# Transformations for Direct Design of 2-D Filters from Appropriate 1-D Functions

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*Abstract:* - In this paper, general transformations for designing 2-D (Two-Dimensional) IIR filters are provided. The present approach can be viewed as an extension of the classical method of magnitude approximation and can be applied in several cases of 2-D filter design. Numerical examples illustrate the validity and the efficiency of the method.

*Key-Words:* - 2-D Filters, FIR Filters, IIR Filters, Multidimensional Systems, Multidimensional Filters, Filter Design, McClellan Transformations

## 1 Introduction

Several techniques for the design of 2-D (Two-Dimensional) filters have been published recently due to the vast applications of 2-D filters in image processing, satellite communications, biomedical imaging, computer vision etc.. Various popular methods have received considerable attention by engineers and research scholars. In general the design of 2-D FIR filters includes *a Fourier method* that uses Fourier analysis, where appropriate window Functions can also eliminate Gibbs' oscillations, *a Transformations' method* which is based on McClellan Transformations from appropriate 1-D filters [1],[2] and *an optimization method* i.e. the minimization of an appropriate norm, [1],[2]. Similarly, the design of 2-D IIR filters includes also transformations, Mirror Image Polynomials, SVD (Singular Value Decomposition) and Optimization, [1],[2]. Several Authors have published works on optimization-based 2-D filter design while a great number of papers are dedicated to transformations and mainly to McClellan Transformations, [3]÷[18]. McClellan Transformations were introduced in [3] and have been used for the last forty years in many theoretical topics and engineering applications. Harn and Shenoi pointed out in [5] and Nguyen and Swamy reported in [6], that till now a transformation for IIR filter design analogous to McClellan transformation does not exist due to the requirements of 2-D stability.

The purpose of this paper is to find such transformations and an attempt is made in section II.

This paper examines this transformation as well as its generalization to the general 2-D filters design. The usefulness of the proposed transforms is verified through two examples in section III. Finally, there is a conclusion.

## 2 The Transformation and its Generalizations

Let us consider the 1-D function of the prototype low-pass Butterworth filter (cut-off frequency $\Omega=1$)

$$y = |H(j\Omega)|^2 = \frac{1}{1+\Omega^{2n}}$$
(1)

The question here is "could it be possible after some appropriate transformation, for this function to derive the stable transfer function of a 2-D low-pass filter without the usual Transformations (McClellan Transformation) or the popular optimization techniques?"

An easy transformation could be for example $\Omega = \omega_1^2 + \omega_2^2$, where $\omega_1, \omega_2$ are the frequencies of the discrete 2-D filter. However, in such a case, a filter with magnitude $|H(j\omega_1, j\omega_2)| = \frac{1}{1+(\omega_1^2+\omega_2^2)^{2n}}$ can not be implemented since it does not correspond to a transfer function that would be a rational function of $z_1^{-1}, z_2^{-1}$. Another transformation could be, for instance, $\Omega = \sin^2(\omega_1) + \sin^2(\omega_2)$. The problem in this case is that

due to $\sin^2(\varphi) = \frac{1-\cos(2\varphi)}{2}$ the period of $\omega_1$ (and $\omega_2$) would be $\pi$ and not $2\pi$ as the 2-D filter design demands.

The transformation $\Omega = \sin^2(\frac{\omega_1}{2}) + \sin^2(\frac{\omega_2}{2})$ or the more general $\Omega = \sin^{2p}(\frac{\omega_1}{2}) + \sin^{2p}(\frac{\omega_2}{2})$ seem more logical, where p is a positive integer.

Obviously, the low frequencies of $\Omega$ are depicted to a region of low frequencies of $\omega_1, \omega_2$ - details will be presented in the next paragraphs.

In all the previous cases, the 2-D filter will be a non-causal 2-D system. The non-causality in 2-D systems is not a problem, since the dimensions are spatial and do not correspond to time.

