Aircrafts' Altitude Measurement Using Pressure Information: Barometric Altitude and Density Altitude

TEODOR LUCIAN GRIGORIE
Avionics Department
University of Craiova
107 Decebal Street, 200440 Craiova
ROMANIA
lgrigore@elth.ucv.ro
http://www.elth.ucv.ro

LIVIU DINCA
Avionics Department
University of Craiova
107 Decebal Street, 200440 Craiova
ROMANIA
ldinca@elth.ucv.ro
http://www.elth.ucv.ro

JENICA-ILEANA CORCAU
Avionics Department
University of Craiova
107 Decebal Street, 200440 Craiova
ROMANIA
jcorcau@elth.ucv.ro
http://www.elth.ucv.ro

OTILIA GRIGORIE
Carol I, High School
2 Ioan Maiorescu Street, 200418 Craiova
ROMANIA
ogrigorie@yahoo.com

Abstract: - The paper is a review of the pressure method used in the aircrafts' altitude measurement. In a short introduction the basic methods used in aviation for altitude determination are nominated, and the importance of the barometric altitude is pointed. Further, the atmosphere stratification is presented and the general differential equation, which gives the dependence of the static pressure by the altitude, is deduced. The barometric and the hypsometric formulas for the first four atmospheric layers are developed both in the analytical and numerical forms. Also, the paper presents a method to determinate the density altitude with an electronic flight instrument system. A brief review of the flight altitudes is performed, and the calculus relations of the density altitude are developed. The first two atmospheric layers (0÷11 Km and 11÷20 Km) are considered. For different indicated barometric altitudes an evaluation of the density altitude, as a function of non-standards temperature variations and of dew point value, is realized.

Key-Words: - standard atmosphere; atmospheric layers; barometric formula; density-altitude; evaluation

1 Introduction

Seen like one of the most important parameters for the flight safety, the altitude can be defined as the distance between the aircraft centre of mass and the point when the local vertical intersects the surface of Earth [1]. The measurement of the altitude can be made directly, with the altimeters, or indirectly, situation in which it is calculated by means of complementary systems, such as, for example, Air Data Computer (ADC), Inertial Navigation System (INS) or Global Positioning System (GPS) [2]÷[5]. Showing altitude information on board can be done by the measuring system, either through an Electronic Flight Instrument System (EFIS). If it is determined using a GPS than it can be defined as aircraft elevation from the surface of the reference geoid WGS 84 (World Geodetic System) [6].

Relative to the position of the ground point taken as a reference or to the corrections introduced in the measuring system, the flight altitude behaves different names [5]:

- true altitude (H_{nm}): is real altitude of aircraft above mean sea level (MSL);
- relative altitude (H_r): is the altitude reported at an airfield level on which the aircraft performs take off or landing manoeuvres; its value depends on the altimeter adjustment.
- absolute altitude (H_a): in this case the reference point is considered at the intersection of the local vertical and the Earth surface, so its calculation takes into account the overflowed forms of relief. This altitude is considered to have the greatest importance for flight safety.
- barometric altitude (pressure altitude) (H_b): this is the altitude indicated by the altimeter when this is tuned on a basic pressure, so the reference point is positioned on a baric surface. If the reference pressure is 760 mmHg it corresponds to the mean sea level in the International Standard Atmosphere (ISA) [7].
- density altitude (H_ρ): is the barometric altitude corrected for non-standard temperature variations. This corresponds to the altitude at which the density of air is equal to ISA air density evaluated for the current flight conditions.
- indicated altitude (H): is the altitude displayed on the dashboard.

Besides the altitude indicators, on board the aircraft are used also different systems for measuring and transmitting information of altitude. Usually called altitude transmitters, these systems provide information relating the altitude to the on board automated equipment used in flying, navigation and not only. The transmitted information is under the form of electrical signals, in digital or analogue format.

Altitude measurement can be accomplished using several methods; some of them are often implemented simultaneously on board the aircraft: trilateration method (GPS system), inertial method (INS), radio method and barometric method [1]÷[5].

Barometric method is most commonly used on board to determine the altitude of flight. This is based on the dependence of the static pressure p by altitude. The altitude measured with this method is called the barometric altitude (H_b), and the measurement devices are called barometric altimeters. This kind of altitude can be defined as the altitude indicated when the altimeter is set on a pressure basic value; in this case the reference point on the Earth surface is positioned on a baric surface. If the reference pressure is 760 mmHg then it corresponds to the mean sea level as it is defined by the International Standard Atmosphere (ISA) [7].

A particularly important role that barometric altitude plays is to provide information to the Air Traffic Control systems (ATC) in order to avoid the collisions between aircrafts. This is achieved through the transponder (*Trans*mitter-res*ponder*), which is a dedicated radio equipment representing the on board part of the Air Traffic Control Radar Beacon System (ACTRBS). The transponder sends a response in the form of a pulse at 1090 MHz when it is interrogated with a radio frequency signal of 1030 MHz by the secondary surveillance radar (SSR) of the ATC. Developed for the civil air traffic control, the SSR is based on military technology Identification Friend or Foe (IFF); currently the two technologies are compatible [8].

Civil flights operating modes of the transponder are A, C and S. The mode A contains the identification of aircraft and is based on a 4-digit code containing numbers between 0 and 7 assigned by ATC and set by pilot. The mode C transmits the *barometric altitude* information. This information is generally received by transponder from a system of barometric altitude coding, which has a barometric pressure sensor connected to the static pressure bus on aircraft. In most cases the information on the identification and barometric altitude of the aircraft are transmitted together; the mode of the transponder is known as 3A/C or A/C. Each of the 2 modes is represented by 12 bits, which means that the maximum number of ID codes available for assignment at a time is 4096 [8]. The third mode S has been designed in order to avoid over-transponder interrogation areas crowded with too many radar stations [9]. Furthermore, a transponder heaving this mode allows the communication of the carrying aircraft with other aircraft equipped with a Traffic Alert and Collision Avoidance System (TCAS) [10]. On the basis of barometric altitude and distance obtained from other aircraft, TCAS may determine by extrapolation, independently of ATC, if it is possible a collision between the two aircrafts. This mode allows the representation of the identification code on 24 bits, which represents 16.777.214 individual codes. Thus, each aircraft can be assigned with a unique identification code that can be generated according to the country in which it was registered and the registration number. So this aircraft can be addressed selectively by ground stations or other aircrafts.

