Parallel Difference Method On Diffusion Equations

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Abstract: In this paper, we present a high order unconditionally stable implicit scheme for diffusion equations. Based on the scheme a class of parallel alternating group explicit method is derived, and stability analysis is given. Then we present another parallel alternating group explicit iterative method, and finish the convergence analysis. Numerical experiments show that the two methods are of higher accuracy than the original alternating group method.

Key–Words: diffusion equation, parallel computation, finite difference, iterative method, alternating group

1 Introduction

In this paper, we will consider the following initial boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1,\ 0 \le t \le T \\ u(x,0) = f(x), \\ u(0,t) = g_1(t),\ u(1,t) = g_2(t). \end{cases} \tag{1}$$

In scientific and engineering computing, we need to solve large system of equations by numerical methods. Finite difference method is one of the most frequently used numerical methods in solving differential equations [1-4]. As we all know, Most of explicit methods are short in stability and accuracy, while implicit methods usually have good stability, but are complex in computing, and need to solve large equation set in the cost of large memory spaces and CPU cycles. Thus it is necessary to construct methods with the advantages of explicit methods and implicit methods, that is, simple for computation and good stability. Recently with the development of parallel computer many scientists payed much attention to the finite difference methods with the property of parallelism. D. J. Evans presented an AGE method in [5] originally. The AGE method is used in computing by applying the special combination of several asymmetry schemes to a group of grid points, and then the numerical solutions at the group of points can be worked out in many groups independently. Furthermore, by alternating use of asymmetry schemes at adherent grid points and different time levels, the AGE method can lead to the counteraction of truncation error partly. We notice the original AGE method has only two order accuracy for spatial step. The AGE method is soon applied to convection-diffusion equations in [6] and telegraph equation in [7]. Z. B. Lin presented a class of alternating segment explicit-implicit scheme in [8]. Based on the concept of AGE method, a class of parallel second- order domain splitting method for diffusion equations is presented in [9,10]. In [11-13], the concept of the AGE method is applied to solve semi-linear and nonlinear equations. T.Z.Fu presented a second order exponential AGE method for convection-diffusion equations in [14]. To our knowledge, AGE methods of fourth order accuracy have been scarcely presented.

Based on the situations mentioned above, we will construct two parallel methods with four order accuracy in spatial step.

Results about existence and uniqueness of theoretic solution for parabolic equations can be found in [15-18].

We organize the rest of this paper as follows:

In section 2, we present an $O(\tau^2 + h^4)$ order unconditionally stable symmetry six-point implicit scheme for solving (1) at first. Then we give a group of asymmetry schemes, and an alternating group explicit (AGE) method will be constructed based on the schemes. Stability analysis for the AGE method are given in section 3. In section 4, we will construct another alternating group explicit iterative (AGEI) method. Convergence analysis is given for the AGEI method in section 5. In section 6, Results of numerical example are presented. Some conclusions are given at the end of the paper.

2 The Alternating Group Explicit (AGE) Method

The domain $\Omega : (0,1) \times (0,T)$ will be divided into $(m \times N)$ meshes with spatial step size h=$\frac{1}{m}$ in x direction and the time step size τ=$\frac{T}{N}$. Grid points are denoted by (x_i, t_n) or (i, n), $x_i = ih(i = 0,\ 1, \cdots, m)$, $t_n = n\tau(n = 0,\ 1, \cdots, \frac{T}{\tau})$. The numerical solution of (1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$. Let $r = \frac{\tau}{2h^2}$.

We approach (1) at $(i,\ n)$ with center-difference scheme:

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\tau} = \frac{1}{4h^2}[(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1})$$
$$+2(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + (u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1})] \quad (2)$$

Applying Taylor formula to the scheme at (x_i, t_n), we have

$$(\frac{\partial u}{\partial t})_i^n + \frac{\tau^2}{6}(\frac{\partial^3 u}{\partial t^3})_i^n = (\frac{\partial^2 u}{\partial x^2})_i^n + \frac{h^2}{12}(\frac{\partial^4 u}{\partial x^4})_i^n + O(h^4)$$

Considering $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^4 u}{\partial x^4}$, we have $(\frac{\partial u}{\partial t})_i^n + \frac{\tau^2}{6}(\frac{\partial^3 u}{\partial t^3})_i^n = (\frac{\partial^2 u}{\partial x^2})_i^n + \frac{h^2}{12}(\frac{\partial^2 u}{\partial t^2})_i^n$. Then we approach $(\frac{\partial^2 u}{\partial t^2})_i^n$ with $\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2}$. Combining with (2) we have the following scheme:

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\tau} = \frac{1}{4h^2}[(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1})$$
$$+2(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + (u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1})]$$
$$-\frac{h^2}{12}(\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2}) \quad (3)$$

The truncation error of (3) can easily be obtained as $O(\tau^2 + h^4)$.

We use fourier method to analyze the stability of (3).

Lemma 1[19] if b and c are real numbers, and $\lambda_1,\ \lambda_2$ are the roots of $\lambda^2 - b\lambda - c = 0$, then we have $|\lambda_i| < 1,\ i = 1,2$ if and only if $|b| \le 1 - c < 2$.