By introducing $\Omega = \sin^{2p}(\frac{\omega_1}{2}) + \sin^{2p}(\frac{\omega_2}{2})$ from our prototype function of (1) we take

$$|H(j\omega_1, j\omega_2)| = \frac{1}{1+(\sin^{2p}(\frac{\omega_1}{2}) + \sin^{2p}(\frac{\omega_2}{2}))^{2n}}$$
(2)

So, using $\sin^2(\varphi) = \frac{1-\cos(2\varphi)}{2}$, one gets

$$|H(j\omega_1, j\omega_2)| = \frac{1}{1+\left(\left(\frac{1-\cos(\omega_1)}{2}\right)^p + \left(\frac{1-\cos(\omega_1)}{2}\right)^p\right)^{2n}}$$

Considering now that our filter is of zero-phase, one can write

$$H(j\omega_1, j\omega_2) = \frac{1}{1+\left(\left(\frac{1-\cos(\omega_1)}{2}\right)^p + \left(\frac{1-\cos(\omega_1)}{2}\right)^p\right)^{2n}}$$

that can be implemented because

$$\cos(\omega_1) = \frac{z_1^{-1} + (z_1^{-1})^{-1}}{2}, \ \cos(\omega_2) = \frac{z_2^{-1} + (z_2^{-1})^{-1}}{2}$$

Therefore

$$H(z_1^{-1}, z_2^{-1}) = \frac{1}{1+\left(\left(\frac{2-z_1^{-1}-(z_1^{-1})^{-1}}{4}\right)^p + \left(\frac{2-z_2^{-1}-(z_2^{-1})^{-1}}{4}\right)^p\right)^{2n}}$$

(3)

which can easily implemented provided that this is a BIBO (Bounded Input Bounded Output Filter). BIBO Stability of (3) can be proven easily taking into account that we have a non-causal 2-D system. So, a necessary and sufficient condition is ([28])

$B(z_1^{-1}, z_2^{-1}) \neq 0$ for all $z_1^{-1}$, $z_2^{-1}$ with $|z_1^{-1}| = 1$, $|z_2^{-1}| = 1$

where

$$B(z_1^{-1}, z_2^{-1}) = 1 + \left( \left( \frac{2 - z_1^{-1} - (z_1^{-1})^{-1}}{4} \right)^p + \left( \frac{2 - z_2^{-1} - (z_2^{-1})^{-1}}{4} \right)^p \right)^{2n}$$

However, this can be proven easily, since $z_1^{-1} = e^{i\vartheta_1}$ and $z_2^{-1} = \rho e^{i\vartheta_2}$.

So, $B(z_1^{-1}, z_2^{-1}) = 1 + \left( \cos^{2p}(\vartheta_1/2) + \cos^{2p}(\vartheta_2/2) \right)^{2n} \neq 0$

which is obviously $\neq 0$

Therefore the BIBO Stability of the filter in (3) has been proved.

A simple extension of (1) can be

$$y = |H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^{2n}\Omega^{2n}} \tag{4}$$

where $\varepsilon > 0$ and cut-off frequency $\varepsilon^{-1}$.

Under the transformation $\Omega = \sin^{2p}(\frac{\omega_1}{2}) + \sin^{2p}(\frac{\omega_2}{2})$, and following the same steps, one takes the stable 2-D filter

$$H(z_1^{-1}, z_2^{-1}) = \frac{1}{1 + \varepsilon^{2n} \left( \left( \frac{z_1^{-1} + (z_1^{-1})^{-1}}{2} \right)^p + \left( \frac{z_2^{-1} + (z_2^{-1})^{-1}}{2} \right)^p \right)^{2n}}$$

## 3 Numerical Examples and Design

### Example III.1

Consider (1) with $n = 8$, the magnitude $y = \frac{1}{1 + \Omega^{16}}$ is depicted in Fig.1

