NURBS Curve Shape Modification and Fairness Evaluation

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Abstract: - For the purpose of evaluation, a NURBS curve is used, because it is commonly used in the areas of CAD/CAM and Computer Graphics. A curve with a monotone radius of curvature distribution is considered as a fair curve in the area of Computer Aided Aesthetic Design (CAAD). But no official standards have been established. Therefore, a criterion for a fair curve is proposed. A quintic NURBS curve, the first derivative of a quintic NURBS curve, curvature vector, curvature, and radius of curvature are expressed. The concept of radius of curvature specification to modify the shape of a NURBS curve is illustrated. The difference between the NURBS curve radius of curvature and the specified radius of curvature is minimized by introducing the least-squares method to modify the shape of the NURBS curve. As curve fairness evaluation, radius of curvature distribution is used as an alternative characteristic of a curve. Algebraic functions such as linear, quadratic, cubic, quartic, quintic, and six degrees are applied to the radius of curvature distribution of the designed curve to specify the radius of curvature. Then, the shape of the curve is modified according to the specified radius of curvature distribution. In this manner, six NURBS curves whose radius of curvature are these algebraic functions are generated, and are predefined. Using the correlation matching, the similarity is evaluated by comparing the radius of curvature distribution of the designed curve with those of six NURBS curves predefined. The highest similarity curve to the designed curve among these predefined curves is selected. The similarity evaluated of the selected curve is determined as fairness of the designed curve.

Key-Words: - curve shape modification, fair curve, radius of curvature specification, correlation matching, fairness evaluation

1 Introduction

In Computer Aided Aesthetic Design (CAAD) [1], designers evaluate the quality of a designed curve by looking at its curvature or radius of curvature plots. If the quality of a designed curve does not meet designer's demands, they usually modify the control points of the curve interactively. If the variation of the radius of curvature of the curve is monotone, this curve is considered to be a fair curve [2]. But the definition of a fair curve is ambiguous and no official standards are given. Therefore, in this paper we have tried to establish criterion for a fair curve.

A NURBS curve, which is commonly used in the field of CAD・CAM and Computer Graphics, is used as an expression of a freeform curve. A quadratic NURBS curve is used as an expression of a quadratic curve using its weights. In this study, a quadratic curve is not used to express the shape of a curve. Therefore, the weights of a NURBS curve are not used. A cubic NURBS curve is widely used, but in this study, radius of curvature ranging over multi segments of a NURBS curve are modified based on the specified radius of curvature. A smooth radius of curvature continuity is needed. Therefore, a quintic NURBS curve is used in this study.

Positions and gradients are given to the NURBS curve equations and first derivative equations of the NURBS curve respectively. Then, a NURBS curve is generated. Afterwards, if necessary, the shape of this NURBS curve is modified according to the specified radius of curvature distribution.

Fair curve expression and fairness evaluation are described. As a measure of curve fairness evaluation, radius of curvature distribution is used as an alternative characteristic of a curve. Evaluation of whether the designed curve is fair or not is accomplished by comparing of the designed curve to a curve whose radius of curvature is monotone.

To specify the radius of curvature, six NURBS curves whose radius of curvature are followed to algebraic functions such as linear, quadratic, cubic, and up to six degrees are predefined based on the designed curve. Then, by introducing the correlation matching, the similarities of the designed curve to these six predefined curves are examined. Among these predefined curves, the highest similarity curve to the designed curve is selected as an ideal fair NURBS curve. Then, the fairness of the designed curve to its ideal fair curve is evaluated.

Fair curve generation algorithms related to curvature by modifying the control points have been published. These make monotone curvature [3], use a clothoidal curve for specifying the curvature [4], and automate a curve fairing algorithm for B-spline curves [5, 6]. Fair curve generation algorithms related to energy function have been published. These are smoothing of cubic parametric splines by energy function [7], finding the unfair portion of a curve using energy function [8], and introducing a low-pass filter to energy function [9]. Fair curve generation algorithms related to curvature by specifying curvature distribution have also been published [10].

Section 2 of this paper describes a quintic NURBS curve, the first derivative of a quintic NURBS curve, curvature vector, curvature, and radius of curvature. In section 3, generation of a quintic NURBS curve which passes through given point sequence and generation of a quintic NURBS curve using the given points and gradients are described. In section 4, NURBS curve shape modification based on the specified radius of curvature is described. Section 5 describes fair curve expression and fairness evaluation giving examples. A criterion for a fair curve is proposed as fairness.

2 NURBS Curve Expression

A quintic NURBS curve is used in this study. The objective of freeform curve design is to design the framework of surface patches.

2.1 NURBS Curve Expression

A quintic NURBS curve consists of $n-5$ segments $(n \geq 6)$ is composed of n control points such as $\boldsymbol{q_0}, \boldsymbol{q_1}, \cdots, \boldsymbol{q}_{n-1}$ and n weights such as $\omega_0, \omega_1, \cdots, \omega_{n-1}$ as in Eq.(1). In this study, the weights of this NURBS curve are not used.

$$\boldsymbol{R}(t) = \frac{\sum_{i=0}^{n-1} N_{i,6}(t) \cdot \omega_i \cdot \boldsymbol{q}_i}{\sum_{i=0}^{n-1} N_{i,6}(t) \cdot \omega_i}, \tag{1}$$

where $N_{i,6}(t)\,(i = 0,1,\cdots,n-1)$ are NURBS basis functions.

