Alternating Group Explicit-Implicit Method And Crank-Nicolson Method For Convection-Diffusion Equation

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Abstract: Based on the concept of alternating group and domain decomposition, we present a class of alternating group explicit-implicit method and an alternating group Crank-Nicolson method for solving convection-diffusion equation. Both of the two methods are effective in convection dominant cases. The concept of the construction of the methods is also be applied to 2D convection-diffusion equations. Numerical results show the present methods are superior to the known methods in [6,11,16] .

Key–Words: convection-diffusion equation, finite difference, parallel computation, exponential-type transformation, alternating group

1 Preface

Convection-diffusion equation is widely used in describing many physical phenomena such as fluid flowing, river and atmosphere pollution and so on. Researches on numerical finite difference methods for it are popular [1-4]. Many finite difference methods have been presented so far, which are classified into three categories: The explicit, implicit and semi-implicit methods. Most of explicit methods are short in stability and accuracy, while implicit methods are unadaptable for parallel computing, and need to solve large system of equations. D. J. Evans presented an alternating group explicit method (AGE) by the specifical combination of several asymmetric schemes in [5,6]. Because of the parallelism and absolute stability, the AGE method is widely cared and developed by many authors such as Baolin Zhang, Zhiyue Zhang etc in [7-13], while Rohallah Tavakoli derived a class of domain-split method based on the AGE method for 1D and 2D diffusion equation in [14-15]. Most of the developed methods have the same advantage of good stability and parallelism, but have difficulty of computation in the case of small ε. Zhenfu Tian presented a new group explicit method using an exponential-type transformation in [16] , which has advantage of solving the convection-diffusion equation with small ε, but the accurate needs to be increased. Furthermore we notice that AGE method for 2D convection-diffusion equations have been scarcely presented.

Results about existence and uniqueness of theoretic solution for parabolic equations can be found in [17-20].

The construction of this paper is as follows: In section 2 of this paper, an exponential-type transformation [16] is used to get the integral conservative form of convection-diffusion equation, and a class of new unconditionally alternating group explicit conservative finite difference method(AGE-I) is derived using the saul'yev asymmetric schemes and the classical explicit-implicit schemes. An alternating group Crank-Nicolson method(AGC-N) is derived in section 3. In section 4, we applied the concept of the construction of the alternating group method to 2D convection-diffusion equations. Stability analysis is given in section 5. In section 6, numerical experiments on stability and accuracy are presented.

2 The Alternating Group Explicit-Implicit(AGE-I) Method

In this section, We will consider the following convection-diffusion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + k\dfrac{\partial u}{\partial x} = \varepsilon \dfrac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, \quad 0 \le t \le T \\ u(x,0) = f(x), \\ u(0,t) = g_1(t), u(1,t) = g_2(t). \end{cases} \tag{1}$$

The domain $\Omega : (0,1) \times (0,T)$ will be divided into $(m \times N)$ meshes with spatial step size $h = \dfrac{1}{m}$ in x direction and the time step size τ=$\dfrac{T}{N}$. Grid points are denoted by (x_i, t_n), $x_i = ih(i = 0,\ 1, \cdots, m)$, $t_n = n\tau(n = 0,\ 1, \cdots, \frac{T}{\tau})$. The numerical solution of (1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$.

According to [12], the equation (1) is equivalent to $e^{-\frac{kx}{\varepsilon}}\frac{\partial u}{\partial t} = \varepsilon\frac{\partial}{\partial x}(e^{-\frac{kx}{\varepsilon}}\frac{\partial u}{\partial x})$. Integral from $x_{i-\frac{1}{2}}$ to $x_{i+\frac{1}{2}}$, then we have $(\frac{\partial u}{\partial t})_i^n \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} e^{-\frac{kx}{\varepsilon}}dx \approx (\varepsilon\frac{\partial}{\partial x}(e^{-\frac{kx}{\varepsilon}}\frac{\partial u}{\partial x}))_i^n h$.

We can establish the following asymmetry finite difference schemes :

$$(e^{\frac{kh}{2\varepsilon}} - e^{-\frac{kh}{2\varepsilon}})\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{k}{h}[e^{-\frac{kh}{2\varepsilon}}(\frac{u_{i+1}^{n+1} - u_i^{n+1}}{2} + \frac{u_{i+1}^n - u_i^n}{2}) - e^{\frac{kh}{2\varepsilon}}(u_i^n - u_{i-1}^n)] \quad (2)$$

$$(e^{\frac{kh}{2\varepsilon}} - e^{-\frac{kh}{2\varepsilon}})\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{k}{h}[e^{-\frac{kh}{2\varepsilon}}(u_{i+1}^{n+1} - u_i^{n+1}) - e^{\frac{kh}{2\varepsilon}}(\frac{u_i^{n+1} - u_{i-1}^{n+1}}{2} + \frac{u_i^n - u_{i-1}^n}{2})] \quad (3)$$

$$(e^{\frac{kh}{2\varepsilon}} - e^{-\frac{kh}{2\varepsilon}})\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{k}{h}[e^{-\frac{kh}{2\varepsilon}}(\frac{u_{i+1}^{n+1} - u_i^{n+1}}{2} + \frac{u_{i+1}^n - u_i^n}{2}) - e^{\frac{kh}{2\varepsilon}}(u_i^{n+1} - u_{i-1}^{n+1})] \quad (4)$$

$$(e^{\frac{kh}{2\varepsilon}} - e^{-\frac{kh}{2\varepsilon}})\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{k}{h}[e^{-\frac{kh}{2\varepsilon}}(u_{i+1}^n - u_i^n) - e^{\frac{kh}{2\varepsilon}}(\frac{u_i^{n+1} - u_{i-1}^{n+1}}{2} + \frac{u_i^n - u_{i-1}^n}{2})] \quad (5)$$

