Quadratic Fluency DA Functions as Non-uniform Sampling Functions for Interpolating Sampled-values

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Abstract: - Interpolation for sampled-values with non-uniform sampling points is required for various cases of signal processing. In such a case, sampling functions are useful to interpolate sampled-values and then to generate signals as a linear combination of the sampling basis weighted by a sequence of the sampled-values. This paper proposes sampling functions for non-uniform sampling points, each of which is composed with piecewise polynomials of degree 2. We name the sampling functions the *fluency DA functions* of degree 2. The *fluency DA functions* generate smooth and undulate signals from a sequence of sampled-values.

Key-Words: - Fluency information theory, Fluency DA functions, Interpolation, Non-uniform sampling functions, Piecewise polynomials

1 Introduction

Multimedia, such as audio, still images and video, which exists in the real world is generally treated as analog signals. In order to treat the analog signals in the computer world, they must be converted into digital signals. The digital signals are converted into analog signals and then original multimedia is reproduced.

Therefore, both of analog-to-digital(A/D) converter and digital-to-analog(D/A) one play important role in signal processing.

In the conventional signal analysis and processing, Information Communication and Technologies(ICT) such as A/D and D/A technologies have been designed in the analytic function space S as subspace of typical Hilbert space L_2, where L_2 is the space spanned by square integrable functions.

Shannon's uniform sampling theorem which guarantees isomorphism between band-limited analog signal space and digital signal one of a sequence of sampled-values is well-known[1],[2] and is also considered in the analytic function space S.

One of authors proposed and established *Fluency Information Theory* [3],[4],[5] that generalizes Shannon's sampling theorem. The *Fluency Information Theory* –based signal analysis and processing are considered in the dual space for the function space spanned by piecewise polynomials

This paper proposes sampling functions for non-uniform sampling points, each of which is composed with piecewise polynomials of degree 2.

We name the sampling functions the *fluency DA functions* of degree 2. The *fluency DA functions* are designed based on geometric criterion of curve. The *fluency DA functions* generate smooth and undulate signals from a sequence of sampled-values.

2 Preliminaries

2.1 Signal Space D_m composed of Piecewise Polynomials of Degree (m-1)

In the conventional signal analysis and processing, Information Communication and Technologies(ICT) such as A/D and D/A technologies have been designed in the analytic functions space S as subspace of typical Hilbert space L_2, where L_2 is the signal space spanned by square integrable functions.

Dirac's delta functions have been often used in making discussion on isomorphism property between analog signals and digital ones. Moreover, sin and cos functions have been also used in DCT-based multimedia coding like as JPEG and MPEG. However, these functions for signal analysis and processing do not belong to L_2.

So, in treating these kinds of functions, it is necessary to expand the conventional signal space L_2 .

If X is a function space we can define its dual space X' to be the set of continuous linear functions T from X to R or C, where R and C are the sets of real and complex numbers, respectively. Such mappings themselves form a normed linear space using the operator norm

$$\|T\| = \sup_{x \in X, x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

If $X \subset Y$, then $Y' \subset X'$, since there are fewer continuous functions on a larger function space. Therefore, a highly restricted Schwartz function space S, which is the set of rapidly decreasing functions, *i.e.*, the functions $x = x(t)$ satisfying the following two conditions for each $k, n \in \{0,1,2,...\}$:

<1> $x \in C^{\infty}(-\infty, \infty)$

<2> $\lim_{|t| \to \infty} \left| t^k \frac{d^n}{dt^n} x(t) \right| = 0,$

has a very large dual space. The dual space S' for the Schwartz function space S is larger than L_2 . However, the dual space S' is too tempered.

We introduce appropriate signal space D_m for the signal analysis, which is composed of piecewise polynomials of degree (m-1) with only (m-2) times continuous differentiability in this paper, where $m \in \{1,2,3,...\}$. In case of m=1, the signal pace D_1 is a function space spanned by discontinuous functions. In case of m=2, the signal pace D_2 is a function space spanned by continuous functions which are not differentiable.

It had been shown [5] that the signal space D_m identical with a band-limited function space, which is treated in the Shannon's uniform sampling theorem, when the parameter m tends to infinity. Based on this fact, it became possible to deal with piecewise polynomial function spaces and band-limited function ones as a unified series of signal spaces of which characteristics vary with the parameter of degree of the polynomials. This series is *fluent* in the sense that we can choose a signal space out of the series which matches with each purpose of signal analysis and processing. So it was named as "fluency". The signal spaces D_1, D_2 and D_∞ are identical with the sets of staircase, polygonal and band-limited functions, respectively.

The *Fluency information theory*-based signal analysis and processing are considered in the dual space $D_m^{'}$ for the signal space D_m.