These functions are called the de Boor-Cox [11] recursion formulas, and are recursively defined by the knot sequence $t_0, t_1, t_2, t_3, \cdots, t_{n+5}$ as in Eq.(2). Knot spacing is fixed in this study.

$$\left.\begin{aligned} N_{i,1}(t) &= \begin{cases} 1 & (t_i \leq t < t_{i+1}) \\ 0 & \text{otherwise} \end{cases} \\ N_{i,M}(t) &= \frac{t - t_i}{t_{i+M-1} - t_i} N_{i,M-1}(t) + \frac{t_{i+M} - t}{t_{i+M} - t_{i+1}} N_{i+1,M-1}(t) \end{aligned}\right\}, \tag{2}$$

where $i = 0,1,\cdots,n-1$ and $M = 2,3,\cdots,6$.

If the knot vector contains a sufficient number of repeated knot values, then a division of the form $N_{i,M-1}(t)/(t_{i+M-1} - t_i) = 0/0$ (for some i) may be encountered during the execution of the recursion. Whenever this occurs, it is assumed that $0/0 = 0$ [12].

A quintic NURBS curve consists of one segment with the knot vector $\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$ is expressed as in Eq.(3).

$$\begin{aligned} \boldsymbol{R}(t) = \frac{1}{120}\Big\{ & (1-t)^5 \boldsymbol{q}_0 \\ & + (5t^5 - 20t^4 + 20t^3 + 20t^2 - 50t + 26)\boldsymbol{q}_1 \\ & + (-10t^5 + 30t^4 - 60t^2 + 66)\boldsymbol{q}_2 \\ & + (10t^5 - 20t^4 - 20t^3 + 20t^2 + 50t + 26)\boldsymbol{q}_3 \\ & + (-5t^5 + 5t^4 + 10t^3 + 10t^2 + 5t + 1)\boldsymbol{q}_4 \\ & + t^5 \boldsymbol{q}_5 \Big\} \end{aligned} \tag{3}$$

The first derivative of a quintic NURBS curve shown in Eq.(3) is expressed as in Eq.(4).

$$\begin{aligned} \frac{d\boldsymbol{R}(t)}{dt} = \frac{1}{120}\Big\{ & -5(1-t)^4 \boldsymbol{q}_0 \\ & + (25t^4 - 80t^3 + 60t^2 + 40t - 50)\boldsymbol{q}_1 \\ & + (-50t^4 + 120t^3 - 120t)\boldsymbol{q}_2 \\ & + (50t^4 - 80t^3 - 60t^2 + 40t + 50)\boldsymbol{q}_3 \\ & + (-25t^4 + 20t^3 + 30t^2 + 20t + 5)\boldsymbol{q}_4 \\ & + 5t^4 \boldsymbol{q}_5 \Big\} \end{aligned} \tag{4}$$

Curvature vector is expressed by Eq.(5).

$$\boldsymbol{\kappa}(t) = \frac{\left(\dot{\boldsymbol{R}}(t) \times \ddot{\boldsymbol{R}}(t)\right) \times \dot{\boldsymbol{R}}(t)}{\left(\dot{\boldsymbol{R}}(t)\right)^4}, \tag{5}$$

where $\dot{\boldsymbol{R}}(t)$ is the first derivative of a NURBS curve, and $\ddot{\boldsymbol{R}}(t)$ is the second derivative of a NURBS curve.

Curvature is the magnitude of the curvature vector, therefore curvature is expressed as in Eq.(6).

$$\kappa(t) = \left|\boldsymbol{\kappa}(t)\right| \tag{6}$$

By definition, the curvature of a plane curve is nonnegative. However, in many cases it is useful to ascribe a sign to the curvature [13]. The choosing of the sign is commonly connected with the tangent rotation in moving along the curve in the direction of the increasing parameter. The curvature of the curve is positive when its tangent rotates counter-clockwise, it is negative when its tangent rotates clockwise.

Radius of curvature is the reciprocal number of curvature, therefore, radius of curvature is expressed as in Eq.(7).

$$\rho(t) = \frac{1}{\kappa(t)} \tag{7}$$

2.2 NURBS Curve Display

The curvature of a curve is the most significant descriptor of its shape [14]. To check the shape of a curve by displaying its curvature or radius of curvature plots is widely known. This is simply the graph of $\kappa(t)$ or $\rho(t)$.

Curvature information is plotted using straight lines drawn outward from and perpendicular to the curve, with the line length proportional to the amount of curvature at that spot. Radius of curvature information is plotted using straight lines drawn inward from and perpendicular to the curve, with the line length proportional to the amount of radius of curvature at that spot.

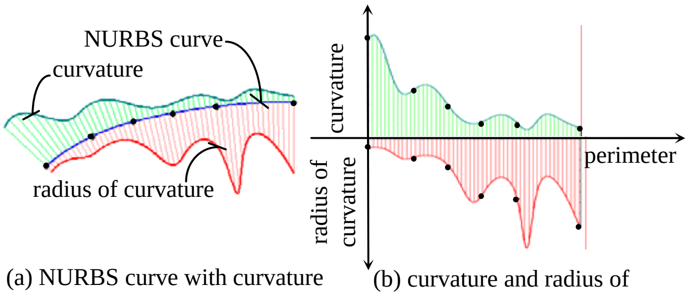
It is hard to distinguish the two curves by just looking at their graphs. If the radius of curvature plots are drawn for both, the difference of the two curves is recognized immediately. Curve shape is judged by looking at the lines coming out from the curve and seeing how their lengths change along the path, not along the parameter. Therefore, curvature or radius of curvature distribution must be drawn to the perimeter of the curve.

