MODERN COMMUNICATIONS USED NEW FORMULATION TO THE PROPORATION OF LAPLACE TRANSFORMS

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Abstract: - Laplace transforms was introduced long time ago, it had explained better afterward, and had many Engineering applications like (control, signals, circuits, etc.), unfortunately it still has a lot of ambiguity and is difficult to be understood by the beginners. In this paper the transformation laws are re-formulated in a brief and unforgettable form, and then a conclusion has been reached for the first time that the transformation laws are symmetric.

Key-Words: - Congestion Control, Signals, Transformation.

1 Introduction

The Laplace transform is defined by:

$Lap = F(s) = \int_0^\infty f(t)e^{-st}dt$, … t > 0 … Always positive

The Laplace transform of f(t) is exist [1,2] , if :

(a) *f(t)* is continuous or at least sectional continuous (piece wise continuous) in any interval $0 \le t \le N$, that is N is Positive.

(b) *f(t)* Must be at most of exponential order for > N. i.e. $f(t) = e^{t^2}$ does not possess a Laplace transform.

(c) The limit $Lim_{t\to\infty}[f(t)] = 0$ for some constant such that $0 < n < 1$. The inverse Laplace transform is denoted by $Lap^{-1}F(s) = f(t)$

Laplace Transform of Some Special Functions:

Some functions are considered to be special because new formulas can be obtained through the use of few basic (or special) laws.

(1) $Lap\, u[t] = 1/s$

(2) $Lap\,[t] = 1/s^2$

(3) $Lap\,[t^n] = \dfrac{n!}{s^{n+1}}$

(4) $Lap\,[e^{-at}] = \dfrac{1}{s+a}$

(5) $Lap\,[\cos bt] = \dfrac{s}{s^2+b^2}$

(6) $Lap\,[\sin at] = \dfrac{a}{S^2+a^2}$

From the above transforms, it can be realized that:

(a) Usually (s) is in the denominator when (t) is in numerator.

(b) The order of (s) is higher than the order of (t) by one.

(c) The Laplace transform of exponential function has reverse sign for the value of a. (see (4) above).

2 Re-formulation of Laplace Transform properties

2.1- Shifting (Translation) Properties (new Formulation)

An exponential function either in *t-domain* or in s-domain transformed into shift in the other domain in, i.e.

$Lap\,\{e^{\pm at} f(t)\} \leftrightarrow F(s \pm a)$, Shift in *s–domain*.

$Lap\,\{f(t-a)\,u(t-a)\} \leftrightarrow e^{-as}F(s)$, Shift in *t–domain*.

i.e. Exponential in a domain ↔ Shift in other domain.

A shift in either *t-domain* or s-domain can be transformed into an exponential function in the other domain. It is obviously shown that the sign is changed when the shift exists in *s-domain*, but isn't changed when the shift exists in *t-domain.*

Example 1: Find $Lap\,\{e^{-t}\cos(2.t)\}$

Since $Lap\,\{\cos 2.t\} = \frac{s}{s^2+4}$

Then $Lap\,\{e^{-t}\cos 2.t\} = \frac{s+1}{(s+1)^2+4}$, "notice the change in sign"

It is clear that exponential function in time domain transformed into shift in *S-domain*.

Example 2: Find $Lap^{-1}\,\{\frac{s}{(s+3)^2+1}\}$

$$Lap^{-1}\,\{\frac{s}{(s+3)^2+1}\} = Lap^{-1}\,\{\frac{s+3-3}{(s+3)^2+1}\}$$
$$= Lap^{-1}\,\{\frac{s+3}{(s+3)^2+1}\} - Lap^{-1}\,\{\frac{3}{(s+3)^2+1}\}$$
$$= e^{-3t}\cos t - 3e^{-3t}\sin t = e^{-3t}(\cos t - 3\sin t)$$