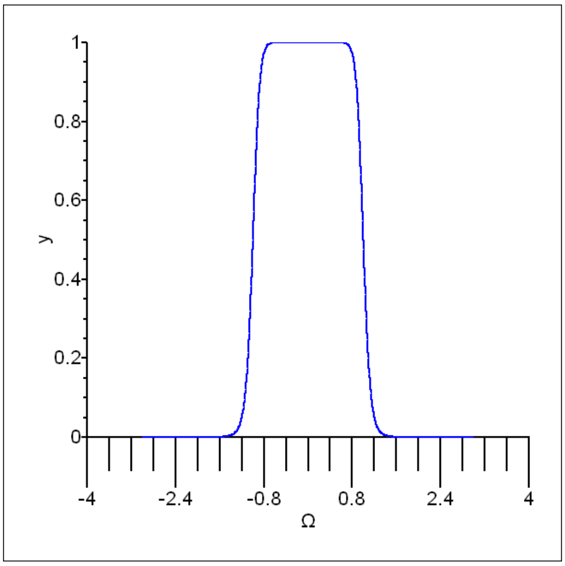


Fig.1 Plot $y = \frac{1}{1 + \Omega^{16}}$

Then, with the transformation $\Omega = \sin^4(\frac{\omega_1}{2}) + \sin^4(\frac{\omega_2}{2})$, i.e. $p = 2$, one gets the zero-phase filter with magnitude response in Fig.2

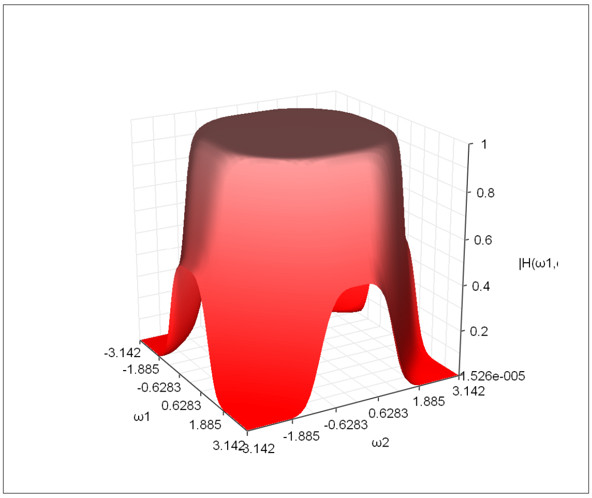


Fig.2. Plot $|H(j\omega_1, j\omega_2)| = \frac{1}{1 + (\sin^4(\frac{\omega_1}{2}) + \sin^4(\frac{\omega_2}{2}))^{16}}$

### Example III.2

Consider again (1), with $n = 8$, $y = \dfrac{1}{1+\Omega^{16}}$

Under the transformation $\Omega = \sin^8(\frac{\omega_1}{2}) + \sin^8(\frac{\omega_2}{2})$, i.e. $p$ =4, one gets the zero-phase filter with magnitude response in Fig.3

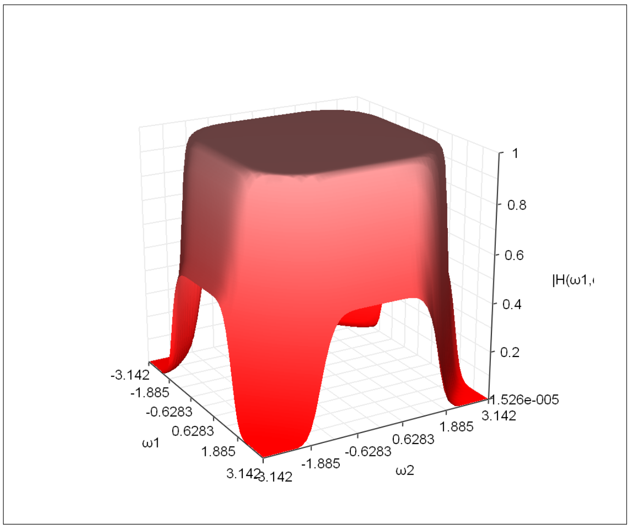


Fig.3. Plot $H(j\omega_1, j\omega_2) = \dfrac{1}{1+(\sin^8(\frac{\omega_1}{2})+\sin^8(\frac{\omega_2}{2}))^{16}}$

By considering $\varepsilon$ =2, we have $y = \dfrac{1}{1+\varepsilon^{16}\Omega^{16}}$

Now the transformation $\Omega = \sin^8(\frac{\omega_1}{2}) + \sin^8(\frac{\omega_2}{2})$

$$y = \frac{1}{1+\varepsilon^{16}(\sin^8(\frac{\omega_1}{2})+\sin^8(\frac{\omega_2}{2}))^{16}}$$ yields