Let $p = e^{-\frac{kh}{2\varepsilon}},\ q = e^{\frac{kh}{2\varepsilon}},\ r = \frac{k\tau}{h(q-p)}$, then we have :

$$(1+\frac{rp}{2})u_i^{n+1} - \frac{rp}{2}u_{i+1}^{n+1} = rqu_{i-1}^n + [1 - r(\frac{p}{2}+q)]u_i^n + \frac{rp}{2}u_{i+1}^n \quad (6)$$

$$-\frac{rq}{2}u_{i-1}^{n+1} + [1 + r(\frac{q}{2}+p)]u_i^{n+1} - rpu_{i+1}^{n+1} = \frac{rq}{2}u_{i-1}^n + (1 - \frac{rq}{2})u_i^n \quad (7)$$

$$-rqu_{i-1}^{n+1} + [1 + r(\frac{p}{2}+q)]u_i^{n+1} - \frac{rp}{2}u_{i+1}^{n+1} = \frac{rp}{2}u_{i+1}^n + (1 - \frac{rp}{2})u_i^n \quad (8)$$

$$(1+\frac{rq}{2})u_i^{n+1} - \frac{rq}{2}u_{i-1}^{n+1} = \frac{rq}{2}u_{i-1}^n + [1 - r(\frac{q}{2}+p)]u_i^n + rpu_{i+1}^n \quad (9)$$

We will also use the following classical explicit-implicit schemes:

$$u_i^{n+1} = rpu_{i+1}^n + [1 - r(p+q)]u_i^n + rqu_{i-1}^n \quad (10)$$

$$-rpu_{i+1}^{n+1} + [1 - r(p+q)]u_i^{n+1} - rqu_{i-1}^{n+1} = u_i^n \quad (11)$$

Let $m - 1 = (2s_1 + 1)s_2$, here s_1 and s_2 are integers. The purpose of the paper is to get the solution of the $(n + 1)$-th and the $(n + 2)$-th time level with the solution of the n-th time level known. Using the schemes mentioned above, we will have four basic point groups:

"GL"group: $2s_2$ inner points are involved, and $(6), (11), \cdots, (11), (7), (8), (11), \cdots, (11), (9)$ are used respectively.

"HR"group: s_2 inner points are involved, and $(6), (11), \cdots, (11), (7)$ are used respectively.

"HL"group: s_2 inner points are involved, and $(8), (10), \cdots, (10), (9)$ are used respectively.

"GR"group: $2s_2$ inner points are involved, and $(6), (10), \cdots, (10), (7), (8), (10), \cdots, (10), (9)$ are used respectively.

Based on the basic point groups above, the alternating group method will be presented as follows:

First at the $(n + 1)$-th time level, we will have $(s_1 + 1)$ point groups. "GL" are used in the first s_1 point groups respectively, while "HR" are used in the last point group. Second at the $(n + 2)$-th time level, we will still have $(s_1 + 1)$ point groups. "HL" are used in the First point group, while "GR" are used in the right s_1 point groups. Let $U^n = (u_1^n,\ u_2^n,\ \cdots,\ u_{m-1}^n)^T$, we can denote the alternating group explicit-implicit method (AGE-I) as follows:

$$\begin{cases} (I + rG_1)U^{n+1} = (I - rG_2)U^n + F_1^n \\ (I + rG_2)U^{n+2} = (I - rG_1)U^{n+1} + F_2^n \end{cases} \quad (12)$$

here F_1^n and F_2^n are known vectors relevant to the boundary.

$$G_1 = \begin{pmatrix} G_{11} & & & & \\ & G_{11} & & & \\ & & \cdots & & \\ & & & G_{11} & \\ & & & & G_{12} \end{pmatrix}_{(m-1)\times(m-1)}$$

$$G_2 = \begin{pmatrix} G_{21} & & & & \\ & G_{22} & & & \\ & & \cdots & & \\ & & & G_{22} & \\ & & & & G_{22} \end{pmatrix}_{(m-1)\times(m-1)}$$

$$G_{11} = \begin{pmatrix} G_{111} & G_{112} \\ G_{113} & G_{114} \end{pmatrix}$$

$$G_{111} = \begin{pmatrix} \frac{p}{2} & -\frac{p}{2} & & & \\ -q & p+q & -p & & \\ & \cdots & \cdots & \cdots & \\ & & -q & p+q & -p \\ & & & -\frac{q}{2} & p+\frac{q}{2} \end{pmatrix}_{s_2\times s_2}$$

$$G_{112} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ -p & 0 & 0 & 0 & 0 \end{pmatrix}_{s_2\times s_2}$$

$$G_{113} = \begin{pmatrix} 0 & 0 & 0 & 0 & -q \\ 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{s_2\times s_2}$$

$$G_{114} = \begin{pmatrix} \frac{p}{2}+q & -\frac{p}{2} & & & \\ -q & p+q & -p & & \\ & \cdots & \cdots & \cdots & \\ & & -q & p+q & -p \\ & & & -\frac{q}{2} & \frac{q}{2} \end{pmatrix}_{s_2\times s_2}$$

$$G_{12} = \begin{pmatrix} \frac{p}{2} & -\frac{p}{2} & & & \\ -q & p+q & -p & & \\ & \cdots & \cdots & \cdots & \\ & & -q & p+q & -p \\ & & & -\frac{q}{2} & p+\frac{q}{2} \end{pmatrix}_{s_2\times s_2}$$

$$G_{21} = \begin{pmatrix} \frac{p}{2}+q & -\frac{p}{2} & & & \\ 0 & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \\ & & 0 & 0 & 0 \\ & & & -\frac{q}{2} & \frac{q}{2} \end{pmatrix}_{s_2\times s_2}$$

$$G_{22} = \begin{pmatrix} G_{221} & G_{222} \\ G_{223} & G_{224} \end{pmatrix}$$

$$G_{221} = \begin{pmatrix} \frac{p}{2} & -\frac{p}{2} & & & \\ 0 & 0 & 0 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & 0 & 0 & 0 \\ & & & -\frac{q}{2} & p+\frac{q}{2} \end{pmatrix}_{s_2\times s_2}$$

$$G_{222} = G_{112},\ G_{223} = G_{113}$$

$$G_{224} = \begin{pmatrix} \frac{p}{2}+q & -\frac{p}{2} & & & \\ 0 & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \\ & & 0 & 0 & 0 \\ & & & -\frac{q}{2} & \frac{q}{2} \end{pmatrix}_{s_2\times s_2}$$

3 The Alternating Group Crank-Nicolson(AGC-N) Method

First we will present the Crank-Nicolson scheme for solving (1):

$$[1+r(\frac{p}{2}+\frac{q}{2})]u_i^{n+1} - \frac{rp}{2}u_{i+1}^{n+1} - \frac{rq}{2}u_{i-1}^{n+1} =$$

$$\frac{rq}{2}u_{i-1}^{n} + [1-r(\frac{p}{2}+\frac{q}{2})]u_i^{n} + \frac{rp}{2}u_{i+1}^{n} \qquad (13)$$