Example 3: Find $Lap\,\{(t-4)u(t-4)\}$

$$Lap\,\{(t-4)u(t-4)\} = e^{-4s}Lap\{t\} = e^{-4s\frac{1}{s^2}}$$

Example 4: Find $Lap\,\{\cos(t-2)u(t-2)\}$

$$Lap\,\{\cos(t-2)u(t-2)\} = e^{-2s}Lap\{\cos(t)\} = e^{-2s\frac{s1}{s^2+1}}$$

Example 5: Find $Lap^{-1}\,\{\frac{2e^{-2s}}{(s+1)(s+2)}\}$

$$Lap^{-1}\,\{\frac{2e^{-2s}}{(s+1)(s+2)}\} = Lap^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right]2e^{-2s}$$

$$= Lap^{-1}\left[\frac{2e^{-2[(s+1)-1]}}{(s+1)} - \frac{2e^{-2[(s+2)-2]}}{(s+2)}\right]$$

$$= Lap^{-1}\left[\frac{2e^{2}\,e^{-2(s+1)}}{(s+1)} - \frac{2e^{4}\,e^{-2(s+2)}}{(s+2)}\right]$$

$$= 2e^2e^{-t}\,Lap^{-1}\left[\frac{e^{-2s}}{s}\right] - 2e^4e^{-2t}Lap^{-1}\left[\frac{e^{-2s}}{s}\right]$$

$$= 2e^{-(t-2)} - u(t-2) - 2e^{-2(t-2)}u(t-2)$$

$$= 2\left[e^{-(t-2)} - e^{-2(t-2)}\right]u(t-2) = 2\left[e^{-(t-2)} - e^{-2(t-2)}\right],\ \ldots\ t > 2$$

2.2 Differentiation Properties (New Formulation)

The multiplication by either t or s in domains causes a differentiation in the other domain i.e.

$Lap\,\{t^n f(t)\} \leftrightarrow (-1)^n \frac{d^n}{ds^n}F(s)$, differentiation in *s-domain*

$Lap\,\{\frac{d^n}{dt^n}f(t)\} \leftrightarrow s^nF(s) - s^{n-1}f(0) - s^{n-2}f(0)\ldots - f^{n-1}(0)$, differentiation in *t-domain*

Notice that the differentiation in S-domain has alternative sign.

i.e. Differentiation in a domain ↔ Multiplication the other domain

Example 1: Find

$$Lap\,\{te^{2t}u(t)\} = -\frac{d}{ds}\frac{1}{(s-2)} = \frac{1}{(s-2)^2}$$

Example 2: Find $Lap\left\{\frac{d}{dt}\cos 3t\right\}$, differentiation in *t-domain*

Let $f(t) = \cos 3t$, then $F(s) = \frac{s}{s^2+9}$

$$\therefore Lap\left\{\frac{d}{dt}\cos 3t\right\} = s\left[\frac{s}{s^2+9}\right] - \cos(0) = \frac{s}{s^2+9} - 1 = -\frac{9}{s^2+9}$$

Example 3: Find $Lap^{-1}\left[\frac{s}{(s^2+4)}\right]$, Using (2.2)

$$= \frac{d}{dt}Lap^{-1}\left[\frac{1}{s^2+4}\right] = \frac{d}{dt}Lap^{-1}\left[\frac{1}{s^2+4}\right]\left[\frac{2}{2}\right] = \frac{d}{dt}\frac{\sin 2.t}{2} = \cos 2.t$$

notice no change in sign (differentiation is in t-domain)

2.3 Integration Properties (New Formulation)

The division by either t or s in its domain transformed into integration in the other domain (the integration must be exist ≠∞),

i.e. (Consider zero initial condition)

$Lap\left\{\frac{f(t)}{t}\right\} \leftrightarrow -\int_s^{\infty} F(s)ds$, integration in *s-domain*

Provided that $Lim_{t\to 0}\frac{f(t)}{t}$ is exists and also integration is exist