The dual space $D_m^{'}$ contains arbitrary derivatives of certain discontinuous functions. Figure 1 shows signal space and its dual space.

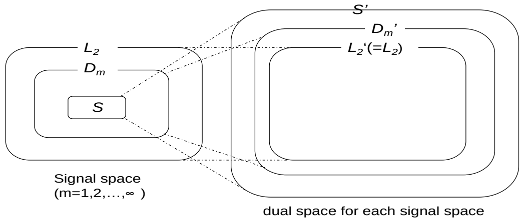

Fig.1 Signal space and its dual space

The signal space D_m and its dual space $D_m^{'}$ are appropriate function spaces for signal analysis and processing.

Let $\{t_k\}_{k=-\infty}^{\infty}$ be sampling points, then the sampling basis in the signal space $D_m^{'}$ is defined by the functions $\{{}_{[DA]}^{m}\psi_k\}_{k=-\infty}^{\infty}$ satisfying

$$u \in D'_m, u(t) = \sum_{k=-\infty}^{\infty} u(t_k) \cdot {}_{[DA]}^{m}\psi_k(t) \quad . \tag{1}$$

Equation (1) gives a representation formula as a linear combination of the sampling basis in $D_m^{'}$ weighted by a sequence of sampled-values $\{u(t_k)\}_{k=-\infty}^{\infty}$. We named each function of the sampling basis $\{{}_{[DA]}^{m}\psi_k\}_{k=-\infty}^{\infty}$ the *Fluency DA function*. In case that the interval between adjacent sampling points is constant, that is, $t_k - t_{k-1} = h$ (h >0), sampling functions in $D_m^{'}$ are called by *uniform Fluency DA functions of degree (m-1)*. In case that the interval is not constant, then those are called by *non-uniform Fluency DA functions of degree (m-1)* in this paper.

2.2 Compactly Supported Uniform Fluency DA functions of Degree 2

We proposed and developed an impulse response that is suitable for DVD-Audio with a maximum sampling rate of 192KHz . It had been designed in the dual space D_3' for the signal space D_3. The impulse response is composed of the compactly supported uniform Fluency DA functions of degree 2 [6]. Practically, DVD-Audio players equipped with the Digital-to-Analog converters designed by the uniform Fluency DA functions of degree 2 have been commercialized. The 53 awards have been received.

There have been many ICT applications [7], [8], [9], [10] designed in the signal space D_3'.

The quadratic uniform Fluency DA function ${}_{[DA]}^{3}\psi_0(t)$ as a sampling function in the signal space D_3' was designed as is satisfied the following 4 conditions <1>,<2>,<3> and <4>.

<1> It is represented by the linear combination of quadratic B-spline functions.

<2> It is only one time continuously differentiable at $(-\infty, \infty)$.

<3> It converges to 0 at the left and right second sampling points from the origin, that is, at $t_{-2}(=-2h)$ and $t_2(=2h)$.

<4> It takes the value of 1 at the origin $t=0$. It takes the value of 0 at sampling points $t = \pm h, \pm 2h$.

Let $\varphi(t)$ denotes the quadratic B-spline function defined as follows [11],[12]:

$$\varphi(t) = \int_{-\infty}^{\infty} \left(\frac{\sin \pi f h}{\pi f h} \right)^3 e^{j2\pi f t} df. \tag{2}$$

The quadratic B-spline function is expressed by piecewise polynomials of degree 2. Then, the compactly supported uniform fluency DA function of degree 2 ${}_{[DA]}^{3}\psi_0(t)$ is represented in the form of linear combination of the function systems $\{\varphi(t - \frac{l}{2}h)\}_{l=-1}^{1}$ as follows:

$${}_{[DA]}^{3}\psi_0(t) = -\frac{h}{2}\phi(t - \tfrac{1}{2}h) + 2h\phi(t) - \frac{h}{2}\phi(t + \tfrac{1}{2}h). \tag{3}$$

The DA function ${}_{[DA]}^{3}\psi_0(t)$ was derived as

$$
{}_{[DA]}^{3}\psi_0(t) =
\begin{cases}
-t^2/(4h^2) - t/h - 1, & -2h \le t < -3h/2, \\
3t^2/(4h^2) + 2t/h + 5/4, & -3h/2 \le t < -h, \\
5t^2/(4h^2) + 3t/h + 7/4, & -h \le t < -h/2, \\
-7t^2/(4h^2) + 1, & -h/2 \le t < 0, \\
-7t^2/(4h^2) + 1, & 0 \le t < h/2, \\
5t^2/(4h^2) - 3t/h + 7/4, & h/2 \le t < h, \\
3t^2/(4h^2) - 2t/h + 5/4, & h \le t < 3h/2, \\
-t^2/(4h^2) + t/h - 1, & 3h/2 \le t < 2h, \\
0, & otherwise.
\end{