

Preservation of Synchronization in Dynamical Systems via Lyapunov Methods

G. FERNÁNDEZ ANAYA
Universidad Iberoamericana
Departamento de Física
y Matemáticas
Prol. Paseo de la Reforma 880,
Lomas de Santa Fe, México,
D. F., C.P. 01219
MÉXICO
guillermo.fernandez@uia.mx

C. RODRÍGUEZ LUCATERO
Universidad Autónoma,
Metropolitana Cuajimalpa
Departamento de Tecnologías
de la Información
Av. Constituyentes 1056,
Col. Lomas Altas, México,
D.F., C.P. 11950
MÉXICO
profesor.lucatero@gmail.com

J.J. FLORES-GODOY &
C. MIRANDA-REYES
Universidad Iberoamericana
Departamento de Física
y Matemáticas
Prol. Paseo de la Reforma 880,
Lomas de Santa Fe,
México, D. F., C.P. 01219
MÉXICO
job.flores@uia.mx,
mirandare@gmail.com

Abstract: In this paper, we use, extend and apply some classic results of the theory of dynamical systems to study the preservation of synchronization in chaotical dynamical systems via Lyapunov method. The obtained results show that synchronization can be preserved after a particular class of changes are made to the linear part of the dynamical system. For illustrative purposes we apply a compound control law to achieve synchronization in a master-slave system. We also show that it is possible to preserve partial synchronization when an additive perturbation is included in the control law. We present numerical simulations to show the effectiveness of our method.

Key-Words: Chaotic Systems, control theory, convergence, stability.

1 Introduction

Chaotic dynamical systems and chaos control is a theme that has been widely developed in the last years. The study of systems with this kind of behavior is well documented and there are several papers where applications are presented, as described in [1]. Also, chaos control and synchronization of systems have been studied in [5, 7]. Studies on preservation of synchronization and chaos structure can be found in [3, 4, 2].

In [4], mathematical tools to ensure synchronization and preservation of hyperbolic points are developed, which in this paper are extended, in particular the stable-unstable manifold theorem.

In [3, 4, 2], local stability is preserved, in the present work, it may be possible to achieve global stability if certain conditions are met.

An extension of the stable-unstable manifold theorem is presented. This extension is based on the modification through matrix multiplication and group action over the Jacobian matrix of the dynamical systems used. In general, we will define a nonlinear function that acts over the linear part of the dynamical system, changing it in such a way that this modification preserves hyperbolic

equilibrium points.

It was proved in [9], [8], that using a composite control law formed by a linear and a nonlinear part, it is possible to synchronize a master-slave system. They also show that the control law ensures ultimate boundedness in the presence of additive perturbations. In this paper we will extend this work and show that the same design procedure can be used to preserve synchronization in a modified master-slave system.

We provide two simulation examples of master-slave systems to illustrate the application of our methodology. The benchmark dynamical systems are Chua's system and the Sprott's Q system. The simulations show how the modified systems still preserve its synchronization properties.

2 Preliminaries

In this section, we present preliminary results needed for the development of our main result. As we know, the local stability of a dynamical system around an equilibrium point is completely related to the sign of the real part of the eigenval-

ues of the matrix that represents the linear part of the system. The main idea of this work is to preserve the stability of a dynamical system after applying a class of transformations. The class of transformations that we study are applied only on the linear part of the dynamical system, these transformations are designed to preserve the characteristics of the original system.

Let \mathbb{C} and \mathbb{R} represent the field of complex and real numbers, respectively. First, we state the following definitions.

Definition 1 ([6]). Given a function $h : \mathbb{C} \rightarrow \mathbb{C}$, we say that $h(\cdot)$ is defined on the spectrum of a matrix $A \in \mathbb{R}^{n \times n}$, if there exist the numbers

$$h(\lambda_k), h'(\lambda_k), \dots, h^{(m_k-1)}(\lambda_k), \quad k = 1, 2, \dots, s, \quad (1)$$

where

$$h^{(m_k-1)}(\lambda_k) = \left. \frac{d^{m_k-1}h(x)}{dx^{m_k-1}} \right|_{x=\lambda_k}. \quad (2)$$

with λ_k the eigenvalues of A with multiplicity m_k ,

Definition 2. An equilibrium point x_0 of a dynamical system

$$\dot{x} = f(x) \quad (3)$$

is said to be hyperbolic, if the Jacobian matrix evaluated at x_0 , $\left. \frac{\partial f}{\partial x} \right|_{x_0} = Df(x_0)$, has eigenvalues with strictly positive and/or strictly negative real part, i. e., the eigenvalues do not lie on the imaginary axis.

The following lemma characterizes some properties that will be used along this work.

Lemma 3. Let $h_1(s), \dots, h_n(s)$ be a set of functions, where $h_k : \mathbb{C} \rightarrow \mathbb{C}$, for $k = 1, \dots, n$. If $h_k(\mathbb{C}^-) \subset \mathbb{C}^-$ and $h_k(\mathbb{C}^+) \subset \mathbb{C}^+$ then

$$1. \left(\sum_{k=1}^n h_k(\mathbb{C}^-) \right) \subset \mathbb{C}^-.$$

$$2. \left(\sum_{k=1}^n h_k(\mathbb{C}^+) \right) \subset \mathbb{C}^+.$$

$$3. h_k(h_l(\mathbb{C}^-)) \subset \mathbb{C}^-, \text{ with } k, l = 1, \dots, n.$$

$$4. h_k(h_l(\mathbb{C}^+)) \subset \mathbb{C}^+, \text{ with } k, l = 1, \dots, n.$$

3 Basic Result

In this section we extend a classical result on properties of dynamical systems, in particular we present the next proposition, which is an extension of the Local Stable-Unstable Manifold Theorem.