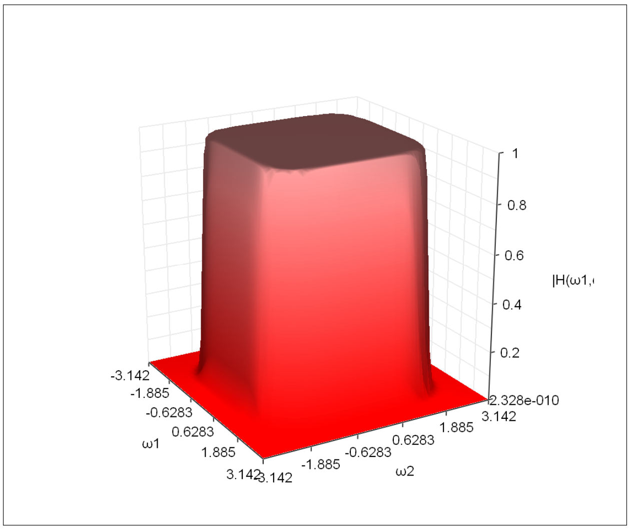


Fig.4.

Plot $H(j\omega_1, j\omega_2) = \dfrac{1}{1+2^{16}(\sin^8(\frac{\omega_1}{2})+\sin^8(\frac{\omega_2}{2}))^{16}}$

with cut-off frequencies to be given by

$$\varepsilon^{16}(\sin^8(\frac{\omega_1}{2})+\sin^8(\frac{\omega_2}{2}))^{16} = 1$$

or equivalently

$$\sin^8(\frac{\omega_1}{2})+\sin^8(\frac{\omega_2}{2}) = \frac{1}{2}$$

or $\left\|(\sin(\frac{\omega_1}{2}), \sin(\frac{\omega_2}{2}))\right\|_8 = \sqrt[0.8]{.5} \approx 0.917$

On the other hand

$$\left\|(\sin(\frac{\omega_1}{2}), \sin(\frac{\omega_2}{2}))\right\|_8 \approx \max\left(\left|\sin(\frac{\omega_1}{2})\right|, \left|\sin(\frac{\omega_2}{2})\right|\right)$$

i.e. the cut-off frequencies can be defined by

$$\omega_1, \omega_2 = 2 \cdot \arcsin(0.917) = 2.3209$$

The previous filters are symmetrical in $\omega_1, \omega_2$ thanks to $\Omega = \sin^{2p}(\frac{\omega_1}{2}) + \sin^{2p}(\frac{\omega_2}{2})$. Introducing now the real parameters *a, b,* one takes the transformation

$\Omega = a^{2p}\sin^{2p}(\frac{\omega_1}{2}) + b^{2p}\sin^{2p}(\frac{\omega_2}{2})$ that provide the (non-causal) filter

$$H(z_1^{-1}, z_2^{-1}) = \frac{1}{1+\varepsilon^{2n}\left(a^{2p}\left(\frac{2-z_1^{-1}-(z_1^{-1})^{-1}}{4}\right)^p + b^{2p}\left(\frac{2-z_2^{-1}-(z_2^{-1})^{-1}}{4}\right)^p\right)^{2n}}$$

(5)

**Example III.3**

Starting from $y = \frac{1}{1+\varepsilon^{16}\Omega^{16}}$ ($n$ = 8) and using $\Omega = a^{2p}\sin^{2p}(\frac{\omega_1}{2}) + b^{2p}\sin^{2p}(\frac{\omega_2}{2})$, with $p$ =4, compute the parameters $\varepsilon$, $a$, and $b$, in (5), such that cut-off frequencies to be $\omega_{10} = \pm\pi/2$, $\omega_{20} = \pm\pi/3$.

Solution: Our filter must be

$$y = \frac{1}{1+\varepsilon^{16}\left(a^8\sin^8(\frac{\omega_1}{2}) + b^8\sin^8(\frac{\omega_2}{2})\right)^{16}}$$

Without loss of generality the constant $\varepsilon$ is incorporated to $a$ and $b$. So, one can set $\varepsilon$=1. The cut-off frequencies are found from the equation

$$\left(a^8\sin^8(\frac{\omega_1}{2}) + b^8\sin^8(\frac{\omega_2}{2})\right)^{16} = 1$$

Or equivalently

$$a^8\sin^8(\frac{\omega_1}{2}) + b^8\sin^8(\frac{\omega_2}{2}) = 1$$

Since,

$$\left\|\left(a\sin(\frac{\omega_1}{2}), b\sin(\frac{\omega_2}{2})\right)\right\|_8 \approx \max\left(\left|a\sin(\frac{\omega_1}{2})\right|, \left|b\sin(\frac{\omega_2}{2})\right|\right)$$

