Traveling Wave Solutions For The (2+1) Dimensional Boussinesq Equation And The Two-Dimensional Burgers Equation By $(\frac{G'}{G})$-expansion method

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Abstract: In this paper, we demonstrate the effectiveness of the $(\frac{G'}{G})$-expansion method by seeking more exact solutions of the (2+1) dimensional Boussinesq equation and the two-dimensional Burgers equation. By the method, the two nonlinear evolution equations are separately reduced to non-linear ordinary differential equations (ODE) by using a simple transformation. As a result, the traveling wave solutions are obtained in three arbitrary functions including hyperbolic function solutions, trigonometric function solutions and rational solutions. When the parameters are taken as special values, we also obtain the soliton solutions of the fifth-order Kdv equation. The method appears to be easier and faster by means of a symbolic computation system.

Key–Words: $(\frac{G'}{G})$-expansion method, Traveling wave solutions, (2+1) dimensional Boussinesq equation, two-dimensional Burgers equation, exact solution, evolution equation, nonlinear equation

1 Introduction

The nonlinear phenomena exist in all the fields including either the scientific work or engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on. It is well known that many non-linear evolution equations (NLEEs) are widely used to describe these complex phenomena. Research on solutions of NLEEs is popular. So, the powerful and efficient methods to find analytic solutions and numerical solutions of nonlinear equations have drawn a lot of interest by a diverse group of scientists. Many efficient methods have been presented so far such as in [1-7].

In this paper, we pay attention to the analytical method for getting the exact solution of some NLEES. Among the possible exact solutions of NLEEs, certain solutions for special form may depend only on a single combination of variables such as traveling wave variables. In the literature, Also there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions. Some of these approaches are the homogeneous balance method [8,9], the hyperbolic tangent expansion method [10,11], the trial function method [12], the tanh-method [13-15], the non-linear transform method [16], the inverse scattering transform [17], the Backlund transform [18,19], the Hirotas bilinear method [20,21], the generalized Riccati equation [22,23], the Weierstrass elliptic function method [24], the theta function method [25-27], the sineCcosine method [28], the Jacobi elliptic function expansion [29,30], the complex hyperbolic function method [31-33], the truncated Painleve expansion [34], the F-expansion method [35], the rank analysis method [36], the exp-function expansion method [37] and so on.

In [38], Mingliang Wang proposed a new method called $(\frac{G'}{G})$-expansion method. Recently several authors have studied some nonlinear equations by this method [39-42]. The value of the $(\frac{G'}{G})$-expansion method is that one can treat nonlinear problems by essentially linear methods. The method is based on the explicit linearization of NLEEs for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Moreover, it transforms a nonlinear equation to a simple algebraic computation. The main merits of the $(\frac{G'}{G})$-expansion method over the other methods are that it gives more general solutions with some free parameters and it handles NLEEs in a direct manner with no requirement for initial/boundary condition or initial trial function at the outset.

Our aim in this paper is to present an application of the $(\frac{G'}{G})$-expansion method to some nonlinear problems to be solved by this method for the first time.

The rest of the paper is organized as follows. In Section 2, we describe the $(\frac{G'}{G})$-expansion method for finding traveling wave solutions of nonlinear evolution equations, and give the main steps of the method. In the subsequent sections, we will apply the method to the (2+1) dimensional Boussinesq equation and the two-dimensional Burgers equation. In section 5, the features of the $(\frac{G'}{G})$-expansion method are briefly summarized.

2 Description of the $(\frac{G'}{G})$-expansion method

In this section we describe the $(\frac{G'}{G})$-expansion method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables $x,\ t$, is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{2.1}$$

or in three independent variables $x,\ y$ and t, is given by

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{yt}, u_{xx}, u_{yy}, ...) = 0, \tag{2.2}$$

where $u = u(x,t)$ or $u = u(x,y,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ or $u = u(x,y,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following, we will give the main steps of the $(\frac{G'}{G})$-expansion method.

Step 1. Suppose that

$$u(x,\ t) = u(\xi), \quad \xi = \xi(x,t) \tag{2.3}$$

or

$$u(x,\ y,\ t) = u(\xi), \quad \xi = \xi(x,y,t) \tag{2.4}$$

The traveling wave variable (2.3) or (2.4) permits us reducing (2.1) or (2.2) to an ODE for $u = u(\xi)$

$$P(u,\ u', u'',\ ...) = 0. \tag{2.5}$$

Step 2. Suppose that the solution of (2.5) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \alpha_m(\frac{G'}{G})^m +\ ... \tag{2.6}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \tag{2.7}$$

$\alpha_m, ..., \lambda$ and μ are constants to be determined later, $\alpha_m \neq 0$. The unwritten part in (2.6) is also a polynomial in $(\frac{G'}{G})$, the degree of which is generally equal to or less than $m-1$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (2.5).