If we replace (10) and (11) with (13) in section 2, then we can derive the alternating group Crank-Nicolson method (AGC-N) as follows:

$$\begin{cases} (I+r\overline{G}_1)U^{n+1} = (I-r\overline{G}_2)U^n + \overline{F}_1^n \\ (I+r\overline{G}_2)U^{n+2} = (I-r\overline{G}_1)U^{n+1} + \overline{F}_2^n \end{cases} \qquad (14)$$

here $\overline{F}_1^n$ and $\overline{F}_2^n$ are also known vectors relevant to the boundary.

$$\overline{G}_1 = \begin{pmatrix} \overline{G}_{11} & & & & \\ & \overline{G}_{11} & & & \\ & & \cdots & & \\ & & & \overline{G}_{11} & \\ & & & & \overline{G}_{12} \end{pmatrix}_{(m-1)\times(m-1)}$$

$$\overline{G}_2 = \begin{pmatrix} \overline{G}_{21} & & & & \\ & \overline{G}_{22} & & & \\ & & \cdots & & \\ & & & \overline{G}_{22} & \\ & & & & \overline{G}_{22} \end{pmatrix}_{(m-1)\times(m-1)},$$

$$\overline{G}_{11} = \begin{pmatrix} \overline{G}_{111} & \overline{G}_{112} \\ \overline{G}_{113} & \overline{G}_{114} \end{pmatrix}$$

$$\overline{G}_{111} = \begin{pmatrix} \frac{p}{2} & -\frac{p}{2} & & & \\ -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} & & \\ & \cdots & \cdots & \cdots & \\ & & -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} \\ & & & -\frac{q}{2} & p+\frac{q}{2} \end{pmatrix}_{s_2\times s_2}$$

$$\overline{G}_{112} = G_{112},\ \overline{G}_{113} = G_{113}$$

$$\overline{G}_{114}=\begin{pmatrix} \frac{p}{2}+q & -\frac{p}{2} & & & \\ -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} & & \\ & \cdots & \cdots & \cdots & \\ & & -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} \\ & & & -\frac{q}{2} & \frac{q}{2} \end{pmatrix}_{s_2\times s_2}$$

$$\overline{G}_{12}=\begin{pmatrix} \frac{p}{2} & -\frac{p}{2} & & & \\ -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} & & \\ & \cdots & \cdots & \cdots & \\ & & -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} \\ & & & -\frac{q}{2} & p+\frac{q}{2} \end{pmatrix}_{s_2\times s_2}$$

$$\overline{G}_{21}=\begin{pmatrix} \frac{p}{2}+q & -\frac{p}{2} & & & \\ -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} & & \\ & \cdots & \cdots & \cdots & \\ & & -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} \\ & & & -\frac{q}{2} & \frac{q}{2} \end{pmatrix}_{s_2\times s_2},$$

$$\overline{G}_2=\begin{pmatrix} \overline{G}_{221} & \overline{G}_{222} \\ \overline{G}_{223} & \overline{G}_{224} \end{pmatrix}$$

$$\overline{G}_{221}=\begin{pmatrix} \frac{p}{2} & -\frac{p}{2} & & & \\ -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} & & \\ & \cdots & \cdots & \cdots & \\ & & -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} \\ & & & -\frac{q}{2} & p+\frac{q}{2} \end{pmatrix}_{s_2\times s_2}$$

$$\overline{G}_{222}=G_{222},\ \overline{G}_{223}=G_{223}$$

$$\overline{G}_{224}=\begin{pmatrix} \frac{p}{2}+q & -\frac{p}{2} & & & \\ -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} & & \\ & \cdots & \cdots & \cdots & \\ & & -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} \\ & & & -\frac{q}{2} & \frac{q}{2} \end{pmatrix}_{s_2\times s_2}$$

4 The alternating group method for 2D convection-diffusion equation

We consider the initial boundary problem of the 2D convection-diffusion equation as below:

$$\begin{cases} \frac{\partial u}{\partial t}+k\frac{\partial u}{\partial x}+k\frac{\partial u}{\partial y}=\varepsilon\frac{\partial^2 u}{\partial x^2}+\varepsilon\frac{\partial^2 u}{\partial y^2}, \\ 0\le x\le 2, 0\le y\le 2, 0\le t\le T, \\ u(x,y,0)=f(x,y), \\ u(0,y,t)=g_1(t), u(2,y,t)=g_2(t). \\ u(x,0,t)=h_1(t), u(2,y,t)=h_2(t). \end{cases} \quad (15)$$

The domain $\Omega:(0,2)\times(0,2)\times(0,T)$ will be divided into $(m\times m\times p)$ meshes with spatial step size h=$\frac{2}{m}$ in x and y direction and the time step size τ=$\frac{T}{p}$. Grid points are denoted by (x_i,y_j,t_n) or by (i,j,n), $x_i=ih, y_j=jh(i,j=0,\ 1,\cdots,m),\ t_n=n\tau(n=0,\ 1,\cdots,\frac{T}{\tau})$. The numerical solution of (15) is denoted by $u_{i,j}^n$, while the exact solution $u(x_i,y_j,t_n)$. r,p,q are defined the same as above.