A NURBS curve with curvature and radius of curvature plots, and curvature and radius of curvature distribution are shown in Fig.1(a), (b) respectively.

A shape modified NURBS curve with curvature and radius of curvature plots, and curvature and radius of curvature distribution are shown in Fig.2(a), (b) respectively.

While both curves shown in Fig.1(a) and Fig.2(a) can hardly be distinguished by just looking at their curves, the curve with curvature and radius of curvature plots tell the two curves apart immediately.

Curvature distribution of a NURBS curve is not monotone as shown in Fig.1(b). But curvature and radius of curvature distribution of a shape modified NURBS curve is monotone as shown in Fig.2(b). This display technique provides designers with the ability to evaluate the quality of a designed curve. Seeing a NURBS curve with curvature and radius of curvature plots gives designers a deeper understanding of their design.

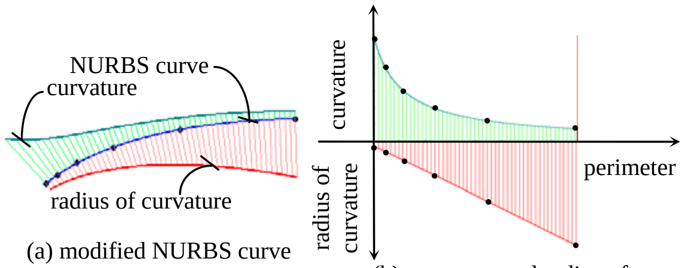


(a) NURBS curve with curvature and radius of curvature plots

(b) curvature and radius of curvature distribution

• Point marks indicate knot position.

Fig.1 NURBS curve, curvature and radius of curvature distribution



(a) modified NURBS curve with curvature and radius of curvature plots

(b) curvature and radius of curvature distribution

• Point marks indicate knot position.

Fig.2 Modified NURBS curve, curvature and radius of curvature distribution

The relation of curvature to radius of curvature is reciprocal. Therefore, it can be seen in Fig.1(b) and Fig.2(b) that in a portion of a curve where the curvature is small, at this portion the radius of curvature will be large. As shown in Fig.2(b), if the radius of curvature to the perimeter is linear, the curvature distribution will be parabolic. On the contrary, if the curvature to the perimeter is linear, radius of curvature distribution will be parabolic.

In case the curve shape is close to a straight line, the radius of curvature becomes infinity. Therefore, a limit value should be assigned to the radius of curvature.

Both curvature distribution and radius of curvature distribution displays are effective to examine the shape of a curve.

3 Generation of a NURBS Curve

In this section, a method to generate a quintic NURBS curve which passes through given points in sequence is shown. Another method to generate a quintic NURBS curve using the given points in sequence with gradients is described.

3.1 Generation of a Quintic NURBS Curve which Passes through Given Point Sequence

Putting zero to the parameter of Eq.(3), Eq.(3) is expressed as Eq.(8) by defining the geometrical knot position corresponding to the knot vector.

$$\boldsymbol{R}_i = \frac{1}{120}(\boldsymbol{q}_i + 26\boldsymbol{q}_{i+1} + 66\boldsymbol{q}_{i+2} + 26\boldsymbol{q}_{i+3} + \boldsymbol{q}_{i+4}), \quad (i = 0,1,2,3,\cdots,m-1) \tag{8}$$

where m is the number of the given points, and $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$, $\cdots$, $\boldsymbol{P}_{m-2}, \boldsymbol{P}_{m-1}$ are the positional vectors of the given points to be assigned to $\boldsymbol{R}_i (i = 0,1,2,3,\cdots,m-1)$ in Eq.(8), and $\boldsymbol{q}_0$, $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$, $\cdots$, $\boldsymbol{q}_{m+2}$, $\boldsymbol{q}_{m+3}$ are the control points of a quintic NURBS curve.

When the control points of a NURBS curve are calculated using Eq.(8), the number of unknowns, which are the positions of the control points, are four more than the number of equations which are expressed by Eq.(8). In this case, by setting the second derivative of the NURBS curve to zero, and setting the fourth derivative of the NURBS curve to zero, unknown variables become known. Therefore, the number of equations will be equal to the number of unknowns. That is, a linear system is determined [15]. Then a NURBS curve is generated by solving this determined system.

In this study, in addition to the given point sequence, gradient at the given points is defined.

Eq.(9) is applied to the gradients by setting the parameter of Eq.(4) as zero.

$$\frac{d\boldsymbol{R}_i}{dt} = \frac{1}{24}(-\boldsymbol{q}_i - 10\boldsymbol{q}_{i+1} + 10\boldsymbol{q}_{i+3} + \boldsymbol{q}_{i+4}), \quad (i = 0,1,2,3,\cdots,n-1) \tag{9}$$

where n is the number of given gradients. The i shown in Eq.(9) corresponds to the i in Eq.(8) and is determined situationally. As a magnitude of the first derivative, one third value of the distance of adjacent given points is assigned as a default value. For further adjustment, the magnitude of the first derivative is determined interactively.

The defined gradients are located at the beginning given point and it's adjacent point, and at the end given point and it's adjacent point in general. In this case, the i are determined as 0, 1, $n-2$, $n-1$ respectively. Using given point sequence and four location specified gradients, a linear system becomes determined. That is, the number of unknowns is equal to the number of equations. The concept of a quintic NURBS curve generation using the given point sequence and four location specified gradients are illustrated in Fig.3. $\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3, \cdots, \boldsymbol{P}_{m-2}, \boldsymbol{P}_{m-1}$ are given points. $\boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_{n-2}$, and $\boldsymbol{d}_{n-1}$ are the four location specified gradients.