Let $w_i^n = (u_i^n, u_i^{n-1})^T$, $p = 1 + \frac{1}{12r}$, $q = 1 - \frac{1}{12r}$, then from (3) we have

$$\begin{pmatrix} -r & 0 \\ 0 & 0 \end{pmatrix} w_{i-1}^{n+1} + \begin{pmatrix} p+2r & 0 \\ 0 & 1 \end{pmatrix} w_i^{n+1}$$
$$+ \begin{pmatrix} -r & 0 \\ 0 & 0 \end{pmatrix} w_{i+1}^{n+1} = \begin{pmatrix} 2r & r \\ 0 & 0 \end{pmatrix} w_{i-1}^n$$
$$+ \begin{pmatrix} p-q-4r & q-2r \\ 1 & 0 \end{pmatrix} w_i^n + \begin{pmatrix} 2r & r \\ 0 & 0 \end{pmatrix} w_{i+1}^n$$

Let $w_i^n = v^n e^{i\alpha x_i}$, then we have

$$\begin{pmatrix} p+2r-2rcos(\alpha h) & 0 \\ 0 & 1 \end{pmatrix} v^{n+1}$$
$$= \begin{pmatrix} p-q-4r+4rcos(\alpha h) & q-2r+2rcos(\alpha h) \\ 1 & 0 \end{pmatrix} v^n$$

Furthermore

$$v^{n+1} = \begin{pmatrix} \frac{\frac{1}{6r} - 8rsin^2(\frac{\alpha h}{2})}{p + 4rsin^2(\frac{\alpha h}{2})} & \frac{1 - \frac{1}{12r} - 4rsin^2(\frac{\alpha h}{2})}{p + 4rsin^2(\frac{\alpha h}{2})} \\ 1 & 0 \end{pmatrix} v^n$$
$$= Tv^n$$

Let λ be the eigenvalue of T, then we have

$$\lambda^2 - \frac{\frac{1}{6r} - 8rsin^2(\frac{\alpha h}{2})}{p + 4rsin^2(\frac{\alpha h}{2})}\lambda - \frac{1 - \frac{1}{12r} - 4rsin^2(\frac{\alpha h}{2})}{p + 4rsin^2(\frac{\alpha h}{2})} = 0$$

The stability of (3) can be obtained under the condition

$$|\frac{\frac{1}{6r} - 8rsin^2(\frac{\alpha h}{2})}{p + 4rsin^2(\frac{\alpha h}{2})}| \le 1 - \frac{1 - \frac{1}{12r} - 4rsin^2(\frac{\alpha h}{2})}{p + 4rsin^2(\frac{\alpha h}{2})} < 2,$$

that is,

$$|\frac{1}{6r} - 8rsin^2(\frac{\alpha h}{2})| \le \frac{1}{6r} + 8rsin^2(\frac{\alpha h}{2})$$
$$< 2 + \frac{1}{6r} + 8rsin^2(\frac{\alpha h}{2}),$$

which is obviously true. So we can get the following theorem:

Theorem 1 The scheme (3) is unconditionally stable.

Based on (3), we present four asymmetry schemes as follows:

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\tau} = \frac{1}{4h^2}[(u_{i+1}^{n+1} - u_i^{n+1} - u_i^n + u_{i-1}^n)$$
$$+2(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + (u_{i+1}^{n-1} - u_i^{n-1} - u_i^n + u_{i-1}^n)]$$
$$-\frac{h^2}{12\tau^2}(u_i^{n+1} - 2u_i^n + u_i^{n-1}) \quad (4)$$

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\tau}=\frac{1}{4h^2}[(u_{i+1}^{n+1}-u_i^{n+1}-u_i^n+u_{i-1}^n)$$
$$+(u_{i+1}^{n+1}-2u_i^{n+1}-u_{i-1}^{n+1})+(u_{i+1}^{n-1}-2u_i^{n-1}+u_{i-1}^{n-1})$$
$$+(u_{i+1}^{n-1}-u_i^{n-1}-u_i^n+u_{i-1}^n)]-\frac{h^2}{12\tau^2}(u_i^{n+1}-2u_i^n+u_i^{n-1}) \tag{5}$$

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\tau}=\frac{1}{4h^2}[(u_{i+1}^{n}-u_i^{n}-u_i^{n+1}+u_{i-1}^{n+1})$$
$$+(u_{i+1}^{n+1}-2u_i^{n+1}-u_{i-1}^{n+1})+(u_{i+1}^{n-1}-2u_i^{n-1}+u_{i-1}^{n-1})$$
$$+(u_{i+1}^{n}-u_i^{n}-u_i^{n-1}+u_{i-1}^{n-1})]-\frac{h^2}{12\tau^2}(u_i^{n+1}-2u_i^n+u_i^{n-1}) \tag{6}$$

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\tau}=\frac{1}{4h^2}[(u_{i+1}^{n}-u_i^{n}-u_i^{n+1}+u_{i-1}^{n+1})$$
$$+2(u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n})+(u_{i+1}^{n}-u_i^{n}-u_i^{n-1}+u_{i-1}^{n-1})]$$
$$-\frac{h^2}{12\tau^2}(u_i^{n+1}-2u_i^n+u_i^{n-1}) \tag{7}$$