Also, extremely useful for flying is so-called Altitude Alerting System. During the ascent or descent aircraft phases the pilot enter the desired altitude in the altitude selector, after that being warned by visual and/or sound signals to have enough time to do the manoeuvres necessary to maintain the respectively ceiling. Also, this system warns the pilot in a situation in which deviate from the required altitude. Such an alerting system uses the same primary information as the altitude measuring system, exactly the static pressure. Its basic operation can be described using the diagram in Fig. 1 [11].

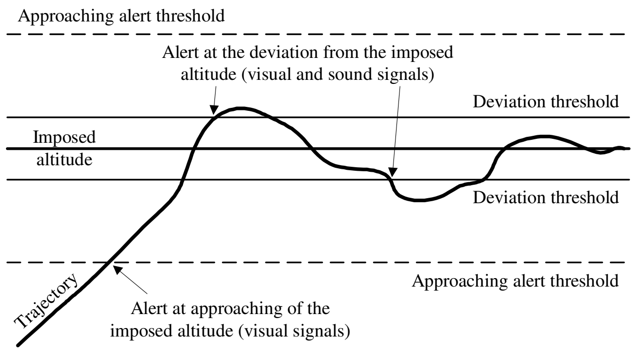


Fig. 1 Basic operation of the altitude alerting systems

When approaching the required altitude, in an ascent or descent phase, usually around 300-400 meters before reaching it, is triggered a visual signal. Before reaching this required ceiling, with approximately 60-100 m, the signal disappears. From the other point of view, if the aircraft deviates from the predefined altitude with 60-100 m (typical) the visual warning signal is accompanied by one sound. An altitude warning should be required to cover the full range of altitudes from which can be operated the aircraft on which it is mounted. The warning threshold values vary from one system to another and from one type of aircraft to another.
One of the altitudes of great importance, calculated from the barometric altitude, is the density altitude (H_ρ). Strongly influenced by changes in temperature and to a lesser extent by changes in air humidity, the real air density can provides to the pilot vital information for the flight safety. It is known that the density of air is the most important factor influencing the performances of aircraft both at the lift forces level, and at the thrust forces level generated by the propulsion systems. Density altitude is such a simple way to give the pilot information on air density, this parameter being calculated such as the pilot can make a coarse assessment of aircraft performance in the current flight conditions [8]. For example, the increase of the temperature or of the humidity of the outdoor environment in which is operated aircraft lead to lower air density [8], [12], which are as a consequence (as defined by the density altitude) the indication of a density altitude higher than the current indicated barometric altitude.
Therefore, the pilot should expect the following effects on the aircraft [8], [12]÷[14]:

- lift decreasing and the need to increase the flight speed to maintain the desired lift;
- decreasing of the propulsion systems power, and, thus, decreasing of the thrust forces;
- runway acceleration decreasing because of the low thrust;
- increasing of the take off distance and decreasing of the climb speed;
- decreasing of the aircraft maximum flight altitude, affecting their capacity to fly above the higher obstacles.

2 Atmosphere Stratification

Since the experimental data collected from the earth's atmosphere have suggested that the dependence of the static pressure by the altitude changes is based on a number of factors (latitude, seasons, weather, switching from day to night ...), was necessary a statistical processing of them to establish a standard dependency relation. This relation must allow the design and calibration of the measuring instruments. After the statistical processing the International Standard Atmosphere (ISA) resulted. It is a model of the earth's atmosphere in terms of pressure, temperature, density and viscosity, on superposed layers in altitude [7]. Therefore, the deduction of mathematical relations which link static the static pressure by altitude (barometric formulas) for each layer, must take into account the dependencies between the air temperature and altitude, and between the air density and altitude

$$T = f_1(H_b),\ \rho = f_2(H_b). \tag{1}$$

According to ISA [7], terrestrial atmosphere can be divided into layers with linear temperature distribution, characterized by the constants and by the frontier parameters in Table 1. The values of temperature gradient in the different layers of the atmosphere suggest that the atmospheric air temperature has not a monotone variation: in the layer 0 the temperature decrease from 15°C to the -56.5°C, in the layer 1 the temperature is constant, while in the layers 2 and 3 the temperature increase but with different gradients.
From the data in Table 1 it results that the air temperature change with altitude can be calculated, by approximation, with relations of the form

$$T = T_i + \tau_i (H_b - H_{bi}) = f_1(H_b), \tag{2}$$

τ_i is the temperature gradient of the i^{th} atmospheric layer, T_i and H_{bi} are the temperature, respectively the altitude, at the lower limit of the same layer.
The relationship linking air density and altitude can be inferred easily approximating the air with an ideal gas and using the equation of state written in the form

$$pV = \nu RT, \tag{3}$$

where the pressure p ([N/m^2]) and the temperature T ([K]) are functions of the barometric altitude, ν is the mol number, V ([m^3]) is the air volume, and R is the gas universal constant (R=8.31432 N·m/(mol·K)).