From (15) we can see $e^{-\frac{kx}{\varepsilon}-\frac{ky}{\varepsilon}}\frac{\partial u}{\partial t}=\varepsilon\frac{\partial}{\partial x}(e^{-\frac{kx}{\varepsilon}-\frac{ky}{\varepsilon}}\frac{\partial u}{\partial x})+\varepsilon\frac{\partial}{\partial y}(e^{-\frac{kx}{\varepsilon}-\frac{ky}{\varepsilon}}\frac{\partial u}{\partial y})$. In order to get the solution of (n+1)-th time level with the solution of n-th time level known, we integral (15) in the domain" $x_{i-\frac{1}{2}}$ to $x_{i+\frac{1}{2}}$,$y_{j-\frac{1}{2}}$ to $y_{j+\frac{1}{2}}$ ", then we can establish the following asymmetry finite difference schemes: (16)-(31).

$$[1+rp]u_{i,j}^{n+1}-\frac{rp}{2}u_{i+1,j}^{n+1}-\frac{rp}{2}u_{i,j+1}^{n+1}=rqu_{i-1,j}^n+\frac{rp}{2}u_{i+1,j}^n+[1-r(p+2q)]u_{i,j}^n+rqu_{i,j-1}^n+\frac{rp}{2}u_{i,j+1}^n \quad (16)$$

$$[1+r(\frac{q}{2}+\frac{3p}{2})]u_{i,j}^{n+1}-\frac{rq}{2}u_{i-1,j}^{n+1}-rpu_{i+1,j}^{n+1}-\frac{rp}{2}u_{i,j+1}^{n+1}=\frac{rq}{2}u_{i-1,j}^n+[1-r(\frac{3q}{2}+\frac{p}{2})]u_{i,j}^n+rqu_{i,j-1}^n+\frac{rp}{2}u_{i,j+1}^n \quad (17)$$

$$[1+r(p+q)]u_{i,j}^{n+1}-rqu_{i-1,j}^{n+1}-\frac{rp}{2}u_{i+1,j}^{n+1}-\frac{rp}{2}u_{i,j+1}^{n+1}=\frac{rp}{2}u_{i+1,j}^n+[1-r(p+q)]u_{i,j}^n+rqu_{i,j-1}^n+\frac{rp}{2}u_{i,j+1}^n \quad (18)$$

$$[1+r(\frac{q}{2}+\frac{p}{2})]u_{i,j}^{n+1}-\frac{rq}{2}u_{i-1,j}^{n+1}-\frac{rp}{2}u_{i,j+1}^{n+1}=rpu_{i+1,j}^n+\frac{rq}{2}u_{i-1,j}^n+[1-r(\frac{3q}{2}+\frac{3p}{2})]u_{i,j}^n+rqu_{i,j-1}^n+\frac{rp}{2}u_{i,j+1}^n \quad (19)$$

$$[1+r(\frac{q}{2}+\frac{3p}{2})]u_{i,j}^{n+1}-\frac{rp}{2}u_{i+1,j}^{n+1}-\frac{rq}{2}u_{i,j-1}^{n+1}-rpu_{i,j+1}^{n+1}=rqu_{i-1,j}^n+\frac{rp}{2}u_{i+1,j}^n+[1-r(\frac{3q}{2}+\frac{p}{2})]u_{i,j}^n+\frac{rq}{2}u_{i,j-1}^n \quad (20)$$

$$[1+r(q+2p)]u_{i,j}^{n+1}-\frac{rq}{2}u_{i-1,j}^{n+1}-rpu_{i+1,j}^{n+1}-\frac{rq}{2}u_{i,j-1}^{n+1}-rpu_{i,j+1}^{n+1}=\frac{rq}{2}u_{i-1,j}^n+[1-rq]u_{i,j}^n+\frac{rq}{2}u_{i,j-1}^n \quad (21)$$

$$[1+r(\frac{3q}{2}+\frac{3p}{2})]u_{i,j}^{n+1}-\frac{rp}{2}u_{i+1,j}^{n+1}-rqu_{i-1,j}^{n+1}-\frac{rq}{2}u_{i,j-1}^{n+1}$$

$$-rpu_{i,j+1}^{n+1}=\frac{rp}{2}u_{i+1,j}^{n}+[1-r(\frac{q}{2}+\frac{p}{2})]u_{i,j}^{n}+\frac{rq}{2}u_{i,j-1}^{n} \quad (22)$$

$$[1+r(p+q)]u_{i,j}^{n+1}-\frac{rq}{2}u_{i-1,j}^{n+1}-\frac{rq}{2}u_{i,j-1}^{n+1}-rpu_{i,j+1}^{n+1}$$
$$=rpu_{i+1,j}^{n}+\frac{rq}{2}u_{i-1,j}^{n}+[1-r(p+q)]u_{i,j}^{n}+\frac{rq}{2}u_{i,j-1}^{n} \quad (23)$$

$$[1+r(p+q)]u_{i,j}^{n+1}-\frac{rp}{2}u_{i+1,j}^{n+1}-\frac{rp}{2}u_{i,j+1}^{n+1}-rqu_{i,j-1}^{n+1}$$
$$=rqu_{i-1,j}^{n}+\frac{rp}{2}u_{i+1,j}^{n}+[1-r(p+q)]u_{i,j}^{n}+\frac{rp}{2}u_{i,j+1}^{n} \quad (24)$$

$$[1+r(\frac{3q}{2}+\frac{3p}{2})]u_{i,j}^{n+1}-\frac{rq}{2}u_{i-1,j}^{n+1}-rpu_{i+1,j}^{n+1}-\frac{rp}{2}u_{i,j+1}^{n+1}$$
$$-rqu_{i,j-1}^{n+1}=\frac{rq}{2}u_{i-1,j}^{n}+[1-r(\frac{q}{2}+\frac{p}{2})]u_{i,j}^{n}+\frac{rp}{2}u_{i,j+1}^{n} \quad (25)$$

$$[1+r(p+2q)]u_{i,j}^{n+1}-\frac{rp}{2}u_{i+1,j}^{n+1}-rqu_{i-1,j}^{n+1}-\frac{rp}{2}u_{i,j+1}^{n+1}$$
$$-rqu_{i,j-1}^{n+1}=\frac{rp}{2}u_{i+1,j}^{n}+[1-rp]u_{i,j}^{n}+\frac{rp}{2}u_{i,j+1}^{n} \quad (26)$$

$$[1+r(\frac{3q}{2}+\frac{p}{2})]u_{i,j}^{n+1}-\frac{rq}{2}u_{i-1,j}^{n+1}-\frac{rp}{2}u_{i,j+1}^{n+1}-rqu_{i,j-1}^{n+1}$$
$$=rpu_{i+1,j}^{n}+\frac{rq}{2}u_{i-1,j}^{n}+[1-r(\frac{q}{2}+\frac{3p}{2})]u_{i,j}^{n}+\frac{rp}{2}u_{i,j+1}^{n} \quad (27)$$