Also $Lap\left\{\int_0^t f(t)dt\right\} \leftrightarrow \frac{F(s)}{s}$, integration in *t-domain*

notice: the limits $\int_0^t$ in *t-domain* and $\int_s^{\infty}$ in *s-domain*

i.e. Integration in a domain $\leftrightarrow$ Division in the other domain

Example 1: Find $Lap^{-1}\left\{Ln\frac{s+1}{s-1}\right\}$

$$Lap^{-1}[Ln(S+1)-Ln(S-1)] = Lap^{-1}\int_s^{\infty}\frac{d}{ds}[Ln(S+1)-Ln(S-1)]$$

$$=-\frac{1}{t}Lap^{-1}\frac{d}{ds}\{Ln(S+1)-Ln(S-1)\}$$

(Notice the elegant use of property 2.2 and 2.3)

$$=\frac{-1}{t}Lap^{-1}\left\{\frac{1}{s+1}-\frac{1}{s-1}\right\}$$

$$=\frac{-1}{t}\left\{e^{-t}-e^{t}\right\}=\frac{2}{t}\frac{e^{t}-e^{-t}}{2}=\frac{2\sin ht}{t}$$

Example 2: Find $Lap\left\{\frac{\sin t}{t}\right\}$

Let $f(t)=\sin t$, then $F(S)=\frac{1}{S^2+1}$

$$\therefore Lap\left\{\frac{\sin t}{t}\right\}=\int_s^{\infty}\frac{1}{S^2+1}=\left[\tan^{-1}S\right]_s^{\infty}=\frac{\pi}{2}-\tan^{-1}s$$

Example 3: Find $Lap\left\{\int_0^t \sin 2t\, dt\right\}$

We have $f(t)=\sin 2t$ then $F(s)=\frac{2}{S^2+4}$

$$\therefore Lap\left\{\int_0^t \sin 2t\, dt\right\}=\frac{1}{s}\left[\frac{2}{s^2+4}\right]$$

3 Convolution Theorem

Let *f(t)* and *g(t)* be a Laplace transformable, then the convolution integral is defined by:

$$Lap^{-1}[F(s)\cdot G(s)]=f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$

Where the symbol* represents the convolution of *F(s). G(s)*

Let $Lap^{-1}F(s)=f(t)$ and $Lap^{-1}G(s)=g(t)$

Then apply the convolution of theorem as:

$$Lap^{-1}[F(s)\cdot G(s)]=f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$

Where the symbol * represents the convolution of *F(s) . G(s)*

Let [*Lap* f_1=*F (s)*] and [*Lap* $f_2(t)=F_2(s)$]

Then the inverse convolution theorem is

$$Lap\{f_1(t)\cdot f_2(t)\}=\frac{1}{2\pi J}\int_{c-jw}^{c+jw}F_1(\tau)\cdot F_2(s-\tau)d\tau$$

$$=\frac{1}{2\pi J}\int_{c-jw}^{c+jw}F_1(s)*F_2(s)\cdot ds$$ … The convolution transformation is bi-directional.

Example 1: Find $Lap^{-1}=\left[\frac{1}{(s-1)^2}\right]$

We have $Lap\{te^{at}\}=\frac{n\,!}{(s-a)^{n+1}}$

$$\therefore Lap^{-1}=\left[\frac{1}{(s-1)^2}\right]=te^{t}$$

Now, solving the same example using convolution theorem

$$Lap^{-1}\left[\frac{1}{(s-1)^2}\right]=Lap^{-1}\left[\frac{1}{(s-1)}-\frac{1}{(s-1)}\right]=Lap^{-1}\{F(s)*G(s)\}$$

where $F(s)=\frac{1}{(s-1)}$, then $f(t)=e^{t}$

and $G(s)=\frac{1}{s-1}$, then $g(t)=e^{t}$

therefore $Lap^{-1}\left[\frac{1}{(s-1)^2}\right]=f(t)\cdot g(t)=e^{t}*e^{t}$

$$=\int_0^t e^{\tau}e^{(1-\tau)}d\tau=\int_0^t e^{t}d\tau=e^{t}\int_0^t d\tau=te^{t}$$ … same result