Step 3. Substituting (2.6) into (2.5) and using second order LODE (2.7), collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of (2.5) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_m, ..., \lambda$ and μ.

Step 4. Assuming that the constants $\alpha_m, ..., \lambda$ and μ can be obtained by solving the algebraic equations in Step 3. Since the general solutions of the second order LODE (2.7) have been well known for us, then substituting $\alpha_m, ...$ and the general solutions of (2.7) into (2.6) we have traveling wave solutions of the nonlinear evolution equation (2.1) or (2.2).

3 Application Of The $(\frac{G'}{G})$-Expansion Method For The (2+1) dimensional Boussinesq Equation

In the following two sections, we will apply the $(\frac{G'}{G})$-expansion method for getting the exact solutions of some nonlinear equations. First we will consider the (2+1) dimensional Boussinesq equation [43]:

$$u_{tt} - u_{xx} - u_{yy} - (u^2)_{xx} - u_{xxxx} = 0 \quad (3.1)$$

In order to obtain the traveling wave solutions of Eq.(3.1), we suppose that

$$u(x,\ y,\ t) = u(\xi),\ \xi = kx + ly + mt + d \quad (3.2)$$

$k,\ l,\ m,\ d$ are constants that to be determined later.

By using (3.2), (3.1) can be converted into an ODE

$$(m^2-k^2-l^2)u''-2k^2(u'^2+uu'')-k^4u^{(4)} = 0 \quad (3.3)$$

Integrating the ODE (3.3) with respect to ξ once, we obtain

$$(m^2 - k^2 - l^2)u' - 2k^2(uu') - k^4u''' = g \quad (3.4)$$

where g is the integration constant that can be determined later.

Suppose that the solution of (3.4) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i(\frac{G'}{G})^i \quad (3.5)$$

where a_i are constants, $G = G(\xi)$ satisfies the second order LODE in the form:

$$G'' + \lambda G' + \mu G = 0 \quad (3.6)$$

where λ and μ are constants.

Balancing the order of uu' and u''' in (3.4), we have $m + m + 1 = m + 3 \ \Rightarrow\ m = 2$. So Eq.(3.5) can be rewritten as

$$u(\xi) = a_2(\frac{G'}{G})^2 + a_1(\frac{G'}{G}) + a_0,\ a_2 \neq 0 \quad (3.7)$$

$a_2,\ a_1,\ a_0$ are constants to be determined later. Then it follows

$$u'(\xi) = -2a_2(\frac{G'}{G})^3 + (-a_1 - 2a_2\lambda)(\frac{G'}{G})^2$$
$$+(-a_1\lambda - 2a_2\mu)(\frac{G'}{G}) - a_1\mu$$

$$u''(\xi) = 6a_2(\frac{G'}{G})^4 + (2a_1 + 10a_2\lambda)(\frac{G'}{G})^3$$
$$+(8a_2\mu + 3a_1\lambda + 4a_2\lambda^2)(\frac{G'}{G})^2$$
$$+(6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2)(\frac{G'}{G})$$
$$+2a_2\mu^2 + a_1\lambda\mu$$

$$u'''(\xi) = -24a_2(\frac{G'}{G})^5 + (-54a_2\lambda - 6a_1)(\frac{G'}{G})^4$$
$$+(-12a_1\lambda - 38a_2\lambda^2 - 40a_2\mu)(\frac{G'}{G})^3$$
$$+(-52a_2\lambda\mu - 7a_1\lambda^2 - 8a_2\lambda^3 - 8a_1\mu)(\frac{G'}{G})^2$$
$$+(-14a_2\lambda^2\mu - a_1\lambda^3 - 16a_2\mu^2 - 8a_1\lambda\mu)(\frac{G'}{G})$$
$$-a_1\lambda^2\mu - 2a_1\mu^2 - 6a_2\lambda\mu^2$$