$$[1+r(\frac{q}{2}+\frac{p}{2})]u_{i,j}^{n+1}-\frac{rp}{2}u_{i+1,j}^{n+1}-\frac{rq}{2}u_{i,j-1}^{n+1}=rqu_{i-1,j}^{n}$$
$$+\frac{rp}{2}u_{i+1,j}^{n}+[1-r(\frac{3q}{2}+\frac{3p}{2})]u_{i,j}^{n}+\frac{rq}{2}u_{i,j-1}^{n}+rpu_{i,j+1}^{n} \quad (28)$$

$$[1+r(p+q)]u_{i,j}^{n+1}-\frac{rq}{2}u_{i-1,j}^{n+1}-rpu_{i+1,j}^{n+1}-\frac{rq}{2}u_{i,j-1}^{n+1}$$
$$=\frac{rq}{2}u_{i-1,j}^{n}+[1-r(p+q)]u_{i,j}^{n}+\frac{rq}{2}u_{i,j-1}^{n}+rpu_{i,j+1}^{n} \quad (29)$$

$$[1+r(\frac{3q}{2}+\frac{p}{2})]u_{i,j}^{n+1}-\frac{rp}{2}u_{i+1,j}^{n+1}-rqu_{i-1,j}^{n+1}-\frac{rq}{2}u_{i,j-1}^{n+1}$$
$$=\frac{rp}{2}u_{i+1,j}^{n}+[1-r(\frac{q}{2}+\frac{3p}{2})]u_{i,j}^{n}+\frac{rq}{2}u_{i,j-1}^{n}+rpu_{i,j+1}^{n} \quad (30)$$

$$[1+rq]u_{i,j}^{n+1}-\frac{rq}{2}u_{i-1,j}^{n+1}-\frac{rq}{2}u_{i,j-1}^{n+1}=rpu_{i+1,j}^{n}$$
$$+\frac{rq}{2}u_{i-1,j}^{n}+[1-r(2p+q)]u_{i,j}^{n}+\frac{rq}{2}u_{i,j-1}^{n}+rpu_{i,j+1}^{n} \quad (31)$$

The sixteen schemes (16)-(31) compose the ”16-point” group, which will be applied to get the solution on 16 grids points $(i,j,n+1)$, $(i+1,j,n+1)$, $(i+2,j,n+1)$, $(i+3,j,n+1)$, $(i,j+1,n+1)$, $(i+1,j+1,n+1)$, $(i+2,j+1,n+1)$, $(i+3,j+1,n+1)$, $(i,j+2,n+1)$, $(i+1,j+2,n+1)$, $(i+2,j+2,n+1)$, $(i+3,j+2,n+1)$, $(i,j+3,n+1)$, $(i+1,j+3,n+1)$, $(i+2,j+3,n+1)$, $(i+3,j+3,n+1)$. Let
$\overline{u}_{i,j}^{n}=(u_{j}^{n},u_{j+1}^{n},u_{j+2}^{n},u_{j+3}^{n})^{T}$,
$u_{j+k}^{n}=(u_{i,j+k}^{n},u_{i+1,j+k}^{n},u_{i+2,j+k}^{n},u_{i+3,j+k}^{n})^{T}$, $k=0,1,2,3$,
$\overline{F}_{i,j}^{n}=(F_{j}^{n},F_{j+1}^{n},F_{j+2}^{n},F_{j+3}^{n})^{T}$,
$F_{j}^{n}=(rqu_{i-1,j}^{n}+rqu_{i,j-1}^{n},rqu_{i+1,j-1}^{n},rqu_{i+2,j-1}^{n},rqu_{i+3,j-1}^{n}+rpu_{i+4,j}^{n})^{T}$,
$F_{j+1}^{n}=(rqu_{i-1,j+1}^{n},0,0,rpu_{i+4,j+1}^{n})^{T}$,
$F_{j+2}^{n}=(rqu_{i-1,j+2}^{n},0,0,rpu_{i+4,j+2}^{n})^{T}$,
$F_{j+3}^{n}=(rqu_{i-1,j+3}^{n}+rpu_{i,j+4}^{n},rpu_{i+1,j+4}^{n},rpu_{i+2,j+4}^{n},rpu_{i+3,j+4}^{n}+rpu_{i+4,j+3}^{n})^{T}$, then the schemes of the basic ”16-point” group can be presented as below:

$$(I+rA_1)\overline{u}_{i,j}^{n+1}=(I-r\overline{A}_1)\overline{u}_{i,j}^{n}+\overline{F}_{i,j}^{n},. \quad (32)$$

$$A_1=\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14}\\ A_{21} & A_{22} & A_{23} & A_{24}\\ A_{31} & A_{32} & A_{33} & A_{34}\\ A_{41} & A_{42} & A_{43} & A_{44}\end{pmatrix}$$

$$A_{11}=\begin{pmatrix} p & -\frac{p}{2} & 0 & 0\\ -\frac{q}{2} & \frac{q}{2}+\frac{3p}{2} & -p & 0\\ 0 & -q & p+q & -\frac{p}{2}\\ 0 & 0 & -\frac{q}{2} & \frac{p}{2}+\frac{q}{2}\end{pmatrix}$$

$$A_{12}=\begin{pmatrix} -\frac{p}{2} & 0 & 0 & 0\\ 0 & -\frac{p}{2} & 0 & 0\\ 0 & 0 & -\frac{p}{2} & 0\\ 0 & 0 & 0 & -\frac{p}{2}\end{pmatrix}$$

$$A_{21}=\begin{pmatrix} -\frac{q}{2} & 0 & 0 & 0\\ 0 & -\frac{q}{2} & 0 & 0\\ 0 & 0 & -\frac{q}{2} & 0\\ 0 & 0 & 0 & -\frac{q}{2}\end{pmatrix}$$

$$A_{22}=\begin{pmatrix} \frac{3p}{2}+\frac{q}{2} & -\frac{p}{2} & 0 & 0\\ -\frac{q}{2} & 2p+q & -p & 0\\ 0 & -q & \frac{3p}{2}+\frac{3q}{2} & -\frac{p}{2}\\ 0 & 0 & -\frac{q}{2} & p+q\end{pmatrix}$$

$$A_{13} = A_{14} = A_{24} = O$$

$$A_{23} = \begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

$$A_{31} = A_{41} = A_{42} = O$$

$$A_{32} = \begin{pmatrix} -q & 0 & 0 & 0 \\ 0 & -q & 0 & 0 \\ 0 & 0 & -q & 0 \\ 0 & 0 & 0 & -q \end{pmatrix}$$