Let

$$\delta_x u_i^n = u_{i+1}^n-u_i^n,$$
$$\delta_{\overline{x}} u_i^n = u_i^n-u_{i-1}^n,$$
$$\delta_t u_i^n = u_i^{n+1}-u_i^n,$$
$$\delta_{\overline{t}} u_i^n = u_i^n-u_i^{n-1},$$
$$\delta_{\hat{t}} u_i^n = u_i^{n+1}-u_i^{n-1},$$

then we rewrite (4)-(7) as:

$$\delta_{\hat{t}} u_i^n = r[\delta_x(u_i^{n+1}+2u_i^n+u_i^{n-1})-4\delta_{\overline{x}}u_i^n]$$
$$-\frac{1}{12r}(\delta_t u_i^n-\delta_{\overline{t}}u_i^n) \tag{8}$$

$$\delta_{\hat{t}} u_i^n = r[\delta_x(2u_i^{n+1}+2u_i^{n-1})-\delta_{\overline{x}}(u_i^{n+1}+2u_i^n+u_i^{n-1})]$$
$$-\frac{1}{12r}(\delta_t u_i^n-\delta_{\overline{t}}u_i^n) \tag{9}$$

$$\delta_{\hat{t}} u_i^n = r[\delta_x(u_i^{n+1}+2u_i^n+u_i^{n-1})-\delta_{\overline{x}}(2u_i^{n+1}+2u_i^{n-1})]$$
$$-\frac{1}{12r}(\delta_t u_i^n-\delta_{\overline{t}}u_i^n) \tag{10}$$

$$\delta_{\hat{t}} u_i^n = r[4\delta_x u_i^n-\delta_{\overline{x}}(u_i^{n+1}+2u_i^n+u_i^{n-1})]$$
$$-\frac{1}{12r}(\delta_t u_i^n-\delta_{\overline{t}}u_i^n) \tag{11}$$

Based on (8)-(11), we will have three basic computing point groups:

"$\kappa 1$"group: four grid points are involved, and $(8),(9),(10),(11)$ are used respectively. Let $U_i^n = (u_i^n, u_{i+1}^n, u_{i+2}^n, u_{i+3}^n)^T$, then we have

$$A_1U_i^{n+1}=B_1U_i^n+C_1U_i^{n-1}+F_i^n \tag{12}$$

here $F_i^n=(4ru_{i-1}^n,0,0,4ru_{i+1}^n)^T$,

$$A_1=\begin{pmatrix} A_{11} & A_{12}\\ A_{13} & A_{14}\end{pmatrix}$$

$$A_{11}=\begin{pmatrix} 1+r+\frac{1}{12r} & -r\\ -r & 1+3r+\frac{1}{12r}\end{pmatrix}$$

$$A_{12}=\begin{pmatrix} 0 & 0\\ -2r & 0\end{pmatrix}$$

$$A_{13}=\begin{pmatrix} 0 & -2r\\ 0 & 0\end{pmatrix}$$

$$A_{14}=\begin{pmatrix} 1+3r+\frac{1}{12r} & -r\\ -r & 1+r+\frac{1}{12r}\end{pmatrix}$$

$$B_1=\begin{pmatrix} B_{11} & B_{12}\\ B_{13} & B_{14}\end{pmatrix}$$

$$B_{11}=\begin{pmatrix} \frac{1}{6r}-6r & 2r\\ 2r & \frac{1}{6r}-2r\end{pmatrix}$$

$$B_{12}=\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$$

$$B_{13}=\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}$$

$$B_{14}=\begin{pmatrix} \frac{1}{6r}-2r & 2r\\ 2r & \frac{1}{6r}-6r\end{pmatrix}$$

$$C_1=\begin{pmatrix} C_{11} & C_{12}\\ C_{13} & C_{14}\end{pmatrix}$$

$$C_{11}=\begin{pmatrix} 1-r-\frac{1}{12r} & r\\ r & 1-3r-\frac{1}{12r}\end{pmatrix}$$

$$C_{12}=\begin{pmatrix} 0 & 0\\ 2r & 0\end{pmatrix}$$

$$C_{13}=\begin{pmatrix} 0 & 2r\\ 0 & 0\end{pmatrix}$$

$$C_{14}=\begin{pmatrix} 1-3r-\frac{1}{12r} & r \\ r & 1-r-\frac{1}{12r}\end{pmatrix}$$

Then the numerical solution at grid nodes (i,n+1), (i+1,n+1), (i+2,n+1), (i+3,n+1) can be obtained explicitly as below:

$$U_i^{n+1}=A_1^{-1}(B_1U_i^n+C_1U_i^{n-1}+F_i^n)$$

"$\kappa 2$"group: two inner points are involved, and (8), (9) are used respectively. Let $\overline{U}_i^n=(u_i^n,u_{i+1}^n)^T$, then we have

$$A_2\overline{U}_i^{n+1}=B_2\overline{U}_i^n+C_2U_i^{n-1}+\overline{F}_i^n \qquad (13)$$

here $\overline{F}_i^n=(4ru_{i-1}^n,2ru_{i+1}^{n+1}+2ru_{i+1}^{n-1})^T$,

$$A_2=\begin{pmatrix} 1+r+\frac{1}{12r} & -r \\ -r & 1+3r+\frac{1}{12r}\end{pmatrix}$$

$$B_2=\begin{pmatrix} \frac{1}{6r}-6r & 2r \\ 2r & \frac{1}{6r}-2r\end{pmatrix}$$

$$C_2=\begin{pmatrix} 1-r-\frac{1}{12r} & r \\ r & 1-3r-\frac{1}{12r}\end{pmatrix}$$

The numerical solution at grid nodes (i,n+1), (i+1,n+1), can be denoted as below:

$$\overline{U}_i^{n+1}=A_2^{-1}(B_2\overline{U}_i^n+C_2U_i^{n-1}+\overline{F}_i^n)$$