Substituting Eq.(3.7) into (3.4) and collecting all terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$$(\frac{G'}{G})^0:\ -m^2a_1\mu + 2k^4a_1\mu^2 + k^2a_1\mu$$
$$+6k^4a_2\lambda\mu^2 + l^2a_1\mu + k^4a_1\lambda^2\mu$$
$$-g + 2k^2a_0a_1\mu = 0$$

$$(\frac{G'}{G})^1 : \ 4k^2a_0a_2\mu + 8k^4a_1\lambda\mu + 14k^4a_2\lambda^2\mu$$
$$+2l^2a_2\mu + 2k^2a_1^2\mu + k^2a_1\lambda$$
$$+2k^2a_0a_1\lambda + l^2a_1\lambda - 2m^2a_2\mu$$
$$+16k^4a_2\mu^2 + 2k^2a_2\mu - m^2a_1\lambda$$
$$+k^4a_1\lambda^3 = 0$$

$$(\frac{G'}{G})^2 : \ l^2a_1 + 2l^2a_2\lambda - m^2a_1$$
$$+4k^2a_0a_2\lambda + 2k^2a_2\lambda + 52k^4a_2\lambda\mu$$
$$+k^2a_1 + 2k^2a_0a_1 + 7k^4a_1\lambda^2$$
$$+8k^4a_1\mu - 2m^2a_2\lambda + 8k^4a_2\lambda^3$$
$$+2k^2a_1^2\lambda + 6k^2a_1a_2\mu = 0$$

$$(\frac{G'}{G})^3 : \ 2k^2a_2 + 12k^4a_1\lambda + 4k^2a_2^2\mu$$
$$-2m^2a_2 + 40k^4a_2\mu + 2k^2a_1^2$$
$$+2l^2a_2 + 6k^2a_1a_2\lambda$$
$$+38k^4a_2\lambda^2 + 4k^2a_0a_2 = 0$$

$$(\frac{G'}{G})^4 : \ 54k^4a_2\lambda + 4k^2a_2^2\lambda + 6k^4a_1$$
$$+6k^2a_1a_2 = 0$$

$$(\frac{G'}{G})^5 : \ 24k^4a_2 + 4k^2a_2^2 = 0$$

Solving the algebraic equations above, yields:

$$a_2 = -6k^2$$
$$a_1 = -6k^2\lambda$$
$$a_0 = -\frac{1}{2}\frac{k^2 + k^4\lambda^2 + 8k^4\mu - m^2 + l^2}{k^2}$$
$$k = k, l = l$$
$$m = m, d = d, g = 0 \tag{3.8}$$

where k, l, m, d are arbitrary constants.

Substituting (3.8) into (3.7), we get that

$$u(\xi) = -6k^2(\frac{G'}{G})^2 - 6k^2\lambda(\frac{G'}{G})$$
$$-\frac{1}{2}\frac{k^2 + k^4\lambda^2 + 8k^4\mu - m^2 + l^2}{k^2}$$
$$\xi = kx + ly + mt + d \tag{3.9}$$

where k, l, m, d are arbitrary constants.

Substituting the general solutions of Eq.(3.6) into (3.9), we can obtain the traveling wave solutions of (3.1) as follows:

Case (a): when $\lambda^2 - 4\mu > 0$

$$u_1(\xi) = \frac{3}{2}k^2\lambda^2 - \frac{3}{2}k^2(\lambda^2 - 4\mu).$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)^2$$
$$-\frac{1}{2}\frac{k^2 + k^4\lambda^2 + 8k^4\mu - m^2 + l^2}{k^2}$$

where

$$\xi = kx + ly + mt + d$$

k, l, m, d, C_1, C_2 are arbitrary constants.

In particular, if

$$C_1 = 1, \ C_2 = 0, \ \mu = 0$$
$$\lambda = 1, \ k = l = m = d = 1,$$

then we have

$$u(x, y, t) = \frac{1}{2} - \frac{3}{2}[\tanh\frac{1}{2}(x + y + t + 1)]^2.$$

Case (b): when $\lambda^2 - 4\mu < 0$

$$u_2(\xi) = \frac{3}{2}k^2\lambda^2 - \frac{3}{2}k^2(4\mu - \lambda^2).$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi} \right)^2$$
$$-\frac{1}{2}\frac{k^2 + k^4\lambda^2 + 8k^4\mu - m^2 + l^2}{k^2}$$

where

$$\xi = kx + ly + mt + d$$

k, l, m, d, C_1, C_2 are arbitrary constants.