Using $\omega_{10} = \pm\pi/2$, $\omega_{20} = \pm\pi/3$ from $\max\left(\left|a\sin(\frac{\omega_{10}}{2})\right|, \left|b\sin(\frac{\omega_{20}}{2})\right|\right) = 1$ we take

$a = \sqrt{2}$, $b = 2$

Hence, our filter is

$$H(z_1^{-1}, z_2^{-1}) = \frac{1}{1+\left(2^4\left(\frac{2-z_1^{-1}-(z_1^{-1})^{-1}}{4}\right)^4 + 2^8\left(\frac{2-z_2^{-1}-(z_2^{-1})^{-1}}{4}\right)^4\right)^{16}}$$

and can be depicted easily in Fig.5 taking

$$\left|H(\omega_1, \omega_2)\right| = \frac{1}{1+\left(2^4\sin^8(\frac{\omega_1}{2}) + 2^8\sin^8(\frac{\omega_2}{2})\right)^{16}}$$

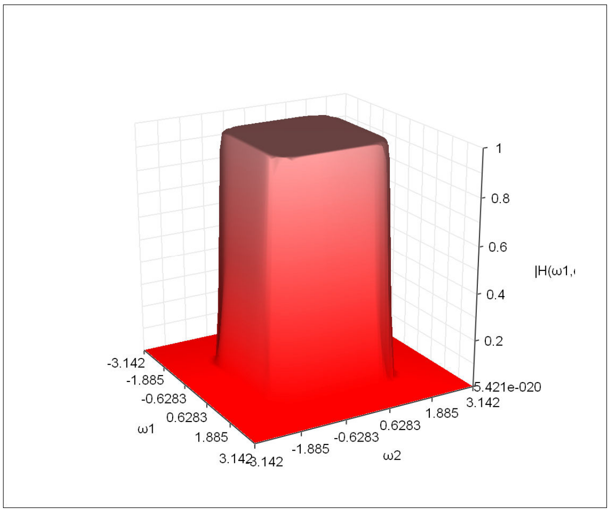


Fig.5: Magnitude response of the filter of Example III.3

In Fig.5, the magnitude response is illustrated if $\omega_2 = 0$

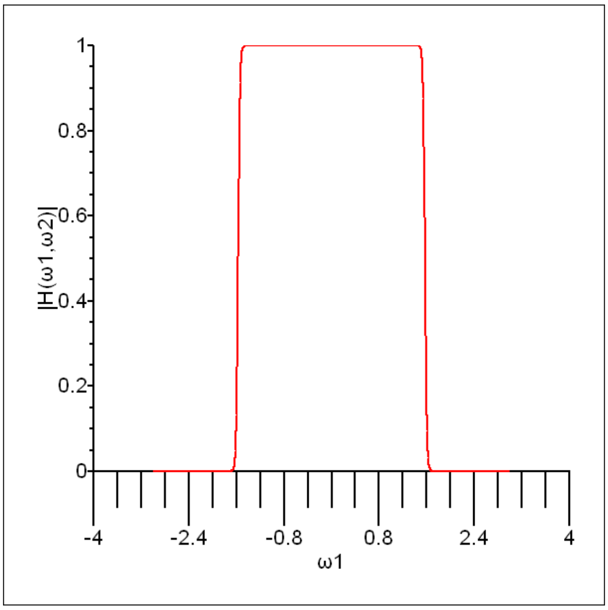


Fig.6: Magnitude response of the filter of Example III.3 if $\omega_2 = 0$

# 4 Conclusion

In this brief, new transformations for designing 2-D (Two-Dimensional) IIR filters are introduced. The resulting non-causal IIR filter is proven to be BIBO stable. Some numerical examples illustrate the validity and usefulness of the proposed transformation. Analogous transformations can be derived for highpass, bandstop and bandpass non-causal 2-D IIR filters.