"$\kappa 3$"group: two inner points are involved, and (10), (11) are used respectively. Let $\widetilde{U}_i^n=(u_i^n,u_{i+1}^n)^T$, then we have

$$A_3\widetilde{U}_i^{n+1}=B_3\widetilde{U}_i^n+C_3U_i^{n-1}+\widetilde{F}_i^n \qquad (14)$$

here $\widetilde{F}_i^n=(2ru_{i-1}^{n+1}+2ru_{i-1}^{n-1},4ru_{i+1}^n)^T$,

$$A_3=\begin{pmatrix} 1+3r+\frac{1}{12r} & -r \\ -r & 1+r+\frac{1}{12r}\end{pmatrix}$$

$$B_3=\begin{pmatrix} \frac{1}{6r}-2r & 2r \\ 2r & \frac{1}{6r}-6r\end{pmatrix}$$

$$C_3=\begin{pmatrix} 1-3r-\frac{1}{12r} & r \\ r & 1-r-\frac{1}{12r}\end{pmatrix}$$

Thus we have:

$$\widetilde{U}_i^{n+1}=A_3^{-1}(B_3\widetilde{U}_i^n+C_3U_i^{n-1}+\widetilde{F}_i^n)$$

Applying the basic point groups above, we construct the alternating group method in the case of two conditions as follows:

In the first condition, we let $m-1=4s$, here s is an integer. First at the (n+1)-th time level, we divide all of the $m-1$ inner grid point into s "$\kappa 1$" groups, and (12) are used in each group. Second at the (n+2)-th time level, we will have $(s+1)$ point groups. "$\kappa 3$" group are applied to get the solution of the left two grid points (1,n+2) and (2,n+2). (12) are used in the following s "$\kappa 1$" groups, while "$\kappa 2$" are used in the right two grid points (m-2,n+2), (m-1,n+2). Let $U^n=(u_1^n,\ u_2^n,\ \cdots,\ u_{m-1}^n)^T$, then we can denote the alternating group method I as follows:

$$\begin{cases} AU^{n+1}=BU^n+CU^{n-1}+F_1^n \\ \widehat{A}U^{n+2}=\widehat{B}U^{n+1}+\widehat{C}U^n+F_2^n \end{cases} \qquad (15)$$

here C_1^n and C_2^n are known vectors relevant to the boundary, while $A,\ B,\ \widehat{A},\ \widehat{B}$ are all $(m-1)\times(m-1)$ matrixes.

$$F_1^n=(4ru_0^n,0,\cdots,0,4ru_m^n)^T$$

$$F_2^n=(2ru_0^{n+2}+2ru_0^n,0,\cdots,0,2ru_m^{n+2}+2ru_m^n)^T$$

$$A=diag(A_1,\ A_1,\ \cdots,\ A_1,\ A_1)$$

$$B=diag(B_3,\ \overline{B}_1,\ \cdots,\ \overline{B}_1,\ B_2)$$

$$C=diag(C_1,\ C_1,\ \cdots,\ C_1,\ C_1)$$

$$\widehat{A}=diag(A_3,\ A_1,\ \cdots,\ A_1,\ A_2)$$

$$\widehat{B}=diag(\overline{B}_1,\ \overline{B}_1,\ \cdots,\ \overline{B}_1,\ \overline{B}_1)$$

$$\widehat{C}=diag(C_3,\ C_1,\ \cdots,\ C_1,\ C_2)$$

Here

$$\overline{B}_1=\begin{pmatrix} \frac{1}{6r}-2r & 2r & 0 & 0 \\ 2r & \frac{1}{6r}-6r & 4r & 0 \\ 0 & 4r & \frac{1}{6r}-6r & 2r \\ 0 & 0 & 2r & \frac{1}{6r}-2r \end{pmatrix}$$

The alternating use of the asymmetry schemes (8)-(11) can lead to partly counteracting of truncation error, and then can increase the numerical accuracy. On the other hand, grouping explicit computation can be obviously obtained. Thus computing in the whole domain can be divided into many sub-domains, and can be worked out with several parallel computers. So the method has the obvious property of parallelism.

In the following we will try to construct the alternating method under the condition of $m-1=4s+2$. First at the (n+1)-th time level, we will have $s+1$ point groups. "$\kappa 2$" are used at the right two inner grid points, while the left $4s$ inner grid points are divided into s groups, and "$\kappa 1$" are used in each group. Second at the (n+2)-th time level, we are still to have $s+1$ point groups. "$\kappa 3$" are used at the left two inner grid points, while the right $4s$ inner grid points are divided into s groups, and "$\kappa 1$" are used in each group. Thus the alternating group method II is established by alternating use of the schemes (8)-(11) in the two time levels:

$$\begin{cases} \widetilde{A}U^{n+1} = \widetilde{B}U^n + \widetilde{C}U^{n-1} + \widetilde{F}_1^n \\ \widetilde{\widetilde{A}}U^{n+2} = \widetilde{\widetilde{B}}U^{n+1} + \widetilde{\widetilde{C}}U^n + \widetilde{F}_2^n \end{cases} \tag{16}$$

here $\widetilde{C}_1^n$ and $\widetilde{C}_2^n$ are known vectors relevant to the boundary, while $\widetilde{A},\ \widetilde{B},\ \widetilde{\widetilde{A}},\ \widetilde{\widetilde{B}}$ are all $(m-1)\times(m-1)$ matrixes.

$$\widetilde{F}_1^n = (4ru_0^n, 0, \cdots, 0, 2ru_m^{n+1} + 2ru_m^{n-1})^T$$

$$\widetilde{F}_2^n = (2ru_0^{n+2} + 2ru_0^n, 0, \cdots, 0, 4ru_m^{n+1})^T,$$

$$\widetilde{A} = diag(A_1,\ A_1,\ \cdots,\ A_1,\ A_2)$$

$$\widetilde{B} = diag(B_3,\ \overline{B}_1,\ \cdots,\ \overline{B}_1,\ B_1)$$

$$\widetilde{C} = diag(C_1,\ C_1,\ \cdots,\ C_1,\ C_2)$$

$$\widetilde{\widetilde{A}} = diag(A_3,\ A_1,\ \cdots,\ A_1,\ A_1)$$

$$\widetilde{\widetilde{B}} = diag(\overline{B}_1,\ \overline{B}_1,\ \cdots,\ \overline{B}_1,\ B_2)$$

$$\widetilde{\widetilde{C}} = diag(C_3,\ C_1,\ \cdots,\ C_1,\ C_1)$$

3 Stability Analysis Of The AGE method

In order to verify the stability of (15) and (16), we present the following lemmas:

Lemma 2[20] Let θ >0, and $G + G^T$ is nonnegative, then $(\theta I + G)^{-1}$exists, and

$$\begin{cases} \|(\theta I + G)^{-1}\|_2 \le \theta^{-1} \\ \|(\theta I - G)(\theta I + G)^{-1}\|_2 \le 1 \end{cases} \tag{17}$$

Lemma 3[20] Let A is a $n \times n$ matrix. λ is the eigenvalue of A, then

$$|\lambda - a_{ss}| \le \sum_{j=1, j\ne s}^{n} |a_{sj}| \tag{18}$$

Theorem 2 if $r < \frac{1}{3}$, then the alternating group method denoted by (15) is stable.

Proof: Let

$$G_1 = diag(G_{11}, G_{11}, \cdots, G_{11}, G_{11})$$

$$G_2 = diag(G_{13}, G_{11}, \cdots, G_{11}, G_{12})$$

$$G_{11} = \begin{pmatrix} 1 & -1 & & \\ -1 & 3 & -2 & \\ & -2 & 3 & -1 \\ & & -1 & 1 \end{pmatrix}$$

$$G_{12} = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix},\ G_{13} = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$$

G_1 and G_2 are obviously nonnegative matrixes. (15) can be rewritten as

$$\begin{cases} (pI + rG_1)U^{n+1} = [(p-q)I - 2rG_2]U^n \\ \qquad + (qI - rG_1)U^{n-1} + F_1^n \\ \\ (pI + rG_2)U^{n+2} = [(p-q)I - 2rG_1]U^{n+1} \\ \qquad + (qI - rG_2)U^n + F_2^n \end{cases} \tag{19}$$

Under the condition of exact boundary value, we have $F_1^n = F_2^n = 0$. Let $V^n = (U^n, U^{n-1})^T$, then it follows

$$\begin{pmatrix} pI + rG_2 & (q-p)I + 2rG_1 \\ O & pI + rG_1 \end{pmatrix} V^{n+2}$$

$$= \begin{pmatrix} qI - rG_2 & O \\ (p-q)I - 2rG_2 & qI - rG_1 \end{pmatrix} V^n$$

that is,

$$V^{n+2} = \begin{pmatrix} pI + rG_2 & (q-p)I + 2rG_1 \\ O & pI + rG_1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} qI - rG_2 & O \\ (p-q)I - 2rG_2 & qI - rG_1 \end{pmatrix} V^n = GV^n$$

here

$$G = \begin{pmatrix} (pI + rG_2)^{-1} & \widehat{G} \\ O & (pI + rG_1)^{-1} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} qI - rG_2 & O \\ (p-q)I - 2rG_2 & qI - rG_1 \end{pmatrix}$$

is the growth matrix.

$$\widehat{G} = -(pI + rG_2)^{-1}[(q-p)I + 2rG_1](pI + rG_1)^{-1}$$

Considering

$$||(pI + rG_i)^{-1}||_2 \le p^{-1} = \frac{1}{1 + \frac{1}{12r}},\ i = 1, 2$$

then from lemma 2 we have:

$$\rho(G) \le max(||(pI + rG_1)^{-1}||_2,\ ||(pI + rG_2)^{-1}||_2).$$
$$max(||(qI - rG_1)||_2,\ ||(qI - rG_2)||_2)$$

$$\le \frac{1}{1+\frac{1}{12r}} max(||qI-rG_1||_2,\ ||qI-rG_2||_2)$$

From the construction of the G_1 and G_2 we can see they are both symmetric matrixes, which shows $||qI-rG_1||_2 = \rho(qI-rG_1)$, $||qI-rG_2||_2 = \rho(qI-rG_2)$.

Let λ be the eigenvalue of $G_i,\ i=1,2$. Then $q+r\lambda$ is the eigenvalue of $qI-rG_i$.

From lemma 3 we have: $\begin{cases} |\lambda-1| \le 1 \\ |\lambda-3| \le 3 \end{cases}$, that is, $0 \le \lambda \le 6$.

In order to arrive $\rho(G) < 1$, we can let $|q-r\lambda| = |1-\frac{1}{12r}-r\lambda| < 1+\frac{1}{12r}$, which is solved with the result $r < \frac{1}{3}$. Then we conclude (15) is stable under the condition of $r < \frac{1}{3}$. So Theorem 1 is proved.

Analogously we have:

Theorem 3 The alternating group method denoted by (16) is stable under the condition of $r < \frac{1}{3}$.

4 The Construction Of AGEI Method

Let $p = 1+\frac{1}{12r},\ q = 1-\frac{1}{12r}$, then from (3) we have $KU^{n+1} = E^n$.