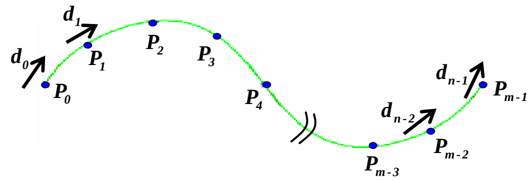


Fig.3 Concept of a quintic NURBS curve generation using the given point sequence and four location specified gradients

As examples of a quintic NURBS curve which passes through the given point sequence, three NURBS curves which simulate filleting curves are shown with their curvature plots in Fig.4. In Fig.4(a) and (b), the i of Eq.(9) are determined as 0, 2, 4, and 6. In Fig.4(c), the i of Eq.(9) are determined as 0, 1, 3, and 4. In the case of a filleting curve note that two gradients are placed at the start and end points of the filleting curve as well as the curve beginning and end points.

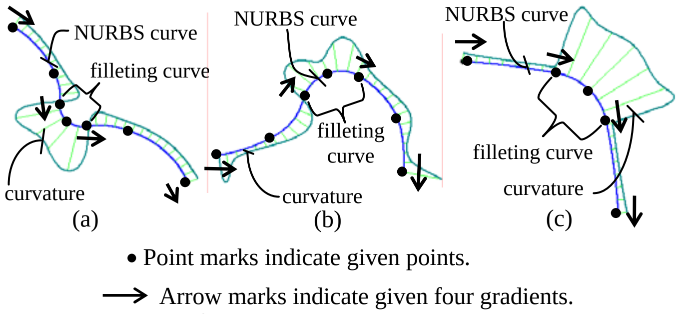


Fig.4 Examples of a NURBS curve which passes through the given point sequence

3.2 Generation of a Quintic NURBS Curve using the Given Points and Gradients

In this sub-section, a NURBS curve generation using the given points with gradients is described.

The concept of generation of a NURBS curve using the given points with gradients is illustrated in Fig.5. $\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3 \cdots, \boldsymbol{P}_{m-2}, \boldsymbol{P}_{m-1}$ is the given point sequence. $\boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \cdots, \boldsymbol{d}_{n-2}, \boldsymbol{d}_{n-1}$ are gradients assigned to the given points in sequence. A NURBS curve which passes through the given points and has the first derivatives at these given points is generated.

A NURBS curve is generated by solving Eq.(8) and Eq.(9) simultaneously by making m in Eq.(8) equal to n in Eq.(9). In this case, the i in Eq.(8) corresponds to the i in Eq.(9). If the number of given points with gradients is 4, the number of NURBS curve equations (Eq.(8)) is 4 and the number of first derivative equations (Eq.(9)) is 4. As a linear system, the total number of equations is 8, whereas the total number of control points of a NURBS curve is 8. Therefore, this linear system is determined. That is, the rank of a coefficient matrix of a linear system is equal to the number of unknowns. The solution to this linear system is exact.

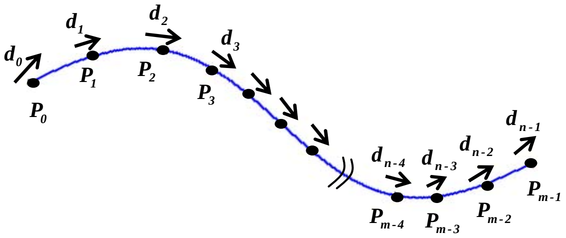


Fig.5 Concept of generation of a NURBS curve using the given points with gradients

But, in case the number of given points with gradients is 3, the number of equations (Eq.(8)) which pass through the given points is 3, and the number of equations of the first derivative (Eq.(9)) is 3. In this case, as a linear system, the total number of equations is 6, whereas the number of control points of the NURBS curve is 7. That is, the number of equations is less than the number of unknowns. Therefore, this linear system is underdetermined [16].

For an underdetermined system, while setting auxiliary function, the linear system is solved under the constraint condition by selecting one solution from infinite number of exact solutions using Lagrange's method of indeterminate multipliers.

In case the number of given points with gradients is 5, the number of equations (Eq.(8)) is 5, and the number of equations of the first derivative (Eq.(9)) is 5. In this case, as a linear system, the total number of equations is 10, whereas the number of control points of the NURBS curve is 9. That is, the number of equations exceeds the number of unknowns. Therefore, this linear system is overdetermined [17].

For an overdetermined system, the differences between the right and left sides of all the equations of the system are minimized. The control points calculated are an approximation.

For a system where the number of given points with gradients is more than 5, the linear system is overdetermined. For these systems, in accordance with the increments of the differences between the number of equations and the number of unknowns, the status of the approximation worsens.

The above mentioned are summarized in Table 1. A determined linear system is shown by the cross hatching.

Table 1 Linear system condition

(I)	(II)	(III)	(IV)
2	6	underdetermined	exact
3	7	underdetermined	exact
4	8	determined	exact
5	9	overdetermined	approximation
6	10	overdetermined	approximation
7	11	overdetermined	approximation
8	12	overdetermined	approximation

(I) number of given points with gradients
(II) number of control points of a NURBS curve
(III) system condition (underdetermined, determined, overdetermined)
(IV) solution status

As an example, in case the number of given points with gradients is 3, that is, m in Eq.(8) and n in Eq.(9) are 3, the NURBS curve generated as an underdetermined system is shown in Fig.6 with its first derivative, which is drawn outward perpendicular to the curve by the straight lines. The length of the line is proportional to the first derivative. This is an unusual way of displaying the first derivatives. Nevertheless, this helps visual recognition of the NURBS curve shape and its first derivative's magnitude variation. The solution to this linear system is exact.