Proposition 4. Let E be an open subset of $\mathbb{R}^{n \times n}$ containing the origin, let f be a continuous differentiable function on E , and let $\phi_{A,t}$ be the flow of the nonlinear system $\dot{x} = f(x) = Ax + g(x)$. Suppose that $f(0) = 0$ and that $A = Df(0)$ (where $Df(0)$ is the Jacobian matrix of f evaluated at 0), has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. Given a function $h : \mathbb{C} \rightarrow \mathbb{C}$, defined on the spectrum of A ; if $h(\mathbb{C}^-) \subset \mathbb{C}^-$ and $h(\mathbb{C}^+) \subset \mathbb{C}^+$, $h(0) = 0$ and h is analytic in an open set containing the origin. There exist a k -dimensional differentiable manifold S_h tangent to the stable subspace E_h^S of the linear system $\dot{x} = h(A)x$ at 0 such that for all $t \geq 0$, $\phi_{h(A),t}(S_h) \subset S_h$ and for all $x_0 \in S_h$

$$\lim_{t \rightarrow \infty} \phi_{h(A),t}(x_0) = 0, \quad (4)$$

where $\phi_{h(A),t}$ is the flow of the nonlinear system $\dot{x} = h(A)x + g(x)$. There also exists a $(n - k)$ -dimensional differential manifold W_h tangent to the unstable subspace E_h^W of the linear system $\dot{x} = h(A)x$ at 0, such that for all $t \leq 0$, $\phi_{h(A),t}(W_h) \subset W_h$ and for all $x_0 \in W_h$

$$\lim_{t \rightarrow -\infty} \phi_{h(A),t}(x_0) = 0. \quad (5)$$

Proof. Let A be a matrix with k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. We take the Schur decomposition of the matrix A , [6],

$$A = UT_AU^\top, \quad (6)$$

where T_A is an upper triangular matrix containing the eigenvalues of A in the diagonal, i. e.,

$$T_A = \begin{bmatrix} \lambda_1 & \star & \cdots & \star \\ 0 & \lambda_2 & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \quad (7)$$

with $\lambda_1, \dots, \lambda_n$ the eigenvalues of A and \star elements of A that may or may not be zero.

We take the power series expansion of $h(x)$

$$h(x) = \sum_{k=0}^{\infty} c_k x^k, \quad (8)$$

substituting (6) in (8) gives

$$\begin{aligned} h(A) &= \sum_{k=0}^{\infty} c_k (UT_A U^\top)^k \\ &= U \left(\sum_{k=0}^{\infty} c_k T_A^k \right) U^\top. \end{aligned} \quad (9)$$

Since the multiplication of triangular matrices is a triangular matrix we have

$$h(A) = U \begin{bmatrix} \sum_{k=0}^{\infty} c_k \lambda_1^k & \sum_{k=0}^{\infty} c_k \lambda_2^k & \cdots & \sum_{k=0}^{\infty} c_k \lambda_n^k \\ 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^{\infty} c_k \lambda_n^k \end{bmatrix} U^\top, \quad (10)$$

which is equivalent to

$$h(A) = U \begin{bmatrix} h(\lambda_1) & \star & \cdots & \star \\ 0 & h(\lambda_2) & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h(\lambda_n) \end{bmatrix} U^\top. \quad (11)$$

Since the function $h(\cdot)$ maps the right-half complex plane and left-half complex plane into themselves, the resultant matrix $h(A)$, has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. Now, the result is a consequence of the stable-unstable manifold theorem and Lemma 3. \square

This result is an extension of Theorem 3.2 from [3], which implies that if we can keep the sign structure of the jacobian matrix, after we apply a certain function to the linear part of a dynamical system, there will still exist a stable and unstable manifolds with the original dimensions. Based on Lemma 3, there exist an infinite family of functions $h(\cdot)$ that can be applied to dynamical systems and preserve hyperbolic equilibrium points and its dynamical properties.

4 Preservation of Control Law and Synchronization

As we mentioned above, one goal of this paper is to preserve synchronization in dynamical systems. To achieve this we applied a function to the linear part of the system and then we use the control law developed in [9], where the control vector u is formed by a linear and a nonlinear part,

$$u = u_L + u_N. \quad (12)$$

Consider the master-slave system

$$\dot{x} = Ax + g(x), \quad (13)$$

$$\dot{\tilde{x}} = A\tilde{x} + g(\tilde{x}) + Bu \quad (14)$$

where $u \in \mathbb{R}^m$ and $x, \tilde{x} \in \mathbb{R}^n$ are the master and slave state vectors, respectively. We also suppose that $g(x)$ is such that for all $x, \tilde{x} \in \chi \subset \mathbb{R}^n$

$$\|g(x) - g(\tilde{x})\| \leq l \|x - \tilde{x}\|, \quad l > 0. \quad (15)$$

Now, it is proved in [9], that using as control (12) with

$$\begin{aligned} u_L &= Ke \\ u_N &= \tau(e)B^\top Pe \end{aligned} \quad (16)$$

where $e = x - \tilde{x}$ is the error between the state variables, K is such that $A - BK$ is Hurwitz and $P = P^\top > 0$ is the solution of the Lyapunov equation

$$(A - BK)^\top P + P(A - BK) = -Q, \quad Q = Q^\top > 0. \quad (17)$$

we can ensure the error between the master-slave system becomes asymptotically zero. If we choose $\tau(e)$ as a constant in equation (16), this control is reduced to a proportional control law, but in general it is a function of the error.