Here $E^n = E_1U^n + E_2U^{n-1} + [2ru_0^n + ru_0^{n-1} + ru_0^{n+1}, 0, \cdots, 0, 2ru_m^n + ru_m^{n-1} + ru_m^{n+1}]^T$.

$$K = \begin{pmatrix} p+2r & -r & & & \\ -r & p+2r & -r & & \\ & \cdots & \cdots & \cdots & \\ & & -r & p+2r & -r \\ & & & -r & p+2r \end{pmatrix}$$

$$E_1 = \begin{pmatrix} \frac{1}{6r}-4r & 2r & & & \\ 2r & \frac{1}{6r}-4r & 2r & & \\ & \cdots & \cdots & \cdots & \\ & & 2r & \frac{1}{6r}-4r & 2r \\ & & & 2r & \frac{1}{6r}-4r \end{pmatrix}$$

$$E_2 = \begin{pmatrix} q-2r & r & & & \\ r & q-2r & r & & \\ & \cdots & \cdots & \cdots & \\ & & r & q-2r & r \\ & & & r & q-2r \end{pmatrix}$$

K, E_1, E_2 are all $(m-1)\times(m-1)$ matrixes.

In order to solve U^{n+1}, we will try to construct an alternating group explicit iterative method so as to avoid solving an implicit equation set.

The alternating group iterative method will be constructed in two cases as follows:

In the first condition, we let $m = 4a+1,\ a$ is an integer. Let $K = \frac{1}{2}(H_1+H_2)$, here

$$H_1 = diag(H_{11}, \cdots, H_{11})_{(m-1)\times(m-1)}$$

$$H_2 = diag(H_{21}, H_{11}, \cdots, H_{11}, H_{21})_{(m-1)\times(m-1)}$$

$$H_{11} = \begin{pmatrix} p+2r & -r & 0 & 0 \\ -r & p+2r & -2r & 0 \\ 0 & -2r & p+2r & -r \\ 0 & 0 & -r & p+2r \end{pmatrix}$$

$$H_{21} = \begin{pmatrix} p+2r & -r \\ -r & p+2r \end{pmatrix}$$

Then the alternating group explicit iterative method (AGEI1) can be constructed as below:

$$\begin{cases} (\rho I + H_1)U_{k+\frac{1}{2}}^{n+1} = (\rho I - H_2)U_k^{n+1} + \widetilde{E}^n \\ (\rho I + H_2)U_{k+1}^{n+1} = (\rho I - H_1)U_{k+\frac{1}{2}}^{n+1} + \widetilde{E}^n \end{cases} k = 0, 1, \cdots \tag{20}$$

Here $\widetilde{E}^n = 2E^n$, k is the iterative parameter.

In the case of $m = 4a+3,\ a$ is an integer. We let $H = \frac{1}{2}(\overline{H}_1 + \overline{H}_2)$, here

$$\overline{H}_1 = diag(H_{11}, \cdots, H_{11}, H_{21})_{(m-1)\times(m-1)}$$

$$\overline{H}_2 = diag(H_{21}, H_{11}, \cdots, H_{11})_{(m-1)\times(m-1)}$$

Then the AGEI2 method can be constructed as below:

$$\begin{cases} (\rho I + \overline{H}_1)U_{k+\frac{1}{2}}^{n+1} = (\rho I - \overline{H}_2)U_k^{n+1} + \widetilde{E}^n \\ (\rho I + \overline{H}_2)U_{k+1}^{n+1} = (\rho I - \overline{H}_1)U_{k+\frac{1}{2}}^{n+1} + \widetilde{E}^n \end{cases} k = 0, 1, \cdots \tag{21}$$

From the construction of matrices $H_1,\ H_2,\ \overline{H}_1,\ \overline{H}_2$ in (20) and (21) we can see that computation can be divided into several groups, which is the same as the AGE method in section 2. Then the parallelism can be obtained obviously.

5 Convergence Analysis of The AGEI Method

Theorem 4 The alternating group explicit iterative method (20) is convergent.

Proof: From the construction of the matrixes we can see that $H_1,\ H_2,\ (H_1+H_1^T),\ (H_2+H_2^T)$ are all nonnegative matrixes. Then from lemma 1 we have

$$\|(\rho I - H_1)(\rho I + H_1)^{-1}\|_2 \le 1$$

$$\|(\rho I - H_2)(\rho I + H_2)^{-1}\|_2 \leq 1$$

From (20), we obtain

$$U_{k+1}^{n+1} = HU_k^{n+1} + (\rho I + H_2)^{-1}$$

$$[(\rho I - H_1)(\rho I + H_1)^{-1}\widetilde{E}^n + \widetilde{E}^n].$$

Here

$$H = (\rho I + H_2)^{-1}(\rho I - H_1)(\rho I + H_1)^{-1}(\rho I - H_2)$$

is growth matrix.

Let

$$\widetilde{H} = (\rho I + H_2)H(\rho I + H_2)^{-1}$$

$$= (\rho I - H_1)(\rho I + H_1)^{-1}(\rho I - H_2)(\rho I + H_2)^{-1}$$

then

$$\rho(H) = \rho(\widetilde{H}) \leq \|\widetilde{H}\|_2 \leq 1$$

which shows the AGEI1 method given by (20) is convergent.

Similarly we have:

Theorem 5 The alternating group explicit iterative method (21) is also convergent.

6 Numerical Experiments

Example 1: We consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2},\ 0 \leq x \leq 1,\ 0 \leq t \leq T \\ u(x,0) = sin(\pi x), \\ u(0,t) = 0,\ u(1,t) = 0. \end{cases} \quad (22)$$

The exact solution of (22) is denoted as below:

$$u(x,t) = e^{-\pi^2 t} sin(\pi x)$$

Let $A.E. = |u_i^n - u(x_i,t_n)|$, $P.E. = \dfrac{|u_i^n - u(x_i,t_n)|}{u(x_i,t_n)}$ denote maximum absolute error and relevant error respectively. We compare the numerical results of the AGE method in this paper with the results from [5] in Table 1-6.