*References:*

[1] Belle A. Shenoi, "Magnitude and Delay Approximation of 1-D and 2-D Digital Filters" , Springer Verlag, Berlin, 1999

[2] Wu-Sheng Lu, Andreas Antoniou, "Two-dimensional digital filters", M. Dekker, New York, 1992

[3] J.H.McClellan, "The Design of two-dimensional digital filters by transformations" in Proc. 7th Annual Princeton Conf.Inf. Sci.Syst. pp247-251, 1973

[4] Charng-Kann Chen;Ju-Hong Lee; McClellan transform based design techniques for two-dimensional linear-phase FIR filters, IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, Vol. 41 No.8, pp.505 – 517, Aug 1994

[5] Harn, L., Shenoi, B., "Design of stable two-dimensional IIR filters using digital spectral transformations" IEEE Transactions on Circuits and Systems, Vol.33 No.5, pp.483 - 490, May 1986

[6] Nguyen, D., Swamy, M. "Approximation design of 2-D digital filters with elliptical magnitude response of arbitrary orientation", IEEE Transactions on Circuits and Systems, Vol. 33. No.6, pp.597-603, Jun 1986.

[7] Nguyen, D. and Swamy, M. "A class of 2-D separable denominator filters designed via the McClellan transform", IEEE Transactions on Circuits and Systems, Vol.33, No.9, pp.874 - 881, Sep.1986

[8] Mersereau, R., Mecklenbrauker, W., Quatieri, T., Jr., "McClellan transformations for two-dimensional digital filtering-Part I: Design" IEEE Transactions on Circuits and Systems, Vol.23, No.7, pp. 405 - 414, Jul 1976

[9] Mecklenbrauker, W., Mersereau, R., "McClellan transformations for two-dimensional digital filtering-Part II: Implementation", IEEE Transactions on Circuits and Systems, Vol.23, No.7, pp. 414-422, Jul 1976

[10] Reddy, M., Hazra, S., "Design of elliptically symmetric two-dimensional FIR filters using the McClellan transformation", IEEE Transactions on Circuits and Systems, Vol.34, No.2, pp.196 - 198, Feb 1987

[11] Mersereau, R., "On the equivalence between the Kaiser-Hamming sharpening procedure and the McClellan transformation for FIR digital filter

design", IEEE Transactions on Acoustics, Speech and Signal Processing, Vol. 27 , No. 4, pp.423 - 424, April 1979

[12] Psarakis, E.Z., Mertzios, V.G.; Alexiou, G.P., "Design of two-dimensional zero phase FIR fan filters via the McClellan transform", IEEE Transactions on Circuits and Systems, Vol. 37 , No. 1, Jan.1990 , pp.10 - 16

[13] Psarakis, E.Z., Moustakides, G.V., "Design of two-dimensional zero-phase FIR filters via the generalized McClellan transform", IEEE Transactions on Circuits and Systems, Vol. 38 , No.11, pp. 1355-1363, Nov.1991

[14] Hung-Ching Lu, Kuo-Hsien Yeh, "2-D FIR filters design using least square error with scaling-free McClellan transformation", IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing, Vol. 47 , No 10, pp.1104 - 1107, Oct.2000

[15] Yeung, K.S., Chan, S.C., "Design and implementation of multiplier-less tunable 2-D FIR filters using McClellan transformation" ISCAS 2002. Vol. pp.V-761 - V-764 vol.5, 2002

[16] Mollova, G., Mecklenbrauker, W.F.G., Three-Dimensional Cone FIR Filters Design using the McClellan Transform", Signals, Systems and Computers, 2007. ACSSC 2007, pp.1116 – 1120, 2007

[17] Mollova, G., Mecklenbrauker, W.F.G., "A Design Method for 3-D FIR Cone-Shaped Filters Based on the McClellan Transformation", IEEE Transactions on Signal Processing, Vol.57, No.2 pp. 551 - 564, Feb. 2009

[18] Jong-Jy Shyu, Soo-Chang Pei, Yun-Da Huang, "3-D FIR Cone-Shaped Filter Design by a Nest of McClellan Transformations and Its Variable Design", IEEE Transactions on Circuits and Systems I: Regular Papers, Vol. 57, No. 7, pp. 1697 – 1707, July 2010 ,

N.E.Mastorakis, "A method for computing the 2-D stability margin based on a new stability test for 2-D systems", Multidimensional Systems and Signal Processing, Vol.10, pp.93-99, 1998.