In case the number of given points with gradients is 4, that is, m in Eq.(8) and n in Eq.(9) is 4, the NURBS curve is generated as a determined system. The solution to this linear system is exact.

Furthermore, in case the number of given points with gradients is 5, that is, m in Eq.(8) and n in Eq.(9) are 5, the NURBS curve is generated as an overdetermined system and is shown with its first derivative vectors in Fig.7. The solution to this linear system is an approximation.

In this manner, a NURBS curve is generated based on the given points with gradients.

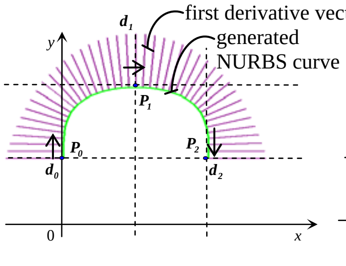


Fig.6 NURBS curve and its first derivative vectors, in case of underdetermined

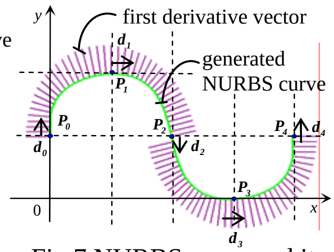


Fig.7 NURBS curve and its first derivative vectors, in case of overdetermined

4 Curve Shape Modification based on the Specified Radius of Curvature

In this section, a method to modify a NURBS curve shape according to the specified radius of curvature distribution to realize an aesthetically pleasing freeform curve is described.

Radius of curvature is suitable, because it conforms to our visual recognition of the shape of the curve. In a case where curve shape is very close to a straight line, the radius of curvature becomes infinity. And also, at the point of inflexion, curvature value becomes zero. Therefore, radius of curvature value becomes infinite. For these reasons, radius of curvature value is converted to curvature value for computation.

The concept of radius of curvature specification and NURBS curve shape modification based on the specified radius of curvature distribution is shown in Fig.8. A NURBS curve and its radius of curvature plots are shown in Fig.8.

Radius of curvature plots shown in Fig.8 are drawn inward from and perpendicular to the curve using straight lines. The length of the line is proportional to the radius of curvature at that spot on the curve. However, the straight lines are not parallel to each other and the beginning points of the individual straight lines are different. Therefore, a curve with a radius of curvature display is suitable to examine the variation of radius of curvature as a whole. But, it is not suitable to examine the length of the straight lines and variation of radius of curvature in detail.

Therefore, considering the parameter of the NURBS curve is different from the perimeter of the curve, the perimeter of a NURBS curve as a straight line is set to the horizontal axis, and the radius of curvature is set to the vertical axis as shown in Fig.9(a). Then, the radius of curvature distribution to the perimeter is drawn. After this, specified radius of curvature is superposed on the current radius of curvature distribution. Linear, quadratic, cubic, quartic, quintic, and six degree algebraic functions are applied as specified radius of curvature to the current radius of curvature distribution to modify the shape of the NURBS curve.

To be more in detail, coefficients of the algebraic function are calculated by introducing the least-squares method using the current radius of curvature distribution. Then, the radius of curvature is specified by the determined algebraic function.

As an example, the linear algebraic function as a specified radius of curvature specification is shown in Fig.9(a). The i th of radius of curvature distribution of a perimetrically represented NURBS curve is denoted as ρ_i, the specified radius of curvature at the same spot is denoted as $\hat{\rho}_i$, the difference δ_i is shown by Eq.(10) and is illustrated in Fig.8 and Fig.9(a).

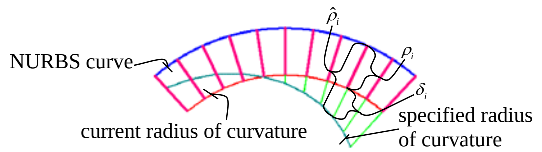
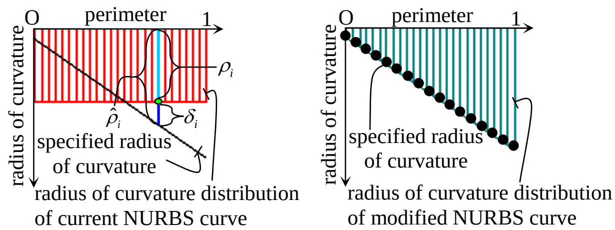


Fig.8 Concept of radius of curvature specification



(a) difference between current radius of curvature and specified radius of curvature

(b) radius of curvature of shape modified NURBS curve and specified radius of curvature (same as in (a))

Fig.9 Concept of curve shape modification based on the specified radius of curvature

$$\delta_i = \rho_i(q_1^x,\ \cdots,\ q_{n-2}^x,\ q_1^y,\ \cdots,\ q_{n-2}^y) - \hat{\rho}_i, \tag{10}$$

where $i = 0,1,2,\cdots,m-1$, m is the number of specified radius of curvature, and n is the number of NURBS curve segments plus 5, which is the degree of the curve.