Table 1 Constants and frontier parameters that characterise the atmospheric layers defined by ISA

No.	Layer	Altitude interval [km]	Lower limit H_{bi} [km]	Temperature gradient τ_i [°/km]	Temperature at lower limit T_i [°C]	Pressure at lower limit p_i [mmHg]
0	Troposphere	0÷11	0	-6.5	+15.0	760
1	Tropopause	11÷20	11	0	-56.5	169.75
2	Stratosphere	20÷32	20	+1.0	-56.5	41.065
3	Stratosphere	32÷47	32	+2.8	-44.5	6.5106
4	Stratopause	47÷51	47	0	-2.5	0.83186
5	Mezosphere	51÷71	51	-2.8	-2.5	0.52847
6	Mezosphere	71÷85	71	-2.0	-58.5	0.03599
7	Mezopause	85÷675	85	-	-86.2	0.00342

Expressing the mol number as being the ratio between the air mass m ([kg]) and the air molar mass μ (μ=0.0289644 kg/mol), and considering that the air density ρ ([kg/m^3]) is calculated with the relation

$$\rho = m/V, \tag{4}$$

the equation (3) can be made in the form

$$p = \rho RT/\mu, \tag{5}$$

that is

$$\rho = p\mu/(RT) = p(H_b)\mu/(RT(H_b)) = f_2(H_b). \tag{6}$$

Being suspended in the earth atmosphere, at small-scale the air may be considered in hydrostatic equilibrium. Thus, separating a layer of air, with infinitesimal thickness dH_b, located in a vertical column of constant section S (Fig. 2 [1]), his equilibrium is characterised by the equation

$$\mathrm{d}\vec{G} + \mathrm{d}\vec{F} = 0, \tag{7}$$

in which $\mathrm{d}\vec{G}$ represents the weight of the air layer, and $\mathrm{d}\vec{F}$ is the resultant of the hydrostatic forces that act on the layer. The force $\mathrm{d}\vec{F}$ is the direct effect of the pressure differences on the lower (p_i) and upper (p_s) parts of the layer.
In scalar form, equation (7) becomes

$$\mathrm{d}G - \mathrm{d}F = 0. \tag{8}$$

Noting with g the gravitation (g=9.80665 m/s^2), the weight dG is expressed by the relation

$$\mathrm{d}G = (\rho \cdot S \cdot \mathrm{d}H_b)g. \tag{9}$$

The hydrostatic force is calculated with

$$\mathrm{d}F = (p_i - p_s)\cdot S = -\mathrm{d}p \cdot S, \tag{10}$$

where the pressure difference dp reflects the negative value of the static pressure with the altitude increase.

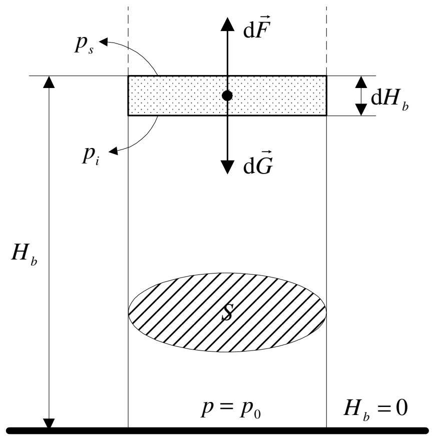


Fig. 2 Hydrostatic equilibrium of an air layer with infinitesimal thickness

With the relations (9) and (10), the equation (8) becomes

$$\rho \cdot g \cdot S \cdot \mathrm{d}H_b = -S \cdot \mathrm{d}p, \tag{11}$$

which leads to

$$\mathrm{d}p = -\rho \cdot g \cdot \mathrm{d}H_b. \tag{12}$$

Substituting the air density, with the expression (6), in equation (12), and separating the variables it results the following differential equation

$$\mathrm{d}p/p = -[\mu g/(RT(H_b))]\cdot \mathrm{d}H_b. \tag{13}$$

3 Barometric and Hypsometric Formulas

Particularising the dependence $T(H_b)$ for each atmospheric layer defined in the ISA, and then integrating numerically the equation (13), formulas which give the dependence of static pressure by altitude $p(H_b)$ (barometric formulas) are obtained. Also, if the inverse dependence $H_b(p)$ is determined, it result the hypsometric formulas for each of the considered layers.

3.1 Barometric and Hypsometric Formulas for Atmospheric Layer 0 (0÷11 km)

According to Table 1 and formula (2), for this layer the temperature dependence of altitude $T(H_b)$ can be calculated by the relation

$$T = T_0 + \tau_0(H_b - H_{b0}), \tag{14}$$

with T_0=15°C=288.15K, τ_0=-6.5°/km, and H_{b0}=0km. In this situation, the equation (13) is expressed as

$$\mathrm{d}p/p = -\{\mu g/[R\cdot(T_0+\tau_0(H_b-H_{b0}))]\}\cdot \mathrm{d}H_b. \quad (15)$$

For this layer, the barometric formula is obtained through the numerical integration of the equation (15) for pressure values between p_0=101325N/m^2=760mmHg and the current value p, and for altitude values between H_{b0}=0km and the current value H_b

$$\int_{p_0}^{p}\frac{\mathrm{d}p}{p} = -\frac{\mu g}{R}\int_{H_{b0}}^{H_b}\frac{\mathrm{d}H_b}{T_0+\tau_0(H_b-H_{b0})}, \quad (16)$$

relation which is equivalent to

$$\int_{p_0}^{p}\frac{\mathrm{d}p}{p} = -\frac{\mu g}{R\tau_0}\int_{H_{b0}}^{H_b}\frac{\mathrm{d}H_b}{H_b-H_{b0}+T_0/\tau_0}. \quad (17)$$

It results

$$\ln(p)\,\Big|_{p_0}^{p} = -(\mu g/(R\tau_0))\cdot\ln(H_b-H_{b0}+T_0/\tau_0)\,\Big|_{H_{b0}}^{H_b}, \quad (18)$$

that is

$$\ln(p/p_0) = -(\mu g/(R\tau_0))\cdot\ln[(H_b-H_{b0}+T_0/\tau_0)/(T_0/\tau_0)]. \quad (19)$$

Eliminating the logarithms in the expression (19), is obtained

$$p/p_0 = \left(1+\tau_0(H_b-H_{b0})/T_0\right)^{-\mu g/(R\tau_0)}, \quad (20)$$

which, finally lead to the standard barometric formula for the layer 0 (0÷11 km)

$$p = p_0\cdot\left(1+\tau_0(H_b-H_{b0})/T_0\right)^{-\mu g/(R\tau_0)}. \quad (21)$$