Theorem 5. Consider a linear function $h(\cdot)$ as in Proposition 4 such that $h(A - BK) = h(A) - h(BK)$, if we have the system

$$\begin{aligned} \dot{x} &= h(A)x + g(x), \\ \dot{\tilde{x}} &= h(A)\tilde{x} + g(\tilde{x}) + \hat{u}, \end{aligned} \quad (18)$$

where $g(x)$ is locally Lipschitz in a domain $x \in \chi \subset \mathbb{R}^n$ and $h(\mathbb{C}^-) \subset \mathbb{C}^-$. Choosing a control law such that

$$\hat{u} = h(BK)e + \tau(e)BB^\top \hat{P}e, \quad (19)$$

with the pair (A, B) stabilizable and $\tau : \mathbb{R}^n \rightarrow [0, \infty)$, locally Lipschitz and positive. Then, the error of the system is asymptotically stable if

$$\lambda_{\min}(\hat{Q}) > 2l\lambda_{\max}(\hat{P}), \quad (20)$$

where

$$(h(A) - h(BK))^\top \hat{P} + \hat{P}(h(A) - h(BK)) = -\hat{Q}, \quad (21)$$

is satisfied with $\hat{Q} > 0$.

Proof. By Proposition 4, $h(A - BK) = h(A) - h(BK)$ is Hurwitz. Therefore, there exists a matrix $\hat{P} = \hat{P}^\top > 0$, such that (21) is satisfied.

Forming the error system $e = x - \tilde{x}$

$$\dot{e} = (h(A) - h(BK))e + g(x) - g(\tilde{x}) - \tau(e)BB^\top \hat{P}e, \quad (22)$$

choosing $v(e) = e^\top \hat{P}e$ as a Lyapunov candidate function and evaluating this along the trajectories of the error system we have

$$\dot{v}(e) = \dot{e}^\top \hat{P}e + e^\top \hat{P}\dot{e}, \quad (23)$$

taking the transpose of \dot{e} , substituting in $\dot{v}(e)$ and using Lemma 3,

$$\begin{aligned} \dot{v}(e) &= e^\top [(h(A - BK))^\top \hat{P} + \hat{P}h(A - BK)]e \\ &\quad + 2e^\top \hat{P}(g(x) - g(\tilde{x})) - 2\tau(e)e^\top \hat{P}BB^\top \hat{P}e \\ &\leq -e^\top \hat{Q}e + 2\|e\| \lambda_{\max}(\hat{P})l\|e\| - 2\tau(e)e^\top \hat{P}BB^\top \hat{P}e \\ &\leq -\|e\|^2 (\lambda_{\min}(\hat{Q}) - 2l\lambda_{\max}(\hat{P})) - 2\tau(e)e^\top \hat{P}BB^\top \hat{P}e \end{aligned} \quad (24)$$

If $\lambda_{\min}(Q) > 2l\lambda_{\max}(P)$ we ensure that $\dot{v}(e) < 0$ regardless of the values of $\tau(e)$ (providing it is positive semidefinite). This implies that the origin of the error system is locally asymptotically stable. If we choose χ arbitrary large, then the error system is semi globally stable. If $\chi = \mathbb{R}^n$ the error system is globally stable. \square

This theorem is a result for preservation of synchronization in nonlinear dynamical systems, it ensures local asymptotical stability or global asymptotical stability depending of the size of χ . We will now prove, that this result can be extended to achieve synchronization when we introduce a perturbation to the system in the control law.

A stronger result similar to the proposition 1 in [9] is the following

Theorem 6. *Suppose that the pair (A, B) is stabilizable with B full column rank and also that $\text{Im}(B) = \text{Im}(g(\cdot, t))$, then under the conditions of Theorem 5, there exist a control law of the form (19) such that the error of the modified system equation (22) is locally asymptotically stable, where $h(A) = MA$ and M being a nonsingular matrix. Furthermore if $g(x)$ is semi-globally Lipschitz (resp. globally Lipschitz), then there exists a control law such that the error of the modified system is semi-globally (resp. globally) asymptotically stable.*

Proof. Notice that B has full column rank and if $\text{Im}(B) = \text{Im}(g(\cdot, t))$, there exists a state-similarity transformation such that we can write the system as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} g_2(x, t) \quad (25)$$

where \bar{B} is nonsingular and $g_2(x, t)$ is locally Lipschitz with Lipschitz constant l . In an explicit way we are going to make the state-similarity transformation as follows; consider $x = Tz$ and $\tilde{x} = T\tilde{z}$

$$\begin{aligned} \tilde{e} &= z - \tilde{z} \\ \tilde{e} &= \tilde{A}\tilde{e} - \tilde{B}\tilde{K}\tilde{e} \\ [T(\tilde{A} - \tilde{B}\tilde{K})T^{-1}]^T \hat{P} + \hat{P}T(\tilde{A} - \tilde{B}\tilde{K})T^{-1} \\ &\Rightarrow (\tilde{A} - \tilde{B}\tilde{K})^T T^T \hat{P}T + T^T \hat{P}T(\tilde{A} - \tilde{B}\tilde{K}) = -T^T \hat{Q}T \\ \hat{\tilde{P}} &= T^T \hat{P}T \\ \hat{\tilde{Q}} &= T^T \hat{Q}T \\ \tilde{A} &= T^{-1}h(A)T = T^{-1}MAT \\ \begin{bmatrix} 0 \\ \tilde{M}\tilde{B}\tilde{K} \end{bmatrix} &= \tilde{B}\tilde{K} \end{aligned}$$

$$\tilde{B}\tilde{K} = h(BK) = MBK$$

Observe that the $\text{Im}(MB) = \text{Im}(B)$ because M is an isomorphism, then $\text{Im}(MB) = \text{Im}(g(\cdot, t))$. Now the new error equation is:

$$\dot{\tilde{e}} = (\tilde{A} - \tilde{B}\tilde{K})\tilde{e} + \begin{bmatrix} 0 \\ I \end{bmatrix} g_2(x, t). \quad (26)$$

