boldsymbol{x}^3=(x_1^3,x_2^3)$ is in-between one. Then the element Γ_j can be expressed as follows

$$x_k(\xi)=N_1(\xi)x_k^1+N_2(\xi)x_k^2+N_3(\xi)x_k^3,\ k=1,2 \tag{10}$$

where

$$N_1(\xi)=\xi(\xi-1)/2,\ N_2(\xi)=\xi(\xi+1)/2,\ N_3(\xi)=(1-\xi)(1+\xi),\ -1\le\xi\le1$$

.

As shown in Fig. 1, the minimum distance d from the field point $\boldsymbol{y}=(y_1,y_2)$ to the boundary element Γ_j is defined as the length of $\overline{\boldsymbol{y}\boldsymbol{x}^p}$, which is perpendicular to the tangential line $\boldsymbol{t}$ and through the projection point $\boldsymbol{x}^p$. Letting $\eta\in(-1,1)$ is the local coordinate of the projection point $\boldsymbol{x}^p$, i.e. $\boldsymbol{x}^p=(x_1(\eta),x_2(\eta))$. Then η is the real root of the following equation

$$x_k'(\eta)(x_k(\eta)-y_k)=0 \tag{11}$$

If the field point y sufficiently approaches the boundary, then Eq. (11) has a unique real root. In fact, setting

$$F(\eta)=x_k'(\eta)(x_k(\eta)-y_k) \tag{12}$$

$$F'(\eta)=x_k'(\eta)x_k'(\eta)+x_k''(\eta)(x_k(\eta)-y_k)=J^2(\eta)+x_k''(\eta)(x_k(\eta)-y_k) \tag{13}$$

where $J(\eta)$ is the Jacobian of the transformation from parabolic element to the line interval $[-1,1]$.

Therefore, when the field point $\boldsymbol{y}$ is sufficiently close to the element, we explicitly have $F'(\eta)>0$.

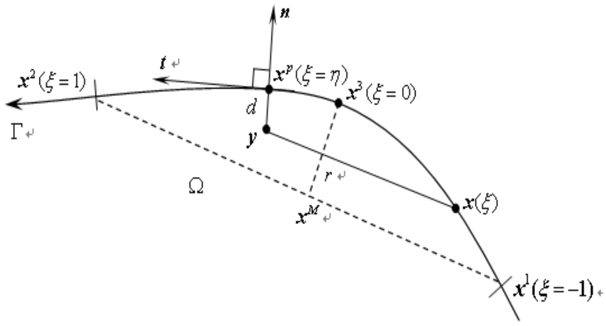

Fig.1. The minimum distance d from the field point $\boldsymbol{y}$ to the boundary element.

The unique real root of Eq. (11) can be evaluated numerically by using the Newton's method or computed exactly by adopting the algebraic root formulas of 3-th algebraic equations. In this paper, two ways are all tested, and practical applications show that both ways can be used to obtain desired results. Furthermore, the Newton's method is more simple and effective, especially if the initial approximation is properly chosen and if we can do this, only two or three iterations are sufficient to approximate the real root. For the root formula of 3-th algebraic equations, let's consider the following algebraic equation

$$ax^3+bx^2+cx+d=0 \tag{14}$$

if there exists only one real root, the analytical solution can be expressed as follows

$$x=-\frac{b}{3a}+\frac{2(\sqrt{s^2+t^2})^{\frac{1}{3}}}{3\sqrt[3]{2}a}\cos\left(\frac{1}{3}\arccos\frac{s}{\sqrt{s^2+t^2}}\right) \tag{15}$$

where $s=-2b^3+9acb-27a^2d$, $t=\sqrt{-4(3ac-b^2)^3-(-2b^3+9acb-27a^2d)^2}$.