In particular, if

$$C_1 = 1,\ C_2 = 0,\ \mu = 0$$
$$\lambda = 1,\ k = l = m = d = 1,$$

then

$$u(x, y, t) = 6[\tan(x + y + t + 1)]^2 - \frac{9}{2}.$$

Case (c): when $\lambda^2 - 4\mu = 0$

$$u_3(\xi) = \frac{3}{2}k^2\lambda^2 - \frac{6k^2C_2^2}{(C_1 + C_2\xi)^2}$$
$$-\frac{1}{2}\frac{k^2 + k^4\lambda^2 + 8k^4\mu - m^2 + l^2}{k^2}$$

where

$$\xi = kx + ly + mt + d$$

k, l, m, d, C_1, C_2 are arbitrary constants.

In particular, if

$$C_1 = C_2 = 1,\ \mu = 1$$
$$\lambda = 2,\ k = l = m = d = 1,$$

then we have

$$u(x, y, t) = -\frac{1}{2} - \frac{6}{(x + y + t + 2)^2}.$$

4 Application Of The $(\frac{G'}{G})$-Expansion Method For The Two-Dimensional Burgers Equation

In this section, we will consider the two-dimensional Burgers equation [44]:

$$u_t - 2uu_x - u_{xx} - u_{yy} - 2vu_y = 0 \qquad (4.1)$$

$$v_t - 2uv_x - v_{xx} - v_{yy} - 2vv_y = 0 \qquad (4.2)$$

Supposing that

$$\xi = kx + \omega y + st \qquad (4.3)$$

By (4.3), (4.1) and (4.2) are converted into ODEs

$$su' - 2kuu' - (k^2 + \omega^2)u'' - 2\omega vu' = 0 \qquad (4.4)$$
$$sv' - 2kuv' - (k^2 + \omega^2)v'' - 2\omega vv' = 0 \qquad (4.5)$$

Suppose that the solution of (4.4) and (4.5) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i (\frac{G'}{G})^i \qquad (4.6)$$

$$v(\xi) = \sum_{i=0}^{n} b_i (\frac{G'}{G})^i \qquad (4.7)$$

where a_i, b_i are constants, $G = G(\xi)$ satisfies the second order LODE in the form:

$$G'' + \lambda G' + \mu G = 0 \qquad (4.8)$$

where λ and μ are constants.

Balancing the order of uu' and vu' in Eq.(4.4), the order of uv' and v'' in Eq.(4.5), then we can obtain

$$2m+1 = m+n+1,\ m+n+1 = n+2 \Rightarrow\ m = n = 1$$

So Eq.(4.6) and (4.7) can be rewritten as

$$u(\xi) = a_1(\frac{G'}{G}) + a_0,\ a_1 \neq 0 \qquad (4.9)$$

$$v(\xi) = b_1(\frac{G'}{G}) + b_0,\ b_1 \neq 0 \qquad (4.30)$$

$a_1,\ a_0,\ b_1,\ b_0$ are constants to be determined later.

Substituting (4.9) and (4.10) into (4.4) and (4.5) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

For Eq.(4.4):

$$(\frac{G'}{G})^0 :\ 2\omega a_1 b_0\mu - a_1\omega^2\lambda\mu - sa_1\mu$$
$$+2ka_1a_0\mu - a_1k^2\lambda\mu = 0$$

$$(\frac{G'}{G})^1 :\ -a_1\omega^2\lambda^2 - sa_1\lambda + 2ka_1a_0\lambda$$
$$-2a_1k^2\mu + 2ka_1^2\mu - a_1k^2\lambda^2$$
$$+2\omega a_1b_1\mu - 2a_1\omega^2\mu + 2\omega a_1b_0\lambda = 0$$

$$(\frac{G'}{G})^2 :\ -sa_1 + 2\omega a_1b_1\lambda - 3a_1k^2\lambda$$
$$+2ka_1^2\lambda + 2\omega a_1b_o - 3a_1\omega^2\lambda$$
$$+2ka_1a_0 = 0$$

$$(\frac{G'}{G})^3 :\ 2ka_1^2 - 2k^2a_1 - 2a_1\omega^2$$
$$+2a_1b\omega_1 = 0$$

For Eq.(4.5):

$$(\frac{G'}{G})^0 :\ -b_1k^2\lambda\mu + 2kb_1a_0\mu + 2\omega b_1b_0\mu$$
$$-sb_1\mu - b_1\omega^2\lambda\mu = 0$$

$$(\frac{G'}{G})^1 :\ -2b_1\omega^2\mu - 2b_1k^2\mu - b_1k^2\lambda^2$$
$$+2kb_1a_1\mu + +2kb_1a_1\lambda + 2\omega b_1^2\mu$$
$$-b_1\omega^2\lambda^2 - sb_1\lambda + 2\omega b_1b_0\lambda = 0$$