Taking into account the numerical values of the constants involved in the previous formula, it results that for altitudes between 0 km and 11 km the pressure values are derived with the relation

$$p = 760\cdot\left(1-H_b/44330.76923\right)^{5.255876}, \quad (22)$$

where the pressure p is expressed in mmHg, and the barometric altitude H_b is expressed in m.
Starting from the equation (20), the standard hypsometric formula for the layer 0 (0÷11 km) is obtained on the form

$$H_b = H_{b0}+(T_0/\tau_0)\cdot[1-(p/p_0)^{-R\tau_0/(\mu g)}], \quad (23)$$

which, with numerical values, becomes

$$H_b = 44330.76923\cdot[1-(p/760)^{0.190263}]. \quad (24)$$

Representing graphically the dependences of the static pressure, respectively of the air temperature, by the altitude, the characteristics in Fig. 3 are obtained. Can be easily observed the pressure exponential decreasing, respectively the temperature linear decreasing with the altitude increasing. Lower limits of pressure and temperature are given in Table 1, and correspond to the altitude of 11 km.

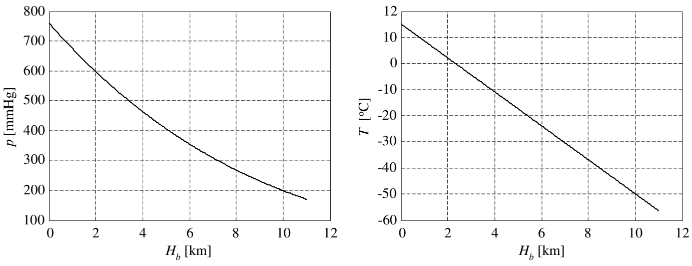


Fig. 3 Pressure and temperature variations with the altitude in the layer 0

3.2 Barometric and Hypsometric Formulas for Atmospheric Layer 1 (11÷20 km)

From Table 1, one can observe that for the atmospheric layer 1 the temperature has a constant value

$$T(H_b) = T_1 = -56.5\,^{\circ}\mathrm{C} = 216.66\,\mathrm{K}. \quad (25)$$

As a consequence, the equation (13) becomes

$$\mathrm{d}p/p = -[\mu g/(RT_1)]\cdot\mathrm{d}H_b. \quad (26)$$

Through numerical integration of the previous equation between the limit p_1=22632.1N/m^2=169.75mmHg and the current value p for pressure, and between the limits H_{b1}=11km and the current value H_b for altitude, it results successively

$$\int_{p_1}^{p}\frac{\mathrm{d}p}{p} = -\frac{\mu g}{RT_1}\int_{H_{b1}}^{H_b}\mathrm{d}H_b, \quad (27)$$

and

$$p = p_1 e^{-\mu g(H_b-H_{b1})/(RT_1)}. \quad (28)$$

The relation (28) represents the standard barometric formula for the layer 1 (11÷20 km). With the numerical values of the involved constants, the previous barometric formula becomes

$$p = 169.75\cdot e^{-1.57681\cdot10^{-4}(H_b-11000)}; \quad (29)$$

the pressure p is calculated in mmHg, while the altitude H_b is expressed in m. The standard hypsometric formula for this layer is obtained as the form

$$H_b = H_{b1}-[RT_1/(\mu g)]\cdot\ln(p/p_1), \quad (30)$$

or, numerically

$$H_b = 11000-6341.912741\cdot\ln(p/169.75). \quad (31)$$

The variation of the static pressure with the altitude for this layer is depicted in Fig. 4.

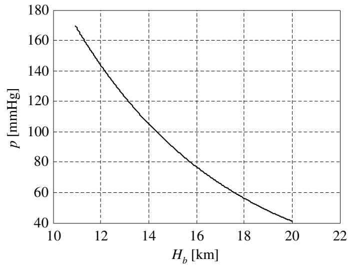


Fig. 4 Pressure variation with the altitude in the layer 1

3.3 Barometric and Hypsometric Formulas for Atmospheric Layer 2 (20÷32 km)

Particularizing the equation (2) for this layer one obtain

$$T = T_2 + \tau_2(H_b - H_{b2}). \qquad (32)$$

According to the ISA (see Table 1), the numerical values of the constants in expression (32) are: T_2=-56.5°C= =216.66K, τ_2=1°/km, and H_{b2}=20km. Therefore, with the dependence $T(H_b)$ in (32), the differential equation (13) that gives the dependence of static pressure by the altitude has the particular expression

$$\mathrm{d}p / p = -\{\mu g / [R \cdot (T_2 + \tau_2(H_b - H_{b2}))]\} \cdot \mathrm{d}H_b. \qquad (33)$$

Through numerical integration of the equation (33) for pressures between p_2=5474.89N/m^2=41.065mmHg and the actual value p, and for altitudes between H_{b2}=20km and the actual value H_b, it results the standard barometric formula for this layer

$$p = p_2[1 + \tau_2(H_b - H_{b2})/T_2]^{-\mu g/(R\tau_2)}. \qquad (34)$$

Considering the values of the numerical constants and of the frontier parameters, the numerical format of the barometric formula is obtained as follow

$$p = 41.065 \cdot [1 + (H_b - 20000)/216660]^{-34.163194}. \qquad (35)$$

Similarly with the previous presented situations, the barometric altitude H_b values are entered in m, while the static pressure p values are obtained in mmHg.

The graphical characteristics that characterise the dependences of the pressure and temperature by altitude, for this layer, are presented in Fig. 5.