Using the procedures described above, we can obtain the value of the real root η. Thus, we have

$$x_k-y_k=x_k-x_k^p+x_k^p-y_k=$$
$$\frac{1}{2}(\xi-\eta)\left[(x_k^1-2x_k^3+x_k^2)(\xi+\eta)+(x_k^2-x_k^1)\right]+x_k(\eta)-y_k \tag{16}$$

By using Eq. (16), the distance square r^2 between the field point $\boldsymbol{y}$ and the source point $\boldsymbol{x}(\xi)$ can be written as

$$r^2(\xi) = (x_k - y_k)(x_k - y_k) = (\xi - \eta)^2 g(\xi) + d^2 \tag{17}$$

where $d^2 = (x_k(\eta) - y_k)(x_k(\eta) - y_k)$,

$$g(\xi) = \frac{1}{4}(x_k^1 - 2x_k^3 + x_k^2)(x_k^1 - 2x_k^3 + x_k^2)(\xi + \eta)^2$$
$$+\frac{1}{2}(x_k^1 - 2x_k^3 + x_k^2)(x_k^2 - x_k^1)(\xi + \eta)$$
$$+h^2 + (x_k^1 - 2x_k^3 + x_k^2)(x_k(\eta) - y_k),$$

where $h = \frac{1}{2}\sqrt{(x_k^2 - x_k^1)(x_k^2 - x_k^1)}.$

Apparently, there is $g(\xi) \geq 0$.

By some simple deductions, the nearly singular integrals in Eq. (5) can be reduced to the following two types

$$\begin{cases} I_1 = \{\int_{-1}^{\eta} + \int_{\eta}^{1}\} f(\xi) \ln\left[(\xi - \eta)^2 g(\xi) + d^2\right] d\xi \\ I_2 = \{\int_{-1}^{\eta} + \int_{\eta}^{1}\} \dfrac{f(\xi)}{\left[(\xi - \eta)^2 g(\xi) + d^2\right]^{\alpha}} d\xi \end{cases} \tag{18}$$

where $f(\cdot)$ is a regular function that consists of shape function, Jacobian and ones which arise from taking the derivative of the integral kernels.

5 The transformation for nearly singular integrals

In Eqs. (9) and (18), if d is very small, the above integrals would present various orders of near singularity. The key to achieving high accuracy is to find an method to calculate these integrals accurately for a small value of d.

The integrals I_1 and I_2 in Eqs. (9) can be reduced to the following integrals by simple deduction

$$\begin{cases} I_1 = \int_0^A g(x) \ln(x^2 + d^2) dx \\ I_2 = \int_0^A \dfrac{g(x)}{(x^2 + d^2)^{\alpha}} dx \end{cases} \tag{19}$$

where A is a constant which is possibly with different values in different element integrals; $g(\cdot)$ is a regular function that consists of shape function, Jacobian and ones which arise from taking the derivative of the integral kernels.

Introducing the following nonlinear transformation

$$x = d(e^{k(1+t)} - 1),\ t \in [-1, 1] \tag{20}$$

where $k = \ln\sqrt{1 + A/d}$.

Substituting (20) into (19), then the integrals I_1 and I_2 can be rewritten as follows

$$\begin{cases} I_1 = dk\int_{-1}^{1} g(t) \ln d^2 e^{k(1+t)} dt + dk\int_{-1}^{1} g(t) \ln[(e^{k(1+t)} - 1)^2 + 1] e^{k(1+t)} dt \\ I_2 = d^{1-2\alpha} k \int_{-1}^{1} \dfrac{g(t) e^{k(1+t)}}{[(e^{k(1+t)} - 1)^2 + 1]^{\alpha}} dt \end{cases} \tag{21}$$

We can observe that $(e^{k(1+t)} - 1)^2 + 1 \geq 1$. Thus, the integrand is fully regular even if the value of d is very small.

Similarly, by some simple deductions, the integrals in Eqs. (18) can be reduced to the following forms

$$\begin{cases} I_1 = \int_0^A f(\xi) \ln\left(\xi^2 g(\xi) + d^2\right) d\xi \\ I_2 = \int_0^A \dfrac{f(\xi)}{\left(\xi^2 g(\xi) + d^2\right)^{\alpha}} d\xi \end{cases} \tag{22}$$

where $f(\cdot)$ is a regular function that consists of shape function, Jacobian and ones which arise from taking the derivative of the integral kernels.

Substituting the nonlinear transformation (20) into Eqs. (22), then we obtain the following equations

$$\begin{cases} I_1 = 2kd \ln d \int_{-1}^{1} f(d(e^{k(1+t)} - 1)) e^{k(1+t)} dt \\ +kd\int_{-1}^{1} f(\xi) \ln\left((e^{k(1+t)} - 1)^2 g(\xi) + 1\right) e^{k(1+t)} dt \\ I_2 = \dfrac{1}{d^{2\alpha - 1}} \int_{-1}^{1} \dfrac{f(\xi)}{\left((e^{k(1+t)} - 1)^2 g(\xi) + 1\right)^{\alpha}} e^{k(1+t)} dt \end{cases} \tag{23}$$

We can see that the denominator always $(e^{k(1+t)} - 1)^2 g(t) + 1 \geq 1$ if $g(x) > 0$ is a regular function as assumed above. Thus, the integrand is fully regular even if the value of d is very small.