$$A_{33} = \begin{pmatrix} p+q & -\frac{p}{2} & 0 & 0 \\ -\frac{q}{2} & \frac{3p}{2}+\frac{3q}{2} & -p & 0 \\ 0 & -q & p+2q & -\frac{p}{2} \\ 0 & 0 & -\frac{q}{2} & \frac{p}{2}+\frac{3q}{2} \end{pmatrix}$$

$$A_{34} = \begin{pmatrix} -\frac{p}{2} & 0 & 0 & 0 \\ 0 & -\frac{p}{2} & 0 & 0 \\ 0 & 0 & -\frac{p}{2} & 0 \\ 0 & 0 & 0 & -\frac{p}{2} \end{pmatrix}$$

$$A_{43} = \begin{pmatrix} -\frac{q}{2} & 0 & 0 & 0 \\ 0 & -\frac{q}{2} & 0 & 0 \\ 0 & 0 & -\frac{q}{2} & 0 \\ 0 & 0 & 0 & -\frac{q}{2} \end{pmatrix}$$

$$A_{44} = \begin{pmatrix} \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} & 0 & 0 \\ -\frac{q}{2} & p+q & -p & 0 \\ 0 & -q & \frac{p}{2}+\frac{3q}{2} & 0 \\ 0 & 0 & -\frac{q}{2} & q \end{pmatrix}$$

$$\overline{A}_1 = \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} & \overline{A}_{13} & \overline{A}_{14} \\ \overline{A}_{21} & \overline{A}_{22} & \overline{A}_{23} & \overline{A}_{24} \\ \overline{A}_{31} & \overline{A}_{32} & \overline{A}_{33} & \overline{A}_{34} \\ \overline{A}_{41} & \overline{A}_{42} & \overline{A}_{43} & \overline{A}_{44} \end{pmatrix}$$

$$\overline{A}_{11} = \begin{pmatrix} p+2q & -\frac{p}{2} & 0 & 0 \\ -\frac{q}{2} & \frac{p}{2}+\frac{3q}{2} & -p & 0 \\ 0 & -q & p+q & -\frac{p}{2} \\ 0 & 0 & -\frac{q}{2} & \frac{3p}{2}+\frac{3q}{2} \end{pmatrix}$$

$$\overline{A}_{12} = \begin{pmatrix} -\frac{p}{2} & 0 & 0 & 0 \\ 0 & -\frac{p}{2} & 0 & 0 \\ 0 & 0 & -\frac{p}{2} & 0 \\ 0 & 0 & 0 & -\frac{p}{2} \end{pmatrix}$$

$$\overline{A}_{21} = \begin{pmatrix} -\frac{q}{2} & 0 & 0 & 0 \\ 0 & -\frac{q}{2} & 0 & 0 \\ 0 & 0 & -\frac{q}{2} & 0 \\ 0 & 0 & 0 & -\frac{q}{2} \end{pmatrix}$$

$$\overline{A}_{22} = \begin{pmatrix} \frac{p}{2}+\frac{3q}{2} & -\frac{p}{2} & 0 & 0 \\ -\frac{q}{2} & q & -p & 0 \\ 0 & -q & \frac{p}{2}+\frac{q}{2} & -\frac{p}{2} \\ 0 & 0 & -\frac{q}{2} & p+q \end{pmatrix}$$

$$\overline{A}_{23} = A_{23},\ \overline{A}_{13} = \overline{A}_{14} = \overline{A}_{24} = O$$

$$\overline{A}_{32} = A_{32},\ \overline{A}_{31} = \overline{A}_{41} = \overline{A}_{42} = O$$

$$\overline{A}_{33} = \begin{pmatrix} p+q & -\frac{p}{2} & 0 & 0 \\ -\frac{q}{2} & \frac{p}{2}+\frac{q}{2} & -p & 0 \\ 0 & -q & p & -\frac{p}{2} \\ 0 & 0 & -\frac{q}{2} & \frac{3p}{2}+\frac{q}{2} \end{pmatrix}$$

$$\overline{A}_{34} = \begin{pmatrix} -\frac{p}{2} & 0 & 0 & 0 \\ 0 & -\frac{p}{2} & 0 & 0 \\ 0 & 0 & -\frac{p}{2} & 0 \\ 0 & 0 & 0 & -\frac{p}{2} \end{pmatrix}$$

$$\overline{A}_{43} = \begin{pmatrix} -\frac{q}{2} & 0 & 0 & 0 \\ 0 & -\frac{q}{2} & 0 & 0 \\ 0 & 0 & -\frac{q}{2} & 0 \\ 0 & 0 & 0 & -\frac{q}{2} \end{pmatrix}$$

$$\overline{A}_{44} = \begin{pmatrix} \frac{3p}{2}+\frac{3q}{2} & -\frac{p}{2} & 0 & 0 \\ -\frac{q}{2} & p+q & -p & 0 \\ 0 & -q & \frac{3p}{2}+\frac{q}{2} & 0 \\ 0 & 0 & -\frac{q}{2} & q+2p \end{pmatrix}$$

our purpose is to get the solution of the $(n+1)-th$ and the $(n+2)-th$ time level with the solution of the $n-th$ time level known. Let $(m-1) = 4s$,s is an integer. then the alternating group method is described as following:

First at the $(n+1)-th$ time level, we will have s^2 point groups. 16 point groups are applied to solve $\overline{u}_{i,j}^{n+1}$;

Second at the $(n+2)-th$ time level, we will have $(s+1)^2$ point groups:

(26), (27), (30), (31) are applied to solve $(u_{1,1}^{n+2}, u_{2,1}^{n+2}, u_{1,2}^{n+2}, u_{2,2}^{n+2})$, which marks "H1" group.

(24), (25), (28), (29) are applied to solve $(u_{m-2,1}^{n+2}, u_{m-1,1}^{n+2}, u_{m-2,2}^{n+2}, u_{m-1,2}^{n+2})$, which marks "H2" group.

(18), (19), (22), (23) are applied to solve $(u_{1,m-2}^{n+2}, u_{2,m-2}^{n+2}, u_{1,m-1}^{n+2}, u_{2,m-1}^{n+2})$, which marks "H3" group.