Example 2: Find $Lap\left[e^{-3t}\int_0^t t\cdot\sin 2.t\, dt\right]$

Let *f(t)*=sin *2.t* , then $F(s)=\frac{2}{s^2+4}$

Therefore $Lap\{t\cdot f(t)\}=-\frac{d}{ds}\frac{2}{s^2+4}$

$$=\frac{-2(-2S)}{(S^2+4)^2}=\frac{4S}{(S^2+4)^2}$$

And $Lap\left[\int_0^t t\, f(t)\,dt\right]=\frac{1}{S}\frac{4S}{(S^2+4)^2}=\frac{4}{(S^2+4)^2}$

Finally $Lap\, e^{-3t}\left\{\int_0^t t\, f(t)dt\right\}=\frac{4}{[(s+3)^2+4]^2}$

Notice the exponential, integration and multiplication in *t-domain*.

<u>Example 3:</u> Find $Lap^{-1}\left\{\frac{e^{-S}+e^{-2S}}{s^2-3s+2}\right\}$

$$=Lap^{-1}\left\{\frac{e^{-S}+e^{-2S}}{(s-2)(s-1)}\right\}$$

$$=Lap^{-1}\left\{\frac{A}{s-2}+\frac{B}{s-1}\right\}\left(e^{-s}+e^{-2s}\right)$$

$$=Lap^{-1}\left\{\frac{1}{s-2}-\frac{1}{s-1}\right\}\left(e^{-s}+e^{-2s}\right)$$

$$=Lap^{-1}\left\{\frac{e^{-s}}{s-2}-\frac{e^{-s}}{s-1}+\frac{e^{-2s}}{s-2}-\frac{e^{-2}}{s-1}\right\}$$

$$=e^{-2(t-1)}u(t-1)-e^{(t-1)}u(t-1)+e^{2(t-2)}u(t-2)-e^{(t-2)}u(t-2),$$

shift in –domain

<u>Example 4:</u> Find $F(s)=Lap\left[\frac{d}{dt}\frac{e^{-2(t-3)}\sin 4(t-3)}{(t-3)}\right]$

$$=e^{-3s}Lap\left[\frac{d}{dt}\frac{e^{-2t}\sin 4t}{t}\right]$$

We have $Lap\,\sin(4.t)=\frac{4}{s^2+16}$

Also $Lap\;\, e^{-2t}\sin(4.t)=\frac{4}{(s+2)^2+16}$

And $Lap\;\,\frac{e^{-2t}\sin(4.t)}{t}\int_S^\infty\frac{4}{(S+2)^2+16}ds$

Then $Lap\;\,\frac{d}{dt}\frac{e^{-2t}\sin(4.t)}{t}=S\cdot\int_S^\infty\frac{4}{(S+2)^2+16}ds$

Finally

$$F(s)=e^{-3s}S\cdot\int_S^\infty\frac{4}{(s+2)^2+16}ds\;Se^{-3s}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{s+2}{4}\right)\right]$$

4 Conclusion

A useful and simple representation of Laplace transform with the most importance properties has been discussed. It is the first time in the text to introduce that the Laplace transform properties are conjugate, and all the transformation laws can be summarized by six words instead of twelve very complicated laws as follows:

Exponential ↔ *Shift*

Multiplication ↔ *Differentiation*

Division ↔ *Integration*

Accordingly, combination of the properties is introduced as:

Exponential + Multiplication + Integration ↔ *Shift + Differentiation + Division*

- The convolution integrals are also Bi-directional.
- During teaching, this new formulation proven double the efficiency of understanding and make very easy to remember the laws.
- It summarizes most of the Laplace transform properties in only six words, even so they are easy to keep in mind and makes unforgettable basis of Laplace transforms.
- It increases the capability of the student to understand the solved example and keep it perfectly in mind.
- This formulation is ideal for engineers as they are interested only in understanding the overall structure of Laplace transform, but not interested in the pure mathematical meaning, than convolution theorem.

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