By following the procedures described above, the near singularity of the boundary integrals has been fully regularized even if the interior point very close to the boundary need to be calculated. The final integral formulations are obtained as shown in Eqs. (21) and (23), which can now be computed straightforward by using the standard Gaussian quadrature.

6 Numerical examples

To verify the method developed above, two simplified test cases are studied in which BEM solutions are compared with the exact solutions.

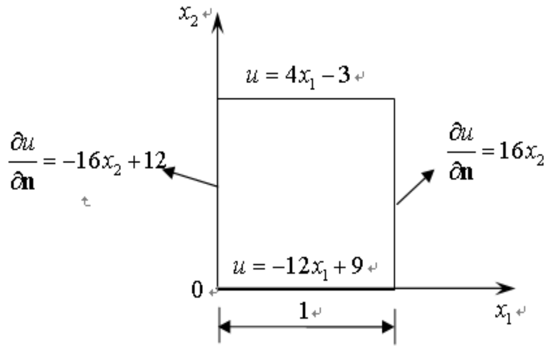

Fig. 2. Thermal transmission in a rectangular domain

Example 1. A prism with square section and infinite length is considered. The prescribed temperature or flux boundary conditions are shown in Fig. 2. From the theory of heat transfer, the analytical temperature solutions are given by $u = 16x_1x_2 - 12x_1 - 12x_2 + 9$.

To solve this problem numerically, the boundary is discretized by 12 uniform linear boundary elements with 24 discontinuous interpolation points. Both the temperatures and the fluxes at interior points are calculated respectively by using the conventional method and the method proposed in this paper.

Fig. 3 presents the temperature results at the interior point $(0.5, a)$. These points are gradually close to the boundary and the conventional BEM cannot be applied to obtain the accurate results. We can see that when the computed points are not too close to the boundary, both the conventional method and the present method are effective and can obtain excellent results, but the results of the conventional method become less satisfactory as the computed points get increasingly close to the boundary, i.e., when the distance from the internal point to the boundary is equal or less 1.0E-2. In contrast, the results of the proposed method are steady and satisfactory even when the computed points are very close to the boundary.

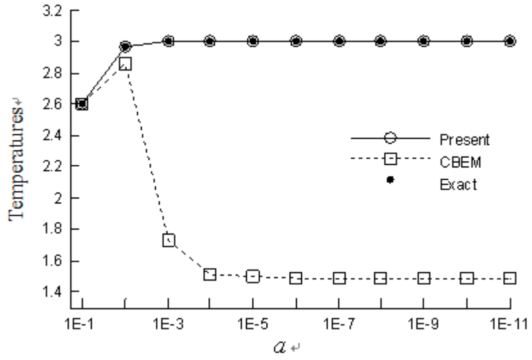

Fig. 3. The temperature results at the interior point $(0.5, a)$

Table 1 Results of fluxes $\partial T / \partial x_1$ at the point on the line $x_2 = 1E-07$

x_1	Exact	CBEM	Relative error (%)	Present	Relative error (%)
0.00001	-0.12000E+02	0.360944E+04	0.301786E+05	-0.119893E+02	0.887571E-01
0.0001	-0.12000E+02	0.353782E+04	0.295818E+05	-0.120023E+02	-0.199678E-01
0.001	-0.12000E+02	0.935638E+03	0.789698E+04	-0.119886E+02	0.949425E-01
0.01	-0.12000E+02	0.563156E+01	0.530703E+02	-0.119891E+02	0.907118E-01
0.1	-0.12000E+02	0.192355E+01	0.116029E+03	-0.120025E+02	-0.209196E-01
0.5	-0.12000E+02	0.600014E+01	0.499987E+02	-0.120019E+02	-0.164738E-01
0.9	-0.12000E+02	0.417189E+01	0.652342E+02	-0.120050E+02	-0.423218E-01
0.99	-0.12000E+02	0.588041E+01	0.509965E+02	-0.120103E+02	-0.865164E-01
0.999	-0.12000E+02	0.307594E+03	0.266328E+04	-0.120103E+02	-0.858660E-01
0.9999	-0.12000E+02	0.117695E+04	0.990794E+04	-0.120086E+02	-0.723638E-01
0.99999	-0.12000E+02	0.120111E+04	0.101092E+05	-0.120063E+02	-0.528396E-01

Table 1 lists the results of fluxes at interior points on the line $x_2 = 1E-07$. These points are very close to the boundary. It is obvious that the results calculated by using the conventional BEM are out of true with the relative errors already greater than 50%. On the other hand, the results of the proposed algorithm are very consistent with the exact solutions with the largest relative error less than 0.1%.