Replacing the matrices A, BK, \hat{P}, \hat{Q} and variables u, x, \tilde{x} and e in the proof of Proposition 1 in [9] by the matrices $\tilde{A}, \tilde{B}\tilde{K}, \hat{\tilde{P}}, \hat{\tilde{Q}}$ and variables \tilde{u}, z, \tilde{z} and \tilde{e} the demonstration follows the same procedure. \square

In practical situations there are always perturbations present. We introduce an additive perturbation at the control input such that the applied control signal \tilde{u} to the slave system is distorted by the perturbation, i. e.,

$$\tilde{u} = \hat{u}(t) + Bd(t). \quad (27)$$

In this case $d(t)$ represents the additive perturbation that we introduce to the system, which we assume to be bounded,

$$\|d(t)\| \leq \delta_0. \quad (28)$$

If this perturbation is present we can not expect to achieve local asymptotic stability, but we will show that it is possible to reach ultimate bounded stability.

Theorem 7. *Consider the dynamical system*

$$\begin{aligned} \dot{x} &= h(A)x + g(x), \\ \dot{\tilde{x}} &= h(A)\tilde{x} + g(\tilde{x}) + \tilde{u}, \end{aligned} \quad (29)$$

and $h(\cdot)$ as in Theorem 5, where (A, B) is stabilizable, $g(x)$ is locally Lipschitz in x in a domain

$x \in \chi$ and $h(\mathbb{C}^-) \subset \mathbb{C}^-$. Then, with the noisy control (27) with a bounded perturbation given by (28), and

$$\hat{u} = h(BK)e + \tau(e)BB^\top \hat{P}e, \quad (30)$$

with $\tau(\cdot) : \mathbb{R}^n \rightarrow [0, \infty)$ locally Lipschitz and positive, and $e = x - \tilde{x}$. Then, the error is locally ultimately bounded within the ball

$$G(e) = \left\{ e : \|e\| \leq \frac{2\kappa(\hat{P}^{\frac{1}{2}})\lambda_{\max}(\hat{P})\|B\|\delta_0}{\lambda_{\min}(\hat{Q}) - 2l\lambda_{\max}(\hat{P})} \right\}, \quad (31)$$

providing that

$$\lambda_{\min}(\hat{Q}) > 2l\lambda_{\max}(\hat{P}), \quad (32)$$

where

$$(h(A) - h(BK))^\top \hat{P} + \hat{P}(h(A) - h(BK)) = -\hat{Q}, \quad (33)$$

with $\hat{Q} > 0$.

If this holds for an arbitrary large set χ , then the error system is semi globally ultimately bounded. If this holds $\forall x, \tilde{x} \in \mathbb{R}^n$, the error system is globally ultimately bounded.

Proof. Since the pair (A, B) is stabilizable, there exists a matrix K such that $A - BK$ is Hurwitz. By Proposition 4, the matrix $h(A - BK) = h(A) - h(BK)$ is also Hurwitz. In consequence, there exists a matrix $\hat{P} > 0$ that satisfies the Lyapunov equation (33).

Now, we form the error system $e = x - \tilde{x}$

$$\dot{e} = h(A) - h(BK)e + g(x) - g(\tilde{x}) - \tau(e)BB^\top \hat{P}e - Bd, \quad (34)$$

choosing $v(e) = e^\top \hat{P}e$ as a Lyapunov candidate function, and evaluating its derivative along the trajectories of the error system we have

$$\dot{v}(e) = \dot{e}^\top \hat{P}e + e^\top \hat{P}\dot{e}, \quad (35)$$

substituting into $\dot{v}(e)$ and using Lemma 3,

$$\begin{aligned} \dot{v}(e) &= e^\top [h(A - BK)^\top \hat{P} + \hat{P}h(A - BK)] e \\ &\quad + 2e^\top \hat{P}(g(x) - g(\tilde{x})) - 2\tau(e)e^\top \hat{P}BB^\top \hat{P}e - 2e^\top \hat{P}Bd \\ &\leq -e^\top \hat{Q}e + 2\|e\|\lambda_{\max}(\hat{P})l\|e\| - 2\tau(e)e^\top \hat{P}BB^\top \hat{P}e \\ &\quad - 2\|e\|\lambda_{\max}(\hat{P})\|B\|\delta_0 - 2\tau(e)e^\top \hat{P}BB^\top \hat{P}e \\ &\leq \|e\| \left\{ -\|e\|\lambda_{\min}(\hat{Q}) + 2l\lambda_{\max}(\hat{P})\|e\| \right\} + 2\lambda_{\max}(\hat{P})\|B\|\delta_0 \\ &\quad - 2\tau(e)e^\top \hat{P}BB^\top \hat{P}e. \end{aligned} \quad (36)$$

Assuming that $\lambda_{\min}(\hat{Q}) > 2l\lambda_{\max}(\hat{P})$ and defining

$$-\lambda_{\min}(\hat{Q}) + 2l\lambda_{\max}(\hat{P}) = -\alpha, \quad (37)$$

yields to

$$\begin{aligned} \dot{v}(e) &\leq \|e\| \left\{ -\alpha\|e\| + 2\lambda_{\max}(\hat{P})\|B\|\delta_0 \right\} - 2\tau(e)e^\top \hat{P}BB^\top \hat{P}e \\ &\leq \|e\| \left\{ -\frac{\alpha\sqrt{v}}{\sqrt{\lambda_{\max}(\hat{P})}} + 2\lambda_{\max}(\hat{P})\|B\|\delta_0 \right\} \\ &\quad - 2\tau(e)e^\top \hat{P}BB^\top \hat{P}e. \end{aligned} \quad (38)$$

Then, for

$$\sqrt{v} > \frac{2\lambda_{\max}(\hat{P})\sqrt{\lambda_{\max}(\hat{P})}\|B\|\delta_0}{\alpha}, \quad (39)$$