(16), (17), (20), (21) are applied to solve $(u_{m-2,m-2}^{n+2}, u_{m-1,m-2}^{n+2}, u_{m-2,m-1}^{n+2}, u_{m-1,m-1}^{n+2})$, which marks "H4" group.

(18),(19),(22),(23),(26),(27),(30),(31) are applied to solve $(u_{1,j}^{n+2}, u_{2,j}^{n+2}, u_{1,j+1}^{n+2}, u_{2,j+1}^{n+2}, u_{1,j+2}^{n+2}, u_{2,j+2}^{n+2}, u_{1,j+3}^{n+2}, u_{2,j+3}^{n+2})$, $j = 3, 7, \cdots, m-6$, which marks "Lx"group.

(16),(17),(20),(21),(24),(25),(28),(29) are applied to solve $(u_{m-2,j}^{n+2}, u_{m-1,j}^{n+2}, u_{m-2,j+1}^{n+2}, u_{m-1,j+1}^{n+2}, u_{m-2,j+2}^{n+2}, u_{m-1,j+2}^{n+2}, u_{m-2,j+3}^{n+2}, u_{m-1,j+3}^{n+2})$, $j = 3, 7, \cdots, m-6$, which marks "Rx"group.

(24),(25),(26),(27),(28),(29),(30),(31) are applied to solve $(u_{i,1}^{n+2}, u_{i+1,1}^{n+2}, u_{i+2,1}^{n+2}, u_{i+3,1}^{n+2}, u_{i,2}^{n+2}, u_{i+1,2}^{n+2}, u_{i+2,2}^{n+2}, u_{i+3,2}^{n+2})$, $i = 3, 7, \cdots, m-6$, which marks "Ly"group.

(16),(17),(18),(19),(20),(21),(22),(23) are applied to solve $(u_{i,m-2}^{n+2}, u_{i+1,m-2}^{n+2}, u_{i+2,m-2}^{n+2}, u_{i+3,m-2}^{n+2}, u_{i,m-1}^{n+2}, u_{i+1,m-1}^{n+2}, u_{i+2,m-1}^{n+2}, u_{i+3,m-1}^{n+2})$, $i = 3, 7, \cdots, m-6$, which marks "Ry"group.

"16 point" group are applied to $\overline{u}_{i,j}^{n+2}, i, j = 3, 7, \cdots, m-6$.

Thus the alternating group method is established by alternating use of the schemes in the two time levels, and computation on each group can be done independently, which shows the method is parallel. Based on the above, the alternating group explicit method can be presented as following:

Let $\overline{U}^n = (\overline{u}_1^n, \overline{u}_5^n, \overline{u}_9^n, \cdots, \overline{u}_{m-4}^n)^T$, $\overline{u}_j^n = (\overline{u}_{1,j}^n, \overline{u}_{5,j}^n, \overline{u}_{9,j}^n, \cdots, \overline{u}_{m-4,j}^n)^T, j = 1, 5, 9, \cdots, m-4$. then we have

$$\left\{\begin{array}{l} (I + r\widehat{G}_1)\overline{U}^{n+1} = (I - r\widehat{G}_2)\overline{U}^n + \widehat{F}_1^n \\ (I + r\widehat{G}_2)\overline{U}^{n+2} = (I - r\widehat{G}_1)\overline{U}^{n+1} + \widehat{F}_2^n \end{array}\right. \quad (33)$$

here $\widehat{F}_1^n$ and $\widehat{F}_2^n$ are vectors known related to boundary. Let $a = (m-1)^2,\ b = 4(m-1)$, then

$$\widehat{G}_1 = \begin{pmatrix} \widehat{G}_{11} & & \\ & \widehat{G}_{11} & \\ & & \widehat{G}_{11} \end{pmatrix}_{a\times a}$$

$$\widehat{G}_{11} = \begin{pmatrix} A_1 & & \\ & A_1 & \\ & & A_1 \end{pmatrix}_{b\times b}$$

$$\widehat{G}_2 = \begin{pmatrix} \widehat{G}_{21} & E & & & \\ F & \widehat{G}_{21} & E & & \\ & ... & ... & ... & \\ & & F & \widehat{G}_{21} & E \\ & & & F & \widehat{G}_{21} \end{pmatrix}_{b\times b}$$

$$\widehat{G}_{21} = \begin{pmatrix} \overline{A}_1 & B & & & \\ C & \overline{A}_1 & B & & \\ & ... & ... & ... & \\ & & C & \overline{A}_1 & B \\ & & & C & \overline{A}_1 \end{pmatrix}_{b\times b}$$

$$B = \begin{pmatrix} B_1 & & & \\ & B_1 & & \\ & & B_1 & \\ & & & B_1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} C_1 & & & \\ & C_1 & & \\ & & C_1 & \\ & & & C_1 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & -q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} O & O \\ E_1 & O \end{pmatrix}$$

$$E_1 = \begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

$$F = \begin{pmatrix} O & F_1 \\ O & O \end{pmatrix}$$

$$F_1 = \begin{pmatrix} -q & 0 & 0 & 0 \\ 0 & -q & 0 & 0 \\ 0 & 0 & -q & 0 \\ 0 & 0 & 0 & -q \end{pmatrix}$$

5 Stability Analysis

Theorem 1 The AGE-I method (12) is unconditionally stable.

Proof: Let n be an even number, then we have

$$U^n = GU^{n-2} + F^{n-2} = G^{\frac{n}{2}}U^0 + \sum_{k=1}^{\frac{n}{2}} F^{n-2k}G^{k-1}$$

here

$$G = (I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1}(I - rG_2).$$

F^{n-k} is definite vector, $F^{n-k} = (I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1}F_1^{n-k} + (I + rG_2)^{-1}F_2^{n-k}.(k = 2, 4, \cdots, n)$. Obviously G_1 and G_2 are both diagonally dominant matrices, which shows G_1 and G_2 and are both nonnegative definite real matrices. From [21] we have:

$$\|(I+rG_1)^{-1}\|_2 \le 1, \|(I-rG_1)(I+rG_1)^{-1}\|_2 \le 1$$

$$\|(I+rG_2)^{-1}\|_2 \le 1, \|(I-rG_2)(I+rG_2)^{-1}\|_2 \le 1.$$

Let $\widetilde{G} = (I + rG_2)G(I + rG_2)^{-1} = (I - rG_1)(I + rG_1)^{-1}(I - rG_2)(I + rG_2)^{-1}$, then we have $\rho(G) = \rho(\widetilde{G}) \le \|\widetilde{G}\|_2 \le 1$. Therefore $\|G^k\| \le M$, here M is an positive number, which shows the method presented by (12) is unconditionally stable.