Table 1: comparison of AGE method and the method in [5] at $m = 41$

	$\tau = 10^{-5}, t = 10000\tau$
$A.E.$	3.672 $\times 10^{-5}$
$A.E.$[5]	1.793 $\times 10^{-4}$
$P.E.$	9.860 $\times 10^{-3}$
$P.E.$[5]	4.829 $\times 10^{-2}$

Table 2: comparison of AGE method and the method in [5] at $m = 41$

	$\tau = 10^{-4}, t = 10000\tau$
$A.E.$	5.094 $\times 10^{-8}$
$A.E.$[5]	1.693 $\times 10^{-7}$
$P.E.$	9.856 $\times 10^{-2}$
$P.E.$[5]	3.438$\times 10^{-1}$

Table 3: comparison of AGE method and the method in [5] at $m = 41$

	$\tau = 10^{-6}, t = 10000\tau$
$A.E.$	8.927 $\times 10^{-6}$
$A.E.$[5]	4.371 $\times 10^{-5}$
$P.E.$	9.860 $\times 10^{-4}$
$P.E.$[5]	4.839 $\times 10^{-3}$

Table 4: comparison of AGE method and the method in [5] at $m = 53$

	$\tau = 10^{-5}, t = 10000\tau$
$A.E.$	3.672 $\times 10^{-5}$
$A.E.$[5]	1.793 $\times 10^{-4}$
$P.E.$	9.860 $\times 10^{-3}$
$P.E.$[5]	4.829 $\times 10^{-2}$

Table 5: comparison of AGE method and the method in [5] at $m = 53$

	$\tau = 10^{-5}, t = 100000\tau$
$A.E.$	5.068 $\times 10^{-9}$
$A.E.$[5]	1.482 $\times 10^{-7}$
$P.E.$	9.804 $\times 10^{-3}$
$P.E.$[5]	2.870$\times 10^{-1}$

Table 6: comparison of AGE method and the method in [5] at $m = 53$

	$\tau = 10^{-6}, t = 100000\tau$
$A.E.$	3.655 $\times 10^{-6}$
$A.E.$[5]	1.076 $\times 10^{-4}$
$P.E.$	9.825 $\times 10^{-4}$
$P.E.$[5]	2.890 $\times 10^{-2}$

Let $||E_1||_\infty = max|u_i^n - u(x_i,t_n)|$, $||E_2||_\infty = max|(u_i^n - u(x_i,t_n))/u(x_i,t_n)|$, $i = 1,2,\cdots,m-1$. We use the iterative error 1×10^{-10} to control the process of iterativeness, and the results of AGEI method are listed in table 7-14.

Table 7: comparison of AGEI method and the method in [5] at $m = 17, \tau = 10^{-4},\ \rho = 1$

	$t = 100\tau$
$\|\|E_1\|\|_\infty$	8.906 $\times 10^{-4}$
$\|\|E_1\|\|_\infty$[5]	3.093 $\times 10^{-2}$
$\|\|E_2\|\|_\infty$	9.872 $\times 10^{-2}$
$\|\|E_2\|\|_\infty$[5]	4.969

Table 8: comparison of AGEI method and the method in [5] at $m = 17, \tau = 10^{-4},\ \rho = 1$

	$t = 200\tau$
$\|\|E_1\|\|_\infty$	8.066×10^{-4}
$\|\|E_1\|\|_\infty$[5]	4.022×10^{-2}
$\|\|E_2\|\|_\infty$	9.869×10^{-2}
$\|\|E_2\|\|_\infty$[5]	8.858

Table 9: comparison of AGEI method and the method in [5] at $m = 17, \tau = 10^{-4},\ \rho = 1$

	$t = 500\tau$
$\|\|E_1\|\|_\infty$	5.994×10^{-4}
$\|\|E_1\|\|_\infty$[5]	5.757×10^{-2}
$\|\|E_2\|\|_\infty$	9.860×10^{-2}
$\|\|E_2\|\|_\infty$[5]	41.443

Table 10: comparison of AGEI method and the method in [5] at $m = 17, \tau = 10^{-4},\ \rho = 1$

	$t = 1000\tau$
$\|\|E_1\|\|_\infty$	3.654×10^{-4}
$\|\|E_1\|\|_\infty$[5]	3.271×10^{-2}
$\|\|E_2\|\|_\infty$	9.847×10^{-2}
$\|\|E_2\|\|_\infty$[5]	169.521

Table 11: comparison of AGEI method and the method in [5] at $m = 17, \tau = 10^{-5},\ \rho = 1$

	$t = 1000\tau$
$\|\|E_1\|\|_\infty$	1.260×10^{-4}
$\|\|E_1\|\|_\infty$[5]	1.439×10^{-2}
$\|\|E_2\|\|_\infty$	1.396×10^{-2}
$\|\|E_2\|\|_\infty$[5]	2.136

Table 12: comparison of AGEI method and the method in [5] at $m = 17, \tau = 10^{-5},\ \rho = 1$

	$t = 2000\tau$
$\|\|E_1\|\|_\infty$	1.522×10^{-4}
$\|\|E_1\|\|_\infty$[5]	4.025×10^{-2}
$\|\|E_2\|\|_\infty$	1.863×10^{-2}
$\|\|E_2\|\|_\infty$[5]	8.864

Table 13: comparison of AGEI method and the method in [5] at $m = 17, \tau = 10^{-5},\ \rho = 1$