$\dot{v}(e) < 0$, defining a set of ultimately boundedness in v . If

$$e \in G(e) := \left\{ e : \|e\| \leq \frac{2\lambda_{\max}(\hat{P})\kappa(\hat{P}^{\frac{1}{2}})\|B\|\delta_0}{\alpha} \right\}, \quad (40)$$

guarantees, for a condition number $\kappa(\hat{P}^{\frac{1}{2}})$ of $\hat{P}^{\frac{1}{2}}$ that $\dot{v}(e) < 0$, which implies that $G(e)$ is ultimately entered. If χ is chosen arbitrarily large, the ultimately boundedness holds for all $x, \tilde{x} \in \chi$, providing that $G(e) \subset \chi$. \square

This theorem ensures ultimately bounded stability inside a region defined by $G(e)$, which is function of δ_0 , $\lambda_{\max}(\hat{P})$ and $\lambda_{\min}(\hat{Q})$. The size of $G(e)$ depends on how small $\lambda_{\max}(\hat{P})$ is and how large $\lambda_{\min}(\hat{Q})$ is, the smaller $\lambda_{\max}(\hat{P})$ and the larger $\lambda_{\min}(\hat{Q})$ are, the smaller the size of $G(e)$ will be.

As it is mention in [9] the condition $\lambda_{\min}(\hat{Q}) > 2l\lambda_{\max}(\hat{P})$ is restrictive, nevertheless it is possible to apply the same technique use in Theorem 6 to relax this condition.

5 Examples of Synchronicity Preservation

In this section, we present two examples of a master-slave system driven by a control law designed according to the last section.

5.1 Sprott's Q system

The dynamical systems used are known as Sprott's Q systems. Let us consider the master system

$$\dot{x}_1 = -x_3, \quad (41)$$

$$\dot{x}_2 = x_1 - x_2, \quad (42)$$

$$\dot{x}_3 = 3.1x_1 + x_2^2 + 0.5x_3, \quad (43)$$

and the slave system

$$\begin{aligned}\dot{\tilde{x}}_1 &= -\tilde{x}_3, \\ \dot{\tilde{x}}_2 &= \tilde{x}_1 - \tilde{x}_2, \\ \dot{\tilde{x}}_3 &= 3.1\tilde{x}_1 + \tilde{x}_2^2 + 0.5\tilde{x}_3 + u_3.\end{aligned}\quad (44)$$

The matrix A that represents the linear part around the origin is

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ 3.1 & 0 & 0.5 \end{bmatrix} \quad (45)$$

and $B = [0, 0, 1]$. To place the poles of the system at $[-2, -2.5, -3]$ the state feedback matrix $K = [-8.9, -3, 7]$. Choosing $Q = I_3$ and solving equation (17) gives

$$P = \begin{bmatrix} 1.4606 & 0.2758 & -0.0646 \\ 0.2758 & 0.4939 & -0.0020 \\ -0.0646 & -0.0020 & 0.0869 \end{bmatrix}. \quad (46)$$

Setting the value of $\tau(e) = e^2$, the initial conditions of the Master System to $x_1(0) = x_2(0) = x_3(0) = 0.05$ and the Slave system initial conditions $\tilde{x}_1 = 0.1, \tilde{x}_2 = \tilde{x}_3 = 0$. We let the master and slave systems evolve without control until $t = 100$ when the control law is engaged.

Figure 1 shows the phase portrait of the master and slave systems which exhibit a chaotic behavior. In Fig. 2 the absolute value of the error between the master and slave systems are plotted in semi-logarithmic scale. It can be seen in Fig. 2 that the absolute value of the error between the two systems converges to zero after the control law is activated at time $t = 100$.

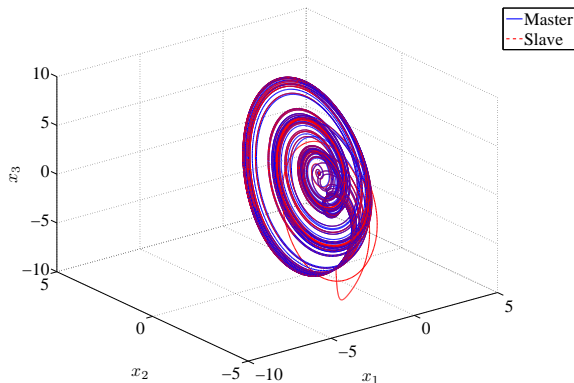


Figure 1: Original Sprott Q attractor showing synchronization between master and slave systems (initial conditions $x_1(0) = x_2(0) = x_3(0) = 0.05$, $\tilde{x}_1(0) = 0.1$, $\tilde{x}_2(0) = \tilde{x}_3(0) = 0$).

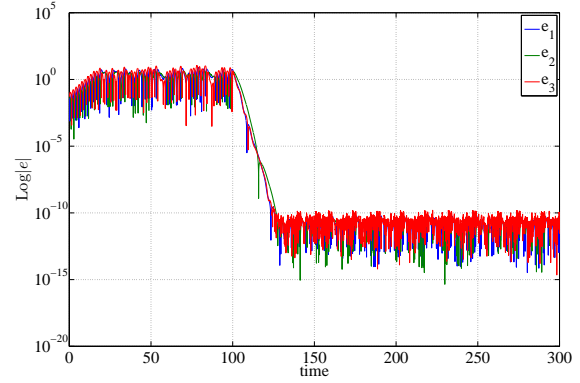


Figure 2: Magnitude of error between master and slave systems ($|e| = |x - \tilde{x}|$).