Starting from the standard barometric formula (32), the standard hypsometric formula for the atmospheric layer 2 are obtained

$$H_b = H_{b2} + (T_2/\tau_2) \cdot [(p/p_2)^{-R\tau_2/(\mu g)} - 1], \qquad (36)$$

which, numerical, becomes

$$H_b = 20000 + 216660 \cdot [(p/41.065)^{-0.029271} - 1]. \qquad (37)$$

3.4 Barometric and Hypsometric Formulas for Atmospheric Layer 3 (32÷47 km)

For this amplitudes interval, the dependence $T(H_b)$ from the equation (2) takes the particular form

$$T = T_3 + \tau_3(H_b - H_{b3}). \qquad (38)$$

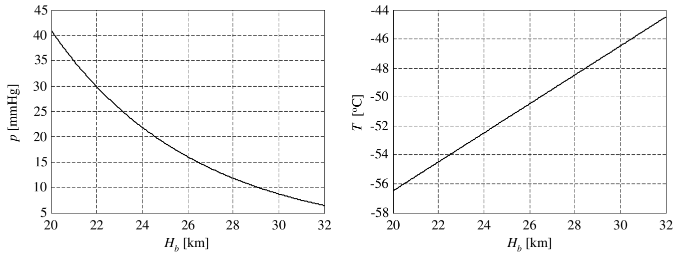


Fig. 5 Pressure and temperature variations with the altitude in the layer 2

The numerical values of the involved constants and parameters are: T_3=-44.5°C=228.65K, τ_3=2.8°/km, H_{b3}=32km. As a consequence, the equation (13) becomes

$$\mathrm{d}p / p = -\{\mu g / [R(T_3 + \tau_3(H_b - H_{b3}))]\} \cdot \mathrm{d}H_b. \qquad (39)$$

Numerical integration of the equation (39), for pressures between p_3=868.019N/m^2=6.5106mmHg and the current value p, and for altitudes between H_{b3}=32km and the current value H_b, leads to the standard barometric formula for the atmospheric layer 3

$$p = p_3[1 + \tau_3(H_b - H_{b3})/T_3]^{-\mu g/(R\tau_3)}, \qquad (40)$$

that is

$$p = 6.5106 \cdot [1 + (H_b - 32000)/81660.714]^{-12.201141}. \qquad (41)$$

The standard hypsometric formula for this layer becomes

$$H_b = H_{b3} + (T_3/\tau_3) \cdot [(p/p_3)^{-R\tau_3/(\mu g)} - 1]. \qquad (42)$$

Taking into account the numerical values of the constants and parameters involved in the equation (42), the numerical format of the previous hypsometric formula can be expressed as follow

$$H_b = 32000 + 81660.714 \cdot [(p/6.5106)^{-0.08196} - 1]. \qquad (43)$$

The graphical characteristics $p(H_b)$ and $T(H_b)$ for this layer are presented in Fig. 6.

The barometric and hypsometric formulas for the others atmospheric layers in Table 1 can be developed using the same model.

As a conclusion, the determination of the barometric altitude is reduced at the static pressure measurement followed by its conversion in altitude units with one of the previous developed hypsometric formulas. The conversion can be made using a board computer, situation in which the altitude is delivered to the pilot through an Electronic Flight Instrument System, or using a barometric altimeter with direct displaying.

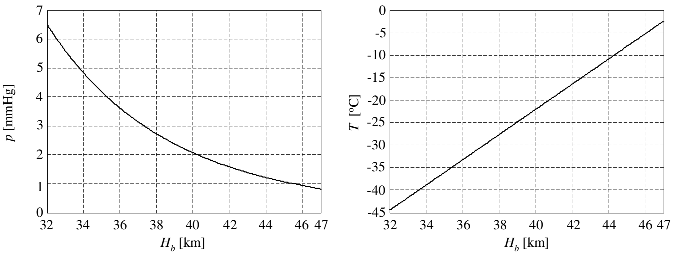


Fig. 6 Pressure and temperature variations with the altitude in the layer 3

4 Vapor Pressure Determination

The density of the atmospheric air, considered dry, can be calculated with the relation

$$\rho = p\mu/(RT), \tag{44}$$

the static pressure p being measured directly. In relation (44) T is the air temperature, μ - molecular mass of the air (μ=0.0289644 kg/mol), and R is the gas universal constant (R=8.31432 N·m/(mol·K)). In real conditions, this density is affected by the humidity in the air, which is considered to be a mixture of dry air molecules and water vapor. So, the measured barometric pressure is in fact the mixture pressure not the dry air pressure. Thus, to determinate the density of the dry air the Dalton partial pressures law must be used. In this way, the mixture pressure is expressed as the sum of the dry air and water vapor p_v pressures

$$p_{am} = p + p_v. \tag{45}$$

It results

$$p = p_{am} - p_v \tag{46}$$

and the relation (44) for the density calculation becomes

$$\rho = (p_{am} - p_v)\mu/(RT). \tag{47}$$

As a consequence, the dry air density calculation supposes to know the actual air pressure (the mixture pressure p_{am}), the water vapor pressure p_v and the local temperature T of the atmospheric air. The Mixture pressure and the temperature of local atmospheric air is determined relatively easily by placing the sensors outside the aircraft. Instead, the vapor pressure determination involves the performing of numerical calculations using information on relative humidity and dew point [15].

According to [15], two relations often used in the calculation of vapor pressure from the dew point:

a) The first relation shows a high accuracy and is based on a polynomial development

$$p_v = \frac{C}{(c_0 + t(c_1 + t(c_2 + t(c_3 + t(c_4 + t(c_5 + t(c_6 + t(c_7 + t(c_8 + t\cdot c_9)))))))))^8}, \tag{48}$$

where t is the dew point expressed in °C, C=610.78, c_0= =0.99999683, c_1=-0.90826951·10^{-2}, c_2=0.78736169·10^{-4}, c_3=-0.61117958·10^{-6}, c_4=0.4388418·10^{-8}, c_5=-0.29883885 ·10^{-10}, c_6=0.2187442·10^{-12}, c_7=-0.1789232·10^{-14}, c_8=0.111 1201·10^{-16}, c_9= -0.30994571·10^{-19}.

b) The second relation, much easier, is generally used in applications that require a lower accuracy

$$p_v = C\cdot 10^{c_1\cdot t/(c_2+t)}, \tag{49}$$

with C=610.78, c_1=7.5 and c_2=237.3. For both relations the pressure p_v measure unity is Pascal.

In Fig. 7 are represented graphically the dependences p_v=$f(t)$ between vapor pressure and dew point, given by the relation (5), respectively the difference ε between the pressure p_v values calculated with relations (48) and (49). To be more suggestive the characteristics were designed for pressures represented in mmHg.

Dependence p_v=$f(t)$ analysis combined with the equation (47) confirming earlier observations that the most disadvantageous circumstances for flying are those with high humidity (positive dew point). For these cases the vapor pressure can reach up to 42 mmHg, the growth being exponential. On the other hand, on observes that the differences between the two methods of calculation are very small (up to 0.012 mmHg). Analyzing both graphs shows that for positive dew points can be used without problems relation (49) in high accuracy applications. Instead, in such applications, for negative values of dew point, especially for those below -20°C, it is necessary to use formula (48) because the relative error caused by the simplified formula is high, reaching up to 4%.

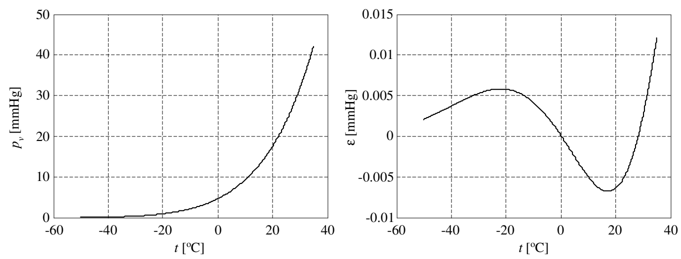


Fig. 7 Evaluation of formulas for calculating the vapor pressure

Once pressure p_v determined, with formula (47) can be calculated the dry air density. Equalling the calculated density with the density of air in ISA is equivalent to combine the relation (44) with the formulas that give the temperature dependence of altitude and with the barometric formulas for the atmospheric layers.

5 Density Altitude

5.1 Density Altitude Evaluation for the Atmospheric Layer 0

For atmospheric layer 0 (0÷11 km), the relation which gives the dependence of the temperature by the altitude is

$$T = T_0 + \tau_0 (H_b - H_{b0}), \tag{50}$$

where T_0=15°C=288.15K (temperature at the lower limit of the layer 0), τ_0=-6.5°/km (temperature gradient of the layer 0), and H_{b0}=0km (altitude at the lower limit of the layer 0). In this situation, the barometric formula, that gives the dependence of the static pressure p by the altitude H_b, is expressed under the form

$$p = p_0 \left(1 + \tau_0 (H_b - H_{b0}) / T_0\right)^{-\mu g/(R\tau_0)}, \tag{51}$$

p_0=101325N/m^2=760mmHg is the static pressure for the altitude H_{b0}. So, for the atmospheric layer 0, the following equations system is obtained

$$\begin{aligned} \rho &= p\mu / RT, \\ T &= T_0 + \tau_0 (H_b - H_{b0}), \\ p &= p_0 \left(1 + \tau_0 (H_b - H_{b0}) / T_0\right)^{-\mu g/(R\tau_0)}. \end{aligned} \tag{52}$$

Successively eliminating the temperature and the pressure between the three equations is obtained the formula for calculating the density altitude in the form (H_b becomes H_ρ)

$$H_\rho = H_{b0} - (T_0 / \tau_0)[1 - \left(\rho R T_0 / (p_0 \mu)\right)^{-R\tau_0/(\mu g + R\tau_0)}], \tag{53}$$

which, numerically, is

$$H_\rho = 4.433076 \cdot 10^4 \cdot [1 - 0.953434 \cdot \rho^{0.234969}]. \tag{54}$$

In previous relation the density values are entered in kg/m^3, and the resulting density altitude values are in m.

Representing graphically the density altitude for the change Δt of environmental temperature outside the aircraft (in the range -30°C÷30°C from the standard values in ISA) and for different values of dew point t (between -50°C÷35°C), the three-dimensional characterristics in Fig. 8 are obtained. These were presented for four values of air pressure considered to be in the mixture with water vapor (p_{am}). These values correspond to altitude readings on barometric altimeter equals to 0m, 3000m, 6000m and 9000m.

The maximum and the minimum values of the density altitude, calculated in the simulated conditions, are presented in Table 2.

From Fig. 8 one observe that the maximum values are obtained for Δt=30°C and t=35°C and the minimum values for Δt=-30°C and t=-50°C. Consistent with the values in Table 2, from Fig. 8 can be remarked also a growing of the area curvature with the increasing of the indicated barometric altitude. According to the tabled maximum and minimum values, the difference between the density altitude and barometric altitude for this layer may even exceed 2500 m superposing certain conditions of temperature and humidity. Successively neglecting the effects of humidity and of non-standard change of environment temperature on density altitude, for the four previously considered cases, for the indicated barometric altitude, result the characteristics in Fig. 9 a., respectively in Fig. 9 b.

Table 2 Maximum and minimum values of the density altitude for the conditions simulated in Fig. 8

No.	Indicated barometric altitude [m]	Maximum value of the density altitude [m]	Minimum value of the density altitude [m]
1	0	1596.8135	-1159.4360
2	3000	4799.5158	1834.7880
3	6000	8106.6341	4828.0847
4	9000	11595.6104	7820.2263

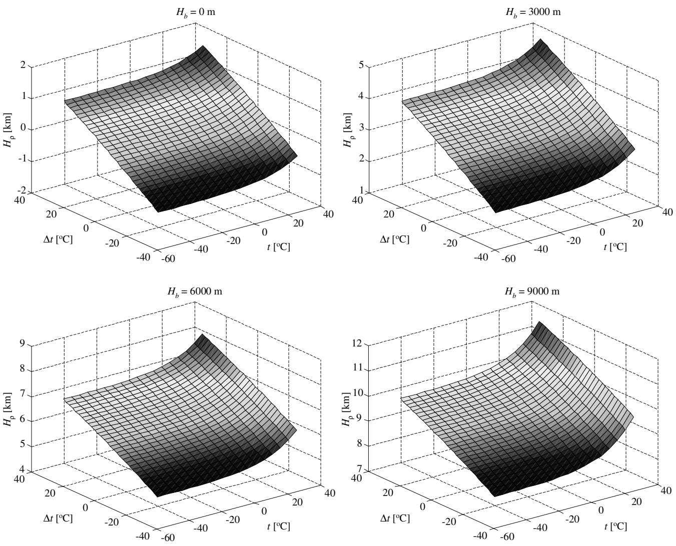
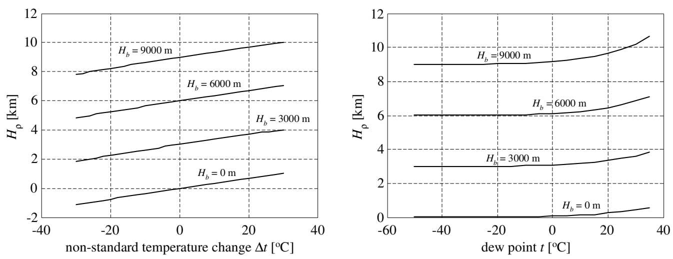


Fig. 8 Density altitude for layer 0 at the temperature and humidity variations



a. b.

Fig. 9 The evaluation of the influences of the non-standard temperature variations and humidity on the density altitude in the layer 0

5.2 Density Altitude Evaluation for the Atmospheric Layer 1

For atmospheric layer 1 (11÷20 km), the relation which gives the dependence of the temperature by the altitude is

$$T = T_1 + \tau_1 (H_b - H_{b1}) = T_1, \qquad (55)$$

where T_1=-56.5°C=216.66K (temperature at the lower limit of the layer 1), τ_1=0°/km (temperature gradient of the layer 1), and H_{b1}=11km (altitude at the lower limit of the layer 1). In this situation the barometric formula becomes

$$p = p_1 e^{-\mu g (H_b - H_{b1})/(RT_1)}, \qquad (56)$$

p_1=22632.1N/m^2=169.75mmHg is the static pressure for the altitude H_{b1}. Starting from the relations (44), (55) and (56), the following equations system is obtained

$$\begin{aligned} \rho &= p\mu / RT, \\ T = T_1 &= -56.5\,^{\circ}\mathrm{C} = 216.66\,\mathrm{K}, \\ p &= p_1 e^{-\mu g (H_b - H_{b1})/(RT_1)}. \end{aligned} \qquad (57)$$

Successively eliminating the temperature and the pressure between the three equations is obtained the formula for calculating the density altitude in the form (H_b becomes H_ρ)

$$H_\rho = H_{b1} - (RT_1/\mu g)\ln[\rho RT_1/(\mu p_1)], \qquad (58)$$

which, numerically, is

$$H_\rho = 11000 - 6341.912741 \cdot \ln(2.747995 \cdot \rho). \qquad (59)$$

The density is entered in kg/m^3, and the values of the density altitude results in m.

Considering the same values for the non-standard variation Δt of the environmental temperature outside the aircraft (-30°C÷30°C), the variation interval -50°C÷35°C for the dew point t at altitudes under 16000 m, and the variation interval -50°C÷25°C for the dew point t at altitudes over 16000 m, it result the three-dimensional characteristics in Fig. 10. For the high altitudes in this layer the dew point values higher than 25°C temperature are irrelevant physically because in these situations the calculated vapor pressure becomes greater than the pressure of the mixture. The characteristics were presented for four values of air pressure considered to be in the mixture with water vapor (p_{am}). These values correspond to altitude readings on barometric altimeter equals to 11000 m, 14000 m, 17000 m and 20000 m.

The maximum and the minimum values of the density altitude, calculated in the simulated conditions, are presented in Table 3.

From Fig. 10 one observe that the maximum values are obtained at Δt=30°C and t=35°C for H_b=11000 m and H_b=14000 m, respectively at Δt=30°C and t=25°C for H_b=17000 m and H_b=20000 m. The minimum values are obtained at Δt=-30°C and t=-50°C for all four situations. According to the maximum and minimum values above table, the difference between the density altitude and barometric altitude for this layer may even exceed 6000 m superposing certain conditions of temperature and humidity.

Table 3 Maximum and minimum values of the density altitude for the conditions simulated in Fig. 10

No.	Indicated barometric altitude [m]	Maximum value of the density altitude [m]	Minimum value of the density altitude [m]
1	11000	13633.6955	10056.6798
2	14000	18048.2587	13057.7078
3	17000	20657.6839	16059.3595
4	20000	26301.9825	19062.0127

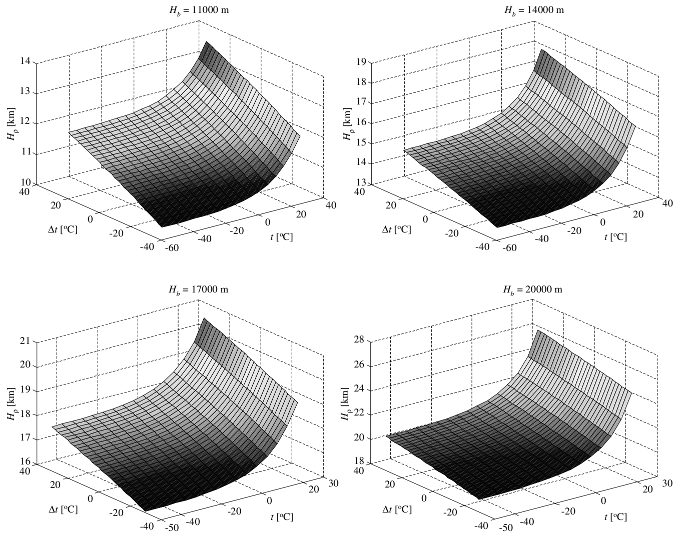


Fig. 10 Density altitude for layer 1 at the temperature and humidity variations

Individual effects of non-standard change of environment temperature and humidity on the density altitude, for the four previously considered cases for the indicated barometric altitude, can be seen in Fig. 11 a., respectively in Fig. 11 b.

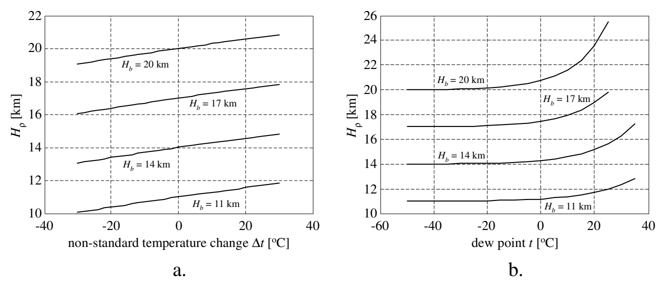


Fig. 11 The evaluation of the influences of the non-standard temperature variations and humidity on the density altitude in the layer 1

6 Conclusions

Firstly, the paper can be considered as a review of the barometric method to calculate the aircrafts' altitude. The basic information considered as input for this method is the static pressure. For the first four atmospheric layers defined by the International Standard Atmosphere (ISA) mathematical relations was developed both for barometric and hypsometric formulas. Also, the analytical, respectively the numerical form, was developed for these equations.

As we can observe from the Fig. 3÷Fig. 6 the dependences of the altitude by the static pressure is exponential but with different forms influenced by the particular variation of the temperature on each layer. The dependences of the temperature by the altitude are linear for the layers 0, 2 and 3, and constant for the layer 1. The values of the atmospheric layers' temperature gradients in Table 1 demonstrate that the air temperature has not a monotone variation: in the layer 0 the temperature decrease from 15°C to the -56.5°C, in the layer 1 the temperature is constant, while in the layers 2 and 3 the temperature increase but with different gradients.

The presented equations are used directly in a barometric altimeter, or are implemented in an Air Data Computer (ADC) or in an Electronic Flight Instrument System (EFIS) to compute the altitude starting from the static pressure information.

On the other hand, a method to calculate the density altitude starting from the static pressure, non-standard air temperature variations and dew point information was presented in the paper. In this way, mathematical relations were developed for the both 0 and 1 atmospheric layers (0÷11km, respectively 11÷20 km).

For the layer 0, values between -30°C÷30°C for non-standard temperature variations, and between -50°C÷35°C for the dew point, was considered. Also, the evaluation was performed for four barometric altitude values: 0m, 3000m, 6000m and 9000m. The maximum deviation of the density altitude from the indicated barometric altitude value is 2595.6104 m, and was obtained for Δt=30°C and t=35°C at H_b=9000m.

For the layer 1, values between -30°C÷30°C for non-standard temperature variations was considered. For the dew point, values between -50°C÷35°C for barometric altitudes fewer than 16000 m, and between -50°C÷25°C for barometric altitudes over 16000 m, was considered. The evaluation was realised for the next values of the indicated barometric altitude: 11000 m, 14000 m, 17000 m, and 20000 m. In this case, the maximum deviation of the density altitude from the indicated barometric altitude value is 6301.9825 m, and was obtained for Δt=30°C and t=25°C at H_b=20000m.

For both layers was depicted the individual effects of the non-standard temperature variation, respectively of the air humidity, on the density altitude (Fig. 9, respectively Fig. 11).

References:

[1] I. Aron, *Aparate de bord pentru aeronave*. Editura Tehnică, Bucureşti, 1984.

[2] T.L. Grigorie, *Sisteme de navigatie inertiala strap-down. Studii de optimizare.* Editura SITECH, Craiova, Romania, 2007.

[3] T.L. Grigorie, D.G. Sandu, I. Aron, *Navigation in non-inertial terrestrial frames using the attitude matrices.* 48th International Symposium ELMAR-2006 focused on Multimedia Systems and Applications, Zadar, Croatia, pp. 205-209, 7-9 June, 2006.

[4] T.L. Grigorie, D.G. Sandu, *Numerical algorithm for navigation in Earth's inertial system.* 47th International Symposium ELMAR-2005 focused on Multimedia Systems and Applications, Zadar, Croatia, pp. 257-262, 8-10 June, 2005.

[5] M. J. Mahoney, *A Discussion of Various Measures of Altitude*. [Online]. Available: http://mtp.jpl.nasa.gov/notes/altitude/altitude.html.

[6] National Imagery and Mapping Agency Technical Report NIMA TR8350.2, *Department of Defence World Geodetic System 1984 - Its Definition and Relationships with Local Geodetic Systems*, 3 January 2000.

[7] International Civil Aviation Organization. *Manual of the ICAO Standard Atmosphere (extended to 80 kilometers* (*262500 feet*)). Third edition 1993.

[8] J. Brandon (2000-2008), *Flight Theory Guide*. [Online]. Available: http://www.auf.asn.au.

[9] Eurocontrol, *Mode S Technical Overview*. [Online]. Available: http://www.eurocontrol.int/msa/public/standard _page/modes_tech_ overview.html.

[10] Eurocontrol, *TCAS II version 7.1.* [Online]. Available: http://www. eurocontrol.int/msa/public/standard_page/ACAS_Upcoming_Changes.html.

[11] SKYbrary, *Altitude Alerter*. [Online]. Available: http://www.skybrary. aero/index.php/Altitude_Alerter.

[12] J. Williams, *Density Altitude - Why airplanes like cool days better*, Flight Training Magazine, July 2003.

[13] J. Williams, *Understanding air density and its effects*. [Online]. Available: http://www.usatoday.com/weather/wdenalt.htm.

[14] Grigorie, T.L., Dinca, L., Corcau, J. *The Density Altitude. Influence Factors and Evaluation*. 6th WSEAS International Conference on Dynamical Systems and Control, Kantaoui, Sousse, Tunisia, May 3-6, 2010.

[15] R. Shelquist, *Air density and density altitude calculations*. [Online]. Available: http://wahiduddin.net/calc/density_altitude .htm, Longmont, Colorado 2 December 2008.