$S(q_1^x,\ \cdots,\ q_{n-2}^x,\ q_1^y,\ \cdots, q_{n-2}^y)$ which is the sum of the squared differences for all specified radius of curvatures in Eq.(11) is minimized by introducing the least-squares method. The radius of curvature expression is non-linear. Therefore, by Taylor's theorem, Eq.(11) is linearlized as in Eq.(12).

$$S(q_1^x,\ \cdots,\ q_{n-2}^x,\ q_1^y,\ \cdots,\ q_{n-2}^y) = \sum_{i=0}^{m-1}\left[\rho_i(q_1^x,\ \cdots,\ q_{n-2}^x,\ q_1^y,\ \cdots,\ q_{n-2}^y) - \hat{\rho}_i\right]^2 \tag{11}$$

$$\begin{aligned} &S(q_1^x+\Delta q_1^x,\ \cdots,\ q_{n-2}^x+\Delta q_{n-2}^x,\ q_1^y+\Delta q_1^y,\ \cdots,\ q_{n-2}^y+\Delta q_{n-2}^y) \\ &= \sum_{i=0}^{m-1}\left[\rho_i(q_1^x,\ \cdots,\ q_{n-2}^x,\ q_1^y,\ \cdots,\ q_{n-2}^y) + \frac{\partial\rho_i}{\partial q_1^x}\Delta q_1^x + \right. \\ &\left. \cdots + \frac{\partial\rho_i}{\partial q_{n-2}^x}\Delta q_{n-2}^x + \frac{\partial\rho_i}{\partial q_1^y}\Delta q_1^y + \cdots + \frac{\partial\rho_i}{\partial q_{n-2}^y}\Delta q_{n-2}^y - \hat{\rho}_i\right]^2 \end{aligned} \tag{12}$$

Eq.(12) is minimized by equating to zero all the partial derivatives of $S(q_1^x+\Delta q_1^x,\ \cdots,\ q_{n-2}^x+\Delta q_{n-2}^x,$ $q_1^y+\Delta q_1^y,\ \cdots,\ q_{n-2}^y+\Delta q_{n-2}^y)$ with respect to Δq_r^x and Δq_r^y $(r=1,\ 2,\ \cdots,\ n-2)$ as in Eq.(13).

$$\left.\begin{aligned} \frac{\partial S}{\partial \Delta q_r^x} &= 0 \quad (r=1,2,\cdots,n-2) \\ \frac{\partial S}{\partial \Delta q_r^y} &= 0 \quad (r=1,2,\cdots,n-2) \end{aligned}\right\} \tag{13}$$

Using these simultaneous linear equations, Δq_r^x and Δq_r^y $(r=1,2,\cdots,n-2)$ are calculated. Then, q_r^x and q_r^y are determined.

This kind of study on the radius of curvature, or the curvature to realize a fair curve is called a constrained non-linear minimization problem [18]. For computation, ρ_i and $\hat{\rho}_i$ are calculated based on the perimeter. Then, the perimeter used is converted to the parameter to calculate the position of the control points of the NURBS curve. Then, a NURBS curve is generated. The total length of the curve which is the perimeter is calculated and rescaled as 1. Repeating these operations, positions of the control points of the NURBS curve are determined while $\delta_i\ (i = 0,1,\cdots,m-1)$ are minimized for the entire perimeter.

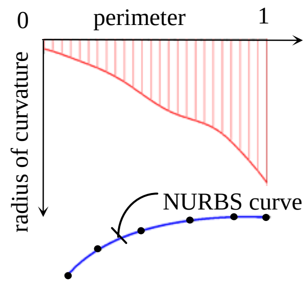
Using the above mentioned method of a linear algebraic function to specify the radius of curvature as shown in Fig.9(a), radius of curvature distribution is changed to the one shown in Fig.9(b), while modifying the shape of the curve. The dotted line shown in Fig.9(b) is a linear algebraic function specifying the radius of curvature distribution shown in Fig.9(a). It is visually recognized that the radius of curvature distribution of the shape modified curve shown in Fig.9(b) matches to the specified radius of curvature.

5 Fair Curve Expression and Fairness Evaluation

In this section, fair curve expression and fairness evaluation are described. A curve with a monotone radius of curvature distribution is considered as a fair curve in the area of Computer Aided Aesthetic Design. But no official standards are given. Therefore, criterion for a fair curve is proposed tentatively.

The shape of a NURBS curve is defined by the number, the location of its control points, and the knot sequence. The designed curve is considered fair, if the variation of radius of curvature is monotone for the same number of control points and the knot sequence.

As curve fairness evaluation, radius of curvature distribution is used as an alternative characteristic of a curve. Algebraic functions such as linear, quadratic, cubic, quartic, quintic, and six degrees are applied to the radius of curvature distribution of the designed curve to specify the radius of curvature. Then, the shape of the curve is modified according to the specified radius of curvature distribution by using the shape modification algorithm mentioned in the previous section. In this manner, six NURBS curves whose radius of curvature are linear, quadratic, cubic, quartic, quintic, and six degrees are generated, and are predefined based on the designed NURBS curve. These six NURBS curves are considered as fair, since their radius of curvature is monotone because their applied algebraic function's variation of the dependent variable is monotone to that of the independent variable. Using the correlation matching, the similarity is evaluated by comparing the radius of curvature distribution of the designed curve with those of six NURBS curves which are predefined. The values of radius of curvature to the perimeter are considered as the components of a multi dimensional vector for the curve. Then, similarity between two curves is expressed by normalizing the dot product of two vectors. This similarity is evaluated as correlation coefficient. The highest similarity curve to the designed curve among these predefined curves is selected. The correlation coefficient of the selected curve is determined as fairness of the designed curve.



• Point marks indicate knot position.