Now we modify the system applying the linear function

$$h(A) = MA, \quad (47)$$

to the linear parts of the system and the control law. We choose M to be positive definite, so it preserves the sign characteristic of the eigenvalues of $(A - BK)$. Since $(A - BK) = UT_{(A-BK)}U^{-1}$, where

$$U = \begin{bmatrix} 0.3123 & 0.4827 & -0.8182 \\ -0.1562 & -0.8235 & -0.5455 \\ 0.9370 & -0.2981 & 0.1818 \end{bmatrix}. \quad (48)$$

If we choose $M = UT_MU^{-1}$ and we set T_M as

$$T_M = \begin{bmatrix} 0.5 & -0.05 & -0.05 \\ 0 & 0.5 & -0.05 \\ 0 & 0 & 1 \end{bmatrix}, \quad (49)$$

M will be given by

$$M = \begin{bmatrix} 0.8597 & 0.2577 & -0.0770 \\ 0.1868 & 0.6156 & -0.0430 \\ -0.0709 & 0.0064 & 0.5247 \end{bmatrix}. \quad (50)$$

Choosing again $\tilde{Q} = I_3$ and solving (21) gives

$$\hat{P} = \begin{bmatrix} 1.5489 & -0.0677 & 0.0853 \\ -0.0677 & 0.8634 & 0.0700 \\ 0.0853 & 0.0700 & 0.1425 \end{bmatrix}. \quad (51)$$

Using $\tau = 30 + 500 \exp(-0.001(|e_1| + |e_2| + |e_3|))$ and the same initial conditions used for the original master-slave system, we solve the modified systems.

Figure 3 shows the synchronization between the master-slave modified system, therefore the original system definitely changes, the new system preserves the synchronization. This can be

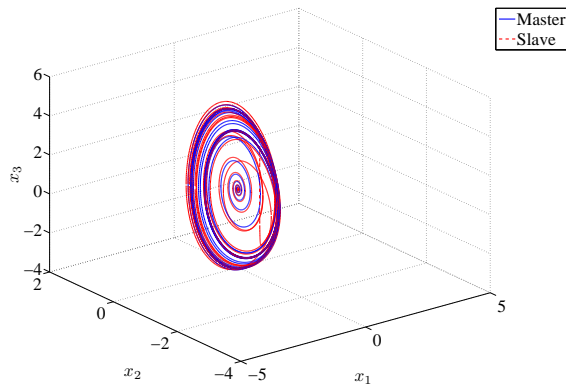


Figure 3: Master-Slave modified system, initial conditions $x_1 = x_2 = x_3 = 0.05$, $\tilde{x}_1 = 0.1$, $\tilde{x}_2 = \tilde{x}_3 = 0$.

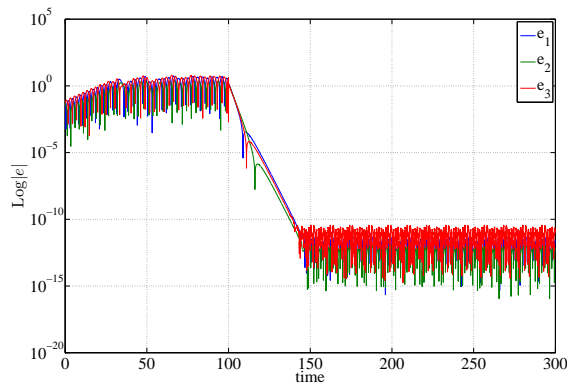


Figure 4: Magnitude of error between master and slave modified systems ($|e| = |\tilde{x} - x|$).

seen in Fig. 4 where the errors between the modified master-slave system are presented, in a semi-logarithmic plot, to empathize the convergence to zero. Here again the control law is activated at $t = 100$.

5.2 Chua's system

Now, we use the well known chaotic system, Chua's circuit, to show in a simulation how synchronization is preserved. Let the Master system be given by

$$\begin{aligned}\dot{x}_1 &= \frac{10}{7}x_1 + 10x_2 - \frac{20}{7}x_1^3, \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -\frac{100}{7}x_2.\end{aligned}\quad (52)$$

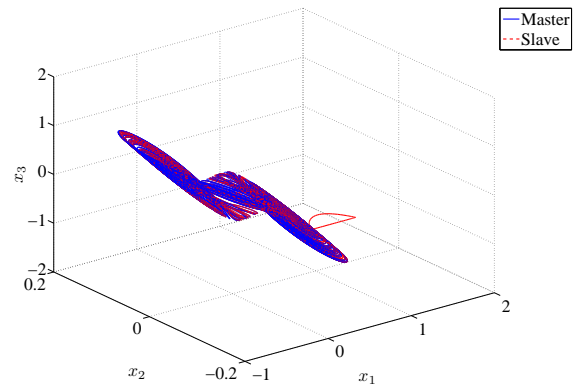


Figure 5: Chua's Master-Slave system, initial conditions $x_1 = 0.02$, $x_2 = 0.05$, $x_3 = 0.04$, $\tilde{x}_1 = \tilde{x}_2 = \tilde{x}_3 = 0$.

and the slave system

$$\begin{aligned}\dot{\tilde{x}}_1 &= \frac{10}{7}\tilde{x}_1 + 10\tilde{x}_2 - \frac{20}{7}\tilde{x}_1^3 + u_1, \\ \dot{\tilde{x}}_2 &= \tilde{x}_1 - \tilde{x}_2 + \tilde{x}_3, \\ \dot{\tilde{x}}_3 &= -\frac{100}{7}\tilde{x}_2.\end{aligned}\quad (53)$$

The matrix A that represents this the linear part of the system is

$$A = \begin{bmatrix} 10/7 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -100/7 & 0 \end{bmatrix}.\quad (54)$$

and the matrix B is $[1, 0, 0]$. We place the poles of the system in $[-1, -1.5, -2]$ thus, the state feedback matrix is $K = [4.9286, -1.2857, 3.29]$. We choose $Q = I_3$ and solving equation (17) gives

$$P = \begin{bmatrix} 11.2795 & 38.9782 & 6.5176 \\ 38.9782 & 141.2133 & 20.9429 \\ 6.5176 & 20.9429 & 4.5911 \end{bmatrix}.\quad (55)$$

Now, we set $\tau(e) = 3.9e3 + 500 \exp(-0.001(|e_1| + |e_2| + |e_3|))$, as suggested by Theorem 6; the initial conditions for the Master system as $x_1 = 0.02$, $x_2 = 0.05$, $x_3 = 0.04$ and the initial conditions for the slave system $\tilde{x}_1 = 0.01$, $\tilde{x}_2 = \tilde{x}_3 = 0$.