	$t = 5000\tau$
$\|\|E_1\|\|_\infty$	2.173×10^{-4}
$\|\|E_1\|\|_\infty$[5]	5.759×10^{-2}
$\|\|E_2\|\|_\infty$	3.576×10^{-2}
$\|\|E_2\|\|_\infty$[5]	41.459

Table 14: comparison of AGEI method and the method in [5] at $m = 17, \tau = 10^{-5},\ \rho = 1$

	$t = 10000\tau$
$\|\|E_1\|\|_\infty$	2.911×10^{-4}
$\|\|E_1\|\|_\infty$[5]	3.272×10^{-2}
$\|\|E_2\|\|_\infty$	7.844×10^{-2}
$\|\|E_2\|\|_\infty$[5]	169.563

Let t_1/t_2 denotes the ratio of running time between the AGE method in this paper and center-difference implicit scheme (2). We finished the numerical experiment in the same conditions.

Table 15: results of comparison at $m = 25$

	$\tau = 10^{-3}, t = 100\tau$	$\tau = 10^{-3}, t = 500\tau$
$A.E.$	5.258×10^{-4}	4.758×10^{-5}
$A.E.$[5]	1.653×10^{-3}	1.548×10^{-4}
$P.E.$	1.587×10^{-2}	6.852×10^{-2}
$P.E.$[5]	4.664×10^{-1}	2.179
t_1/t_2	0.251	0.255

Table 16: results of comparison at $m = 25$

	$\tau = 10^{-4}, t = 100\tau$	$\tau = 10^{-4}, t = 500\tau$
$A.E.$	4.770×10^{-5}	9.610×10^{-6}
$A.E.$[5]	1.117×10^{-4}	5.193×10^{-5}
$P.E.$	6.833×10^{-3}	3.267×10^{-3}
$P.E.$[5]	1.785×10^{-2}	6.787×10^{-2}
t_1/t_2	0.245	0.262

Table 17: results of comparison at $m = 35$

	$\tau = 10^{-3}, t = 100\tau$	$\tau = 10^{-3}, t = 500\tau$
$A.E.$	9.576×10^{-4}	2.881×10^{-5}
$A.E.$[5]	3.513×10^{-3}	3.302×10^{-4}
$P.E.$	3.284×10^{-2}	1.313×10^{-1}
$P.E.$[5]	9.737×10^{-1}	4.626
t_1/t_2	0.158	0.164

Table 18: results of comparison at $m = 35$

	$\tau = 10^{-4}, t = 100\tau$	$\tau = 10^{-4}, t = 500\tau$
$A.E.$	6.112×10^{-6}	2.472×10^{-5}
$A.E.$[5]	5.802×10^{-5}	1.958×10^{-4}
$P.E.$	3.565×10^{-3}	6.549×10^{-3}
$P.E.$[5]	4.249×10^{-2}	
t_1/t_2	0.154	0.173

Example 2: We consider the following nonhomogeneous boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = 0.01\dfrac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1,\ 0 \le t \le T \\ u(x,0) = cos(\pi x), \\ u(0,t) = e^{-0.01\pi^2 t},\ u(1,t) = -e^{-0.01\pi^2 t}. \end{cases} \tag{23}$$

The exact solution of (23) is denoted by

$$u(x,t) = e^{-0.01\pi^2 t} cos(\pi x)$$

Let $||E_1||_\infty = max|u_i^n - u(x_i,t_n)|$, $||E_2||_\infty = max|(u_i^n - u(x_i,t_n))/u(x_i,t_n)|$, $i = 1, 2, \cdots, m-1$. The numerical results is listed in table 3.

Table 19: results of comparison at $m = 49$

	$\tau = 10^{-1}, t = 10\tau$	$\tau = 10^{-1}, t = 20\tau$
$\|\|E_1\|\|_\infty$	4.698 $\times 10^{-4}$	6.315 $\times 10^{-4}$
$\|\|E_1\|\|_\infty$[5]	6.837 $\times 10^{-3}$	2.361 $\times 10^{-2}$
$\|\|E_2\|\|_\infty$	3.395 $\times 10^{-2}$	3.167 $\times 10^{-2}$
$\|\|E_2\|\|_\infty$[5]	7.412 $\times 10^{-1}$	
t_1/t_2	0.114	0.119

Table 20: results of comparison at $m = 49$

	$\tau = 10^{-1}, t = 50\tau$	$\tau = 10^{-1}, t = 100\tau$
$\|\|E_1\|\|_\infty$	7.028 $\times 10^{-4}$	1.015 $\times 10^{-3}$
$\|\|E_1\|\|_\infty$[5]	2.693 $\times 10^{-2}$	8.013 $\times 10^{-2}$
$\|\|E_2\|\|_\infty$	2.674 $\times 10^{-2}$	4.247 $\times 10^{-1}$
$\|\|E_2\|\|_\infty$[5]	2.131	3.256
t_1/t_2	0.131	0.126

From the results of Table 1-20 we can see that the numerical solution for the presented methods are of higher accuracy than the original AGE method in [5], which is obvious even in the case of large τ. Furthermore, for its intrinsic parallelism, the AGE method in this paper can shorten the running computing time in comparison with the fully implicit scheme, and the effect becomes obvious when the amount of grid points increases.

7 Conclusions

In this paper, we present an alternating group explicit (AGE) method and an alternating group explicit iterative(AGEI) method. Then the stability analysis and convergence analysis are done respectively. The AGEI method is suitable for parallel computation in solving large system of equations, and is superior to the original AGE method in [5]. Furthermore, the construction of the two methods can also be applied to other partial differential equations.

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