Fig.10 Designed curve and its radius of curvature distribution

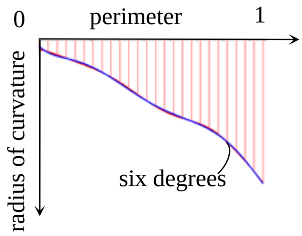


Fig.11 Radius of curvature distribution and a given six degree algebraic function to specify radius of curvature

As an example of fairness evaluation, a NURBS curve and its radius of curvature distribution to the perimeter are shown in Fig.10.

The radius of curvature distribution shown in Fig.10 is specified by six algebraic functions mentioned above. Then, the shape of the curve is modified according to the specified radius of curvature distribution using the shape modification algorithm. In this manner, six NURBS curves whose radius of curvature are linear, quadratic, cubic, quartic, quintic, and six degrees are generated and are predefined based on the designed NURBS curve shown in Fig.10. Afterwards, setting the radius of curvature distributions of the shape modified six curves as references and the radius of curvature distribution of the designed curve shown in Fig.10 as a match, six similarities are evaluated by the correlation matching. The correlation coefficients evaluated are summarized in Table 2. Correlation coefficients in Table 2 expresses the fairness of the designed curve to derivatively generated six curves. From Table 2, the designed curve whose radius of curvature is shown in

Fig.10, is judged to be designed so that the radius of curvature distribution will be six degree. This is shown in Fig.11 together with the radius of curvature distribution shown in Fig.10.

Table 2 Fairness of the designed

algebraic function	linear	quadratic	cubic	quartic	quintic	six
radius of curvature shown in Fig.11	0.99772	0.99888	0.99905	0.99974	0.99981	0.99993

Giving eight sample curves, fairness is examined. Eight designed curves and their radius of curvature distributions are shown in Fig.12 (a), (b), (c), (d), (e), (f), (g), and (h), and are labeled curve A, B, C, D, E, F, G, and H respectively.

Applying the six algebraic functions to the radius of curvature distributions of these eight designed curves respectively, the radius of curvature distributions corresponding to these six algebraic functions are generated. If the radius of curvature is negative, it is considered that this algebraic function is not applicable. The original radius of curvature distributions of eight designed curves and those of modified based on the specified algebraic functions are shown in Fig.13.

The correlation coefficients are evaluated by setting the radius of curvature distribution according to these six algebraic functions as references and the radius of curvature distribution of the eight designed curves shown in Fig.12 as matches. As described above, the correlation coefficient evaluated is considered as the fairness of the designed curve.

The fairness of eight curves shown in Fig.12 is summarized in Table 3.

The correlation coefficient evaluated is expressed in cosine. Therefore, it is not easy to distinguish the small differences. So, the correlation coefficient which shows similarity is expressed in degree and summarized in Table 4.

From Table 4, it is recognized that curve A, curve B, curve E, and curve H are designed so that their radius of curvature distribution will follow quintic function as shown by the yellow hatching. The fairness for curve A is 0.998624, for curve B is 0.999627, for curve E is 0.997770, and for curve H is 0.997263.

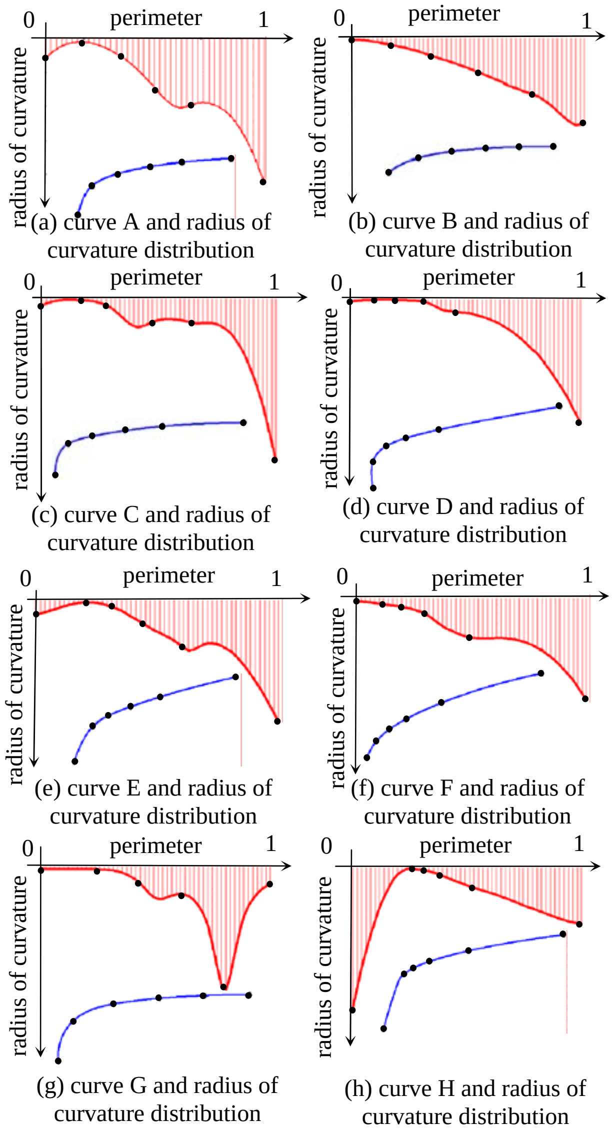
The radius of curvature distribution of these eight sample curves are shown with their closest reference algebraic function in Fig.13.

It is also recognized that curve F is designed so that its radius of curvature distribution will follow six degree function as shown by the green hatching. The fairness for curve F is 0.999800.