Figure 5 shows a phase portrait of the original Chua's Master-Slave system, where we can see this system reaches synchronization. In Fig. 6 the absolute value for the error between the Master and slave system are shown in a semi-logarithmic plot, to empathize the fact that the error converges to zero.

As shown before, we design a matrix M, that is simultaneously triangularizable with $A - BK$

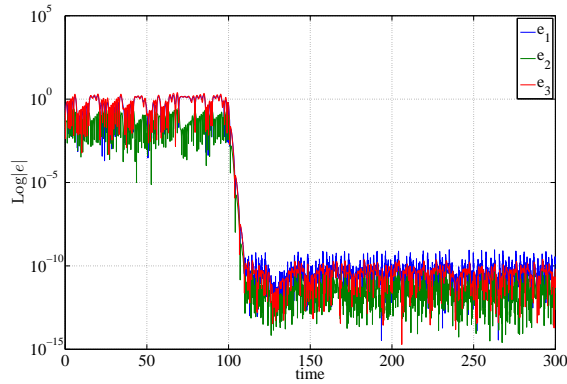


Figure 6: Magnitude of error between master and slave systems in semi-logarithmic scale ($\log |e| = \log |\tilde{x} - x|$).

and that preserves the sign structure of the eigenvalues of $A - BK$. We know that $A - BK = UT_{A-BK}U'$, where U is an unitary matrix

$$U = \begin{bmatrix} -0.7486 & -0.5591 & -0.3564 \\ 0.0919 & 0.4448 & -0.8909 \\ 0.6566 & -0.6997 & -0.2815 \end{bmatrix}, \quad (56)$$

If we let M be given by $M = UT_MU'$ and we choose the upper triangular matrix T_M as

$$T_M = \begin{bmatrix} 0.5000 & 0 & 0.0010 \\ 0 & 0.5000 & -0.0010 \\ 0 & 0 & 1.1000 \end{bmatrix}, \quad (57)$$

M will be given by

$$M = \begin{bmatrix} 0.5763 & 0.1907 & 0.0603 \\ 0.1906 & 0.9766 & 0.1506 \\ 0.0597 & 0.1493 & 0.5472 \end{bmatrix}. \quad (58)$$

Using these values, we solve equation (21)

$$\hat{P} = \begin{bmatrix} 2.9698 & 15.9383 & 0.1208 \\ 15.9383 & 94.7681 & -0.7060 \\ 0.1208 & -0.7060 & 1.0032 \end{bmatrix}. \quad (59)$$

The initial conditions for the modified system are the same that we used for the original system, $x_1 = 0.02, x_2 = 0.05, x_3 = 0.04$ for the Master system and $\tilde{x}_1 = 0.1, \tilde{x}_2 = \tilde{x}_3 = 0$ for the slave system.

Figure 7 shows the trajectories for the solutions of the master and slave systems. In Fig. 8 the error between the master and slave system is plotted in a semi-logarithmic scale so it can be seen that the error converges to zero and the synchronization is achieved.

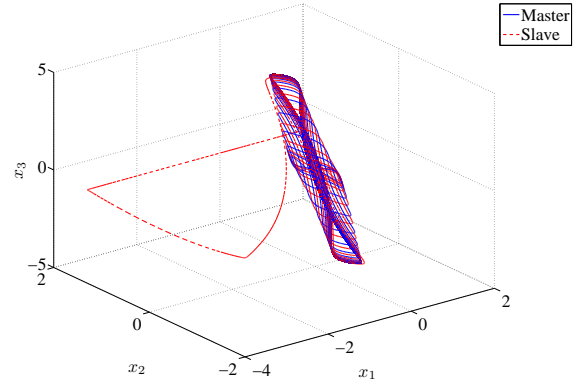


Figure 7: Chua's modified Master-Slave system, initial conditions $x_1 = 0.02, x_2 = 0.05, x_3 = 0.04, \tilde{x}_1 = 0.1, \tilde{x}_2 = \tilde{x}_3 = 0$.

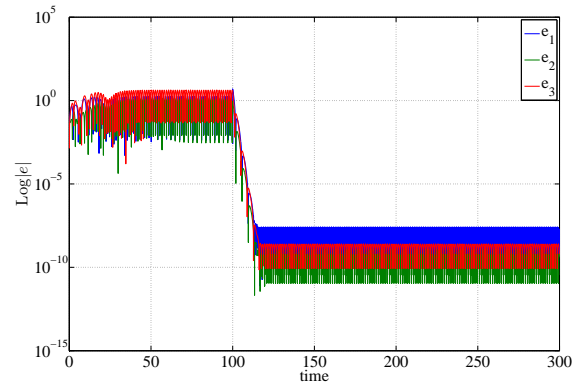


Figure 8: Magnitude of error between master and slave modified system in semi-logarithmic scale ($\log |e| = \log |\tilde{x} - x|$).