Similarly we have:

Theorem 2 The AGC-N method (14) is unconditionally stable.

Theorem 3 The alternating group explicit method denoted by (33) is unconditionally stable.

6 Numerical Experiments

Example 1: We consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} + k\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, \quad 0 \le t \le T \\ u(x,0) = 0, \\ u(0,t) = 0, u(1,t) = 1. \end{cases} \tag{34}$$

The exact solution of the problem above is denoted in [6] as below:

$$u(x,t) = \frac{e^{\frac{kx}{\varepsilon}} - 1}{e^{k\varepsilon} - 1}$$

$$+\sum_{n=1}^{\infty} \frac{(-1)^n n\pi}{(n\pi)^2 + (\frac{k}{2\varepsilon})^2} e^{\frac{k(x-1)}{2\varepsilon}} sin(n\pi x) e^{-[(n\pi)^2\varepsilon + \frac{k^2}{4\varepsilon}]t}$$

Let A.E.=$|u_i^n - u(x_i,t_n)|$ and P.E.=$100 \times \frac{|u_i^n - u(x_i,t_n)|}{u(x_i,t_n)}$ denote maximum absolute error and relevant error respectively, while let $s_2 = 3, m = 16$. We compare the numerical results of (12) and (14) with the results in [6,11,16] in Table 1,2:

Table 1: Numerical results of comparison $k = 1$

	$\tau = 0.01, t = 120\tau, \varepsilon = 1$
A.E.(AGE-I)	8.785×10^{-7}
A.E.(AGC-N)	6.335×10^{-7}
A.E.[16]	1.298×10^{-6}
A.E.[11]	1.161×10^{-6}
A.E.(Evans)	3.799×10^{-5}
P.E.(AGE-I)	3.681×10^{-4}
P.E.(AGC-N)	2.717×10^{-4}
P.E.[16]	5.352×10^{-3}
P.E.[11]	4.183×10^{-3}
P.E.(Evans)	1.739×10^{-2}

Table 2: Numerical results of comparison $k = 1$

	$\tau = 0.0001, t = 100\tau,\ \varepsilon = 0.01$
A.E.(AGE-I)	6.850×10^{37}
A.E.(AGC-N)	3.951×10^{37}
A.E.[16]	1.151×10^{38}
A.E.[11]	9.564×10^{37}
A.E.(Evans)	7.741×10^{38}
P.E.(AGE-I)	8.155×10^{-1}
P.E.(AGC-N)	5.836×10^{-1}
P.E.[16]	5.243
P.E.[11]	4.614
P.E.(Evans)	7.488

Example 2: Consider the equation

$$\frac{\partial u}{\partial t} + k\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}, \quad -L \le x \le L, t > 0$$

with the boundary condition(L=2):

$$u(0,t) = 1.0, u(2,t) = 0.0, t > 0. \tag{35}$$

and the initial condition:

$$u(x,0) = \begin{cases} 1.0, & -2 \le x < 0, \\ 0.5, & x = 0, \\ 0.0, & 0 < x \le 2. \end{cases} \tag{36}$$

The exact solution of the problem above is denoted as below:

$$u(x,t) =$$

$$\frac{1}{2} - \frac{2}{\pi}\sum_{m=1}^{n} sin[(2m-1)\frac{\pi(x-kt)}{L}]\frac{exp(-\varepsilon(2m-1)^2\pi^2 t/L^2)}{2m-1}.$$

We compare the present method with the methods in [6,11,16].

Table 3: Numerical results of comparison at $m = 154, \tau = 10^{-4}, k = 1, \varepsilon = 1$

	$t = 1000\tau$	$t = 2000\tau$	$t = 3000\tau$
P.E.(AGE-I)	6.815×10^{-4}	9.008×10^{-4}	8.507×10^{-3}
P.E.(AGC-N)	3.144×10^{-4}	4.381×10^{-4}	4.187×10^{-3}
P.E.[6](Evans)	4.262×10^{-2}	7.323×10^{-2}	8.952×10^{-1}
P.E.[16]	8.216×10^{-3}	1.357×10^{-1}	3.568×10^{-1}
P.E.[11]	2.931×10^{-3}	5.648×10^{-2}	3.429×10^{-2}

Example 3: Consider the equation

$$\frac{\partial u}{\partial t} + k\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}, \ -2 \le x \le 2, t > 0$$

with the boundary condition:

$$u(-2,t) = u(2,t) = -sin(\pi kt)e^{-\varepsilon\pi^2 t}, t > 0. \quad (37)$$

and the initial condition:

$$u(x,0) = sin(\pi x). \quad (38)$$

The exact solution of the problem above is denoted as below:

$$u(x,t) = sin[\pi(x-kt)]e^{-\varepsilon\pi^2 t}$$

We also compare the numerical results of the present method with the methods in [6,11,16].

Table 4: Numerical results of comparison at $m = 154, \tau = 10^{-4}, k = 1, \varepsilon = 0.01$

	$t = 1000\tau$	$t = 2000\tau$	$t = 3000\tau$
P.E.(AGE-I)	5.586 $\times 10^{-1}$	9.384$\times 10^{-1}$	1.017
P.E.(AGC-N)	1.151 $\times 10^{-1}$	4.612$\times 10^{-1}$	8.235$\times 10^{-1}$
P.E.[6](Evans)	Invalid	Invalid	Invalid
P.E.[16]	Invalid	Invalid	Invalid
P.E.[11]	Invalid	Invalid	Invalid

From the results in Table 1,2,3,4 we can see that the AGE-I method and the AGC-N method are of higher accurate than the methods in [2,7,9] especially when ε is small, that is, convection dominant cases.

7 Conclusions

In this paper, we present two methods for 1D convection-diffusion equations based on the concept of domain decomposition and alternating group. Both of the two methods are effective in convection dominant cases. Numerical results show the two methods are superior to the known methods in [6,11,16]. Furthermore we apply the concept of the construction of the methods to 2D convection-diffusion equations, and derive a new alternating group explicit method. All of the three methods have the property of intrinsic parallelism, and suitable for parallel computation.

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