Table 2 lists the results of fluxes at interior points on the line $x_1 = 0.1$. These points are increasingly close to the boundary. It can be seen that the conventional method and the proposed method are both efficient when $x_2 \le 0.9$, but the conventional method fails when the internal point becomes close to the boundary. However, the results obtained by using the proposed algorithm are excellently consistent with the analytical solutions even in the very unfavorable computational condition $x_2 = 0.99999999$. The relative errors are also given in Table 2, from which we can see that the accuracy of the results of the proposed method are high and steady even when the computed points are very close to the boundary, while the relative errors of the conventional method are relatively too large to be accepted with the computed points increasingly close to the boundary.

With the increase of the discretized boundary elements, the relative errors of the computed temperatures at the interior point (0.5, 1E-9) are shown in Fig. 4, from which we can observe that the convergence speeds of the computed temperatures are still fast when the distance of the computed point to the boundary reaches 1.0E-9.

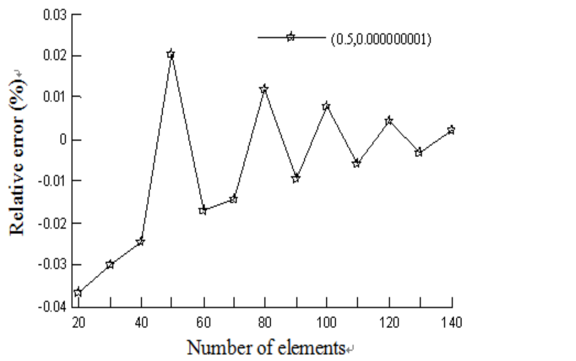

Fig. 4. Convergence curves of temperatures at the interior points (0.5,1E-9)

Table 2 Results of fluxes $\partial T / \partial x_1$ at the point on the line $x_1 = 0.1$

x_2	Exact	CBEM	Relative error (%)	Present	Relative error (%)
0.9	0.24000E+01	0.240000E+01	-0.481096E-11	0.240000E+01	-0.132116E-10
0.99	0.38400E+01	0.656161E+01	-0.708754E+02	0.384000E+01	-0.558581E-11
0.999	0.39840E+01	0.768581E+03	-0.191916E+05	0.398400E+01	-0.401976E-09
0.9999	0.39984E+01	0.135107E+03	-0.327904E+04	0.399840E+01	0.106819E-06
0.99999	0.39998E+01	0.224534E+02	-0.461358E+03	0.399984E+01	0.102793E-04
0.999999	0.39999E+01	0.111148E+02	-0.177872E+03	0.399996E+01	0.441749E-03
0.9999999	0.39999E+01	0.998088E+01	-0.149522E+03	0.398849E+01	0.287542E+00
0.99999999	0.40000E+01	0.986748E+01	-0.146687E+03	0.403389E+01	-0.847291E+00

Example 2. As shown in Fig. 5, this example is concern with a cylinder with infinite length whose inner and outer radii are 1 and 2, respectively. The corresponding boundary conditions are also described in Fig. 5.

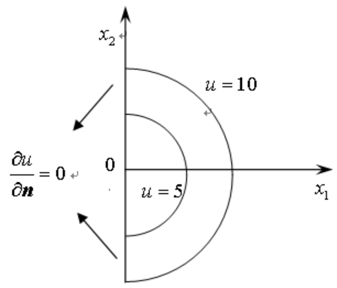

Fig. 5. Heat transmission in a thick cylinder

Six and twelve quadratic elements are divided on the inner and outer surfaces, and four quadratic elements are divided on the two short straight boundaries. Thus the total number of the quadratic elements is 22.