$$(\frac{G'}{G})^2 :\ -3b_1\omega^2\lambda + 2kb_1a_1\lambda + 2\omega b_1b_0$$
$$+2\omega b_1^2\lambda - 3b_1k^2\lambda + 2kb_1a_0 - sb_1 = 0$$

$$(\frac{G'}{G})^3 :\ -2b_1k^2 + 2\omega b_1^2 - 2b_1\omega^2 + 2kb_1a_1 = 0$$

Solving the algebraic equations above yields:

$$a_1 = \frac{k^2 - b_1\omega + \omega^2}{k}$$
$$a_0 = a_0,\ b_1 = b_1,\ b_0 = b_0$$
$$k = k,\ \omega = \omega,\ s = 2\omega b_0 - \omega^2\lambda + 2ka_0 - k^2\lambda \quad (4.31)$$

where a_0, b_1, b_0, k, ω are arbitrary constant, $b_1 \neq 0$.

Substituting (4.11) into (4.9) and (4.10), yields:

$$u(\xi) = \frac{k^2 - b_1\omega + \omega^2}{k}(\frac{G'}{G}) + a_0 \qquad (4.32)$$

$$v(\xi) = b_1(\frac{G'}{G}) + b_0 \qquad (4.13)$$

where

$$\xi = kx + \omega y + (2\omega b_0 - \omega^2\lambda + 2ka_0 - k^2\lambda)t.$$

Substituting the general solutions of (4.8) into (4.12) and (4.13), we have three types of traveling wave solutions of the two-dimensional Burgers equation as follows:

When $\lambda^2 - 4\mu > 0$

$$u_1(\xi) = -\frac{\lambda(k^2 - b_1\omega + \omega^2)}{2k} + \frac{(k^2 - b_1\omega + \omega^2)\sqrt{\lambda^2 - 4\mu}}{2k}.$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right) + a_0$$

$$v_1(\xi) = -\frac{b_1\lambda}{2} + \frac{b_1\sqrt{\lambda^2 - 4\mu}}{2}.$$
$$\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right) + b_0$$

$$\xi = kx + \omega y + (2\omega b_0 - \omega^2\lambda + 2ka_0 - k^2\lambda)t$$

When $\lambda^2 - 4\mu < 0$

$$u_2(\xi) = -\frac{\lambda(k^2 - b_1\omega + \omega^2)}{2k} + \frac{(k^2 - b_1\omega + \omega^2)\sqrt{4\mu - \lambda^2}}{2k}.$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right) + a_0$$

$$v_2(\xi) = -\frac{b_1\lambda}{2} + \frac{b_1\sqrt{4\mu - \lambda^2}}{2}.$$
$$\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right) + b_0$$

$$\xi = kx + \omega y + (2\omega b_0 - \omega^2\lambda + 2ka_0 - k^2\lambda)t$$

When $\lambda^2 - 4\mu = 0$

$$u_3(\xi) = \frac{(k^2 - b_1\omega + \omega^2)(2C_2 - C_1\lambda - C_2\lambda\xi)}{2k(C_1 + C_2\xi)} + a_0$$

$$v_3(\xi) = \frac{-b_1(2C_2 - C_1\lambda - C_2\lambda\xi)}{2(C_1 + C_2\xi)} + b_0$$

$$\xi = kx + \omega y + (2\omega b_0 - \omega^2\lambda + 2ka_0 - k^2\lambda)t$$

5 Conclusions

In this paper, a generalized $(\frac{G'}{G})$-expansion method is used to obtain more general exact solutions of the (2+1) dimensional Boussinesq equation and the two-dimensional Burgers equation. As a result, exact traveling wave solutions with three arbitrary functions are obtained including hyperbolic function solutions, trigonometric function solutions and rational solutions. The arbitrary functions in the obtained solutions imply that these solutions have rich local structures. It may be important to explain some physical phenomena.

The value of the $(\frac{G'}{G})$-expansion method is that one can treat nonlinear problems by essentially linear methods. The method is based on the explicit linearization of NLEEs for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Moreover, it transforms a nonlinear equation to a simple algebraic computation. Compared to the methods used before, one can see that this method is direct, concise and effective. As we can use the MATHEMATICA or MAPLE to find out a useful solution of the algebraic equations resulted, so we can also avoids tedious calculations. This method can also be used to many other nonlinear equations.

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