But it is recognized that curve C, D and G are not designed so that their radius of curvature distribution will follow one of these six algebraic functions.

The radius of curvature distribution of these eight sample curves are shown with their closest reference algebraic function in Fig.13.

As mentioned above, fairness of the designed curve is proposed by correlation coefficient using the radius of curvature distribution.



(a) curve A and radius of curvature distribution
(b) curve B and radius of curvature distribution
(c) curve C and radius of curvature distribution
(d) curve D and radius of curvature distribution
(e) curve E and radius of curvature distribution
(f) curve F and radius of curvature distribution
(g) curve G and radius of curvature distribution
(h) curve H and radius of curvature distribution

● Point marks indicate knot position.

Fig.12 Designed curves and their radius of curvature distributions

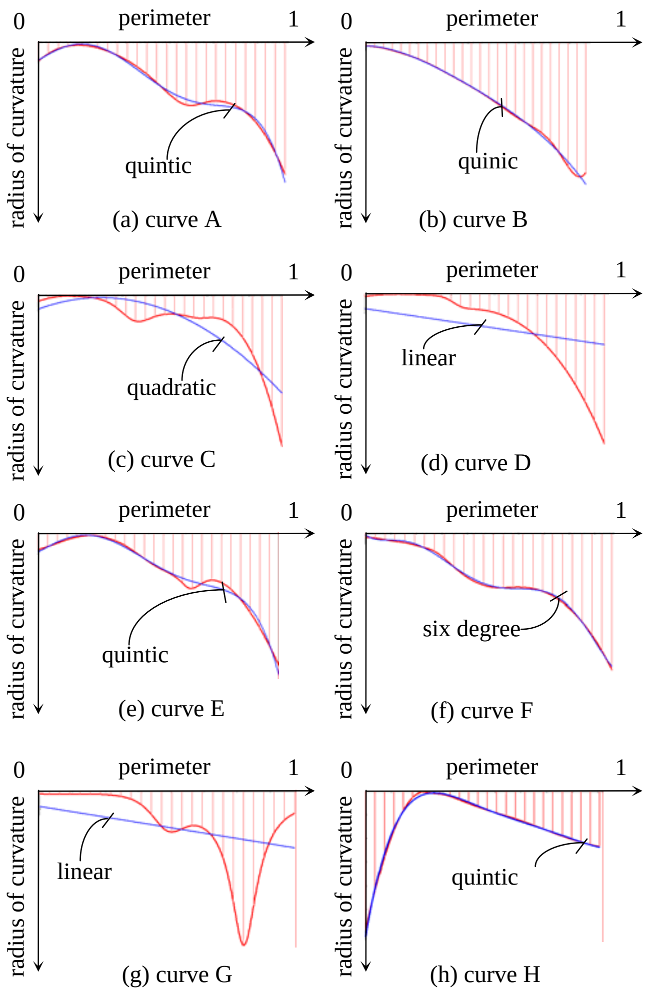


Fig.13 Radius of curvature distribution of individual curve and their closest reference algebraic function

Table 3 Fairness evaluated in cosine of the designed curves

	A	B	C	D	E	F	G	H
linear	0.952	0.966	0.781	0.816	0.923	0.969	0.804	0.729
quadratic	0.985	0.999	0.938	-	0.987	0.990	-	0.917
cubic	0.986	0.999	-	-	0.989	-	-	-
quartic	-	0.999	-	-	-	0.998	-	-
quintic	0.998	0.999	-	-	0.997	0.998	-	0.997
six	0.994	0.999	-	-	0.997	0.999	-	0.992

Table 4 Fairness evaluated in degree of the designed curves

	A	B	C	D	E	F	G	H
linear	17.73	14.80	3857	35.25	22.58	14.26	36.45	43.12
quadratic	9.32	1.95	20.25	-	9.21	7.82	-	23.41
cubic	9.30	1.98	-	-	8.25	-	-	-
quartic	-	1.58	-	-	-	2.82	-	-
quintic	3.00	1.56	-	-	3.82	2.75	-	4.24
six	6.21	1.68	-	-	4.27	1.14	-	7.03

Algebraic functions up to six degrees are applied to the radius of curvature distribution in this study. But the application of algebraic functions over six degrees should be studied as a future work.

6 Concluding Remarks

A quintic NURBS curve, the first derivative of a quintic NURBS curve, curvature vector, curvature, and radius of curvature are expressed.

The concept of radius of curvature specification to modify the shape of a NURBS curve is illustrated. The difference between the NURBS curve radius of curvature and the specified radius of curvature is minimized by introducing the least-squares method to modify the shape of the NURBS curve.

As curve fairness evaluation, radius of curvature distribution is used as an alternative characteristic of a curve. Algebraic functions such as linear, quadratic, cubic, quartic, quintic, and six degrees are applied to the radius of curvature distribution of the designed curve to specify radius of curvature. Then, the shape of the curve is modified according to the specified radius of curvature distribution. In this manner, six NURBS curves whose radius of curvature are these six algebraic functions respectively, are predefined. These six NURBS curves are considered as fair, since their radius of curvature is monotone because their applied algebraic function's variation of the dependent variable is monotone to that of the independent variable.

The similarity is evaluated by comparing the radius of curvature distribution of the designed curve with those of six predefined NURBS curves.

The highest similarity curve to the designed curve among these predefined curves is selected.

The similarity evaluated of the selected curve is determined as fairness of the designed curve.

In the future, we are planning to establish a definition of a fair curve using a lot of curve data that will be gathered.

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