Table 3 Results of fluxes $\partial T / \partial x_1$ at interior point near the inner boundary

x_1	Exact	CBEM	Relative error (%)	Present	Relative error (%)
1.1	0.655770E+01	0.655770E+01	0.220767E-11	0.655770E+01	0.207223E-11
1.01	0.714205E+01	0.131025E+02	-0.834568E+02	0.714205E+01	0.733717E-11
1.001	0.720626E+01	0.718684E+03	-0.987304E+04	0.720626E+01	0.270165E-10
1.0001	0.721275E+01	0.791035E+03	-0.108671E+05	0.721275E+01	-0.191513E-04
1.00001	0.721340E+01	0.791780E+03	-0.108765E+05	0.721921E+01	-0.805597E-01
1.000001	0.721346E+01	0.791780E+03	-0.108765E+05	0.723031E+01	-0.233527E+00
1.0000001	0.721347E+01	0.791780E+03	-0.108765E+05	0.723650E+01	-0.319236E+00

Table 4 Results of fluxes $\partial T / \partial x_1$ at interior point near the outer boundary

x_1	Exact	CBEM	Relative error (%)	Present	Relative error (%)
1.9	0.379656E+01	0.379656E+01	0.311073E-09	0.379656E+01	0.491279E-12
1.99	0.362486E+01	-0.785126E+02	0.226594E+04	0.362486E+01	0.426341E-10
1.999	0.360854E+01	-0.911001E+03	0.253456E+05	0.360854E+01	0.355779E-08
1.9999	0.360691E+01	-0.943684E+03	0.262631E+05	0.360689E+01	0.579243E-03
1.99999	0.360675E+01	-0.944008E+03	0.262733E+05	0.360385E+01	0.805685E-01
1.999999	0.360673E+01	-0.944010E+03	0.262735E+05	0.359706E+01	0.268247E+00
1.9999999	0.360673E+01	-0.944010E+03	0.262735E+05	0.360067E+01	0.168008E+00

The calculated results of the fluxes at internal points, on the line $x_2 = 0$, close to the inner boundary and the outer boundary are shown in Table 3 and Table 4, respectively. Table 3 and Table 4 show that the results obtained by using the conventional BEM are out of true with the relative errors already greater than 80% when the distance between the interior point and the boundary less than 0.1. On the other hand, the results of the proposed algorithm are very consistent with the exact solutions with the largest error less than 0.4% even when the distance of the interior point to the boundary reaches 1.0E-7.

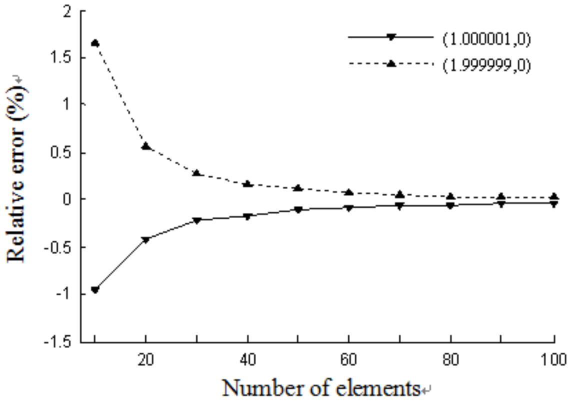

Fig. 6. Convergence curves of fluxes at the interior points

In addition, with the increase of the discretized boundary elements, the relative errors of the computed fluxes at the interior point $(1.000001, 0)$ and $(1.999999, 0)$ are shown in Fig. 6, from which we can observe that the convergence speeds of the computed fluxes are still fast when the distance of the computed point to the boundary reaches 1.0E-6.

7 Conclusions

In this paper, a general nonlinear transformation, based on the direct regularized boundary element method, is adopted and applied to calculating the potential and its derivative at the interior points very close to the boundary in 2D potential problems. With the proposed transformation, the near singularities of the nearly singular integrals can be remove or damp out efficiently, and fairly high accuracy of numerical results is achieved for the nearly singular integrals with stand Gauss quadrature procedures. The proposed transformation is available for linear and quadratic elements. Compared with the conventional BEM, the present algorithm can be used to accurately compute the physical quantities at interior points much closer to the boundary. Numerical examples of the potential problem are presented to test the proposed algorithm, with which excellent results are obtained. The results verify the feasibility and the effectiveness of the present method, and the boundary layer effect has been overcome successfully with the proposed transformation in the applications. The present method is also general and can be applied to other problems in BEM (such as thin-walled structures), which will be discussed later.

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