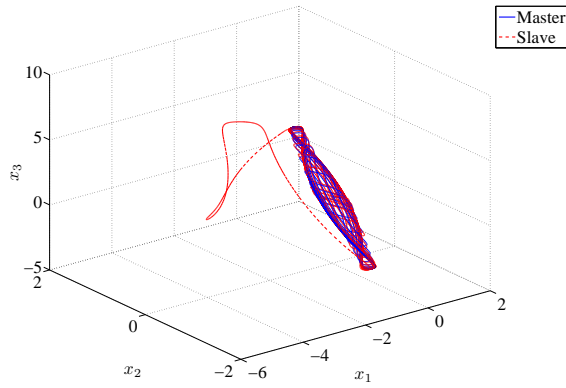


Figure 9: Chua's modified Master-Slave system with perturbation, initial conditions $x_1 = 0.02, x_2 = 0.05, x_3 = 0.04, \tilde{x}_1 = 0.01, \tilde{x}_2 = \tilde{x}_3 = 0$.

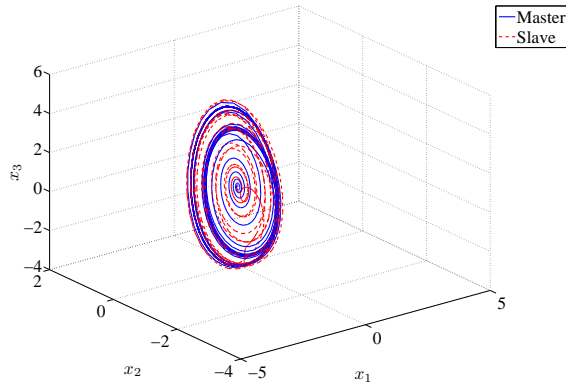


Figure 10: Sprott's Q modified Master-Slave system with perturbations, initial conditions $x_1 = 0.02, x_2 = 0.05, x_3 = 0.04, \tilde{x}_1 = 0.1, \tilde{x}_2 = \tilde{x}_3 = 0$.

6 Partial Synchronization with Perturbations

In this section we use the two modified systems, the modified Sprott's Q and the modified Chua's system, to prove that we can achieve partial synchronization when a sinusoidal function, which represents the perturbation, is introduced to the feedback control of the system. Therefore, the new control law has the extra term $d(t)$ and it will be given by

$$\tilde{u} = h(BK)e + \tau(e)BB'\hat{P}e + Bd(t). \quad (60)$$

For the Sprott's Q system we used $d(t) = 0.1\sin(t)$ and for Chua's system we used $d(t) = 0.2\sin(5t)$. Now we simulate the systems with this new control and using the same parameters that we use for the original systems.

Figure (9) and Fig.(10) show the phase portrait diagrams for the error systems of the modified Chua's system and Sprott's Q systems. In

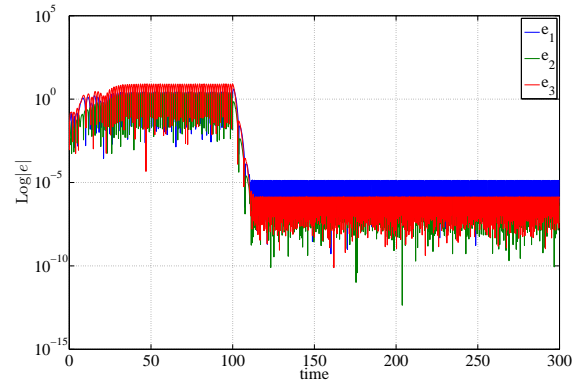


Figure 11: Magnitude of error between master and slave modified system in semi-logarithmic scale ($\log |e| = \log |\tilde{x} - x|$).

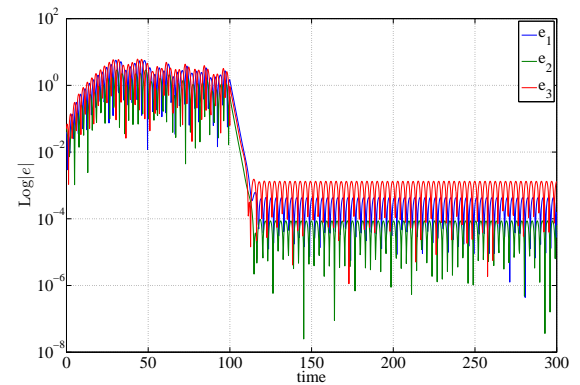


Figure 12: Magnitude of error between master and slave modified system in semi-logarithmic scale ($\log |e| = \log |\tilde{x} - x|$).

this simulations we can see that the error between the state variables do not converges to zero but it remains inside a region ultimately bounded, as was predicted by the theorem.

In Fig. (11) and Fig. (12) is easier to see that the error do not converges asymptotically to zero and it remains bounded as time increases. These plots are in a logarithmic scale and can be compared with Fig. 8 for Chua's modified system and Fig. 4 for Sprott's system, where it can be seen that the error converges to zero when the perturbation is not present.

7 Conclusions

If a dynamical system is changed by applying a function of the kind described in this work, when the structure of the sign of the eigenvalues of the jacobian matrix, evaluated in the hyperbolic points is preserved, the new pair master-slave system also achieves synchronization. We suppose

that the chaotic dynamics that the original system exhibits is also preserved. Although the principal interest of this work was to preserve synchronization after modifying a dynamical system, we have proved that the control law we used along this work, which was proved in other paper to work correctly synchronizing a master-slave system, can be modified in its linear part to make the modified master slave system achieve synchronization via Lyapunov method. We have to point out that the value of the parameter τ is important since the system is really sensible to it.

When a perturbation was introduced to the system, we showed that it is possible to preserve the stability of the error system, but the kind of stability we can ensure is ultimately bounded in contrast of the local asymptotic stability that is obtained when the perturbation in the control law is not present. This ultimate bounded stability means that the error between the state variables, is inside a region defined by $G(e)$, defined in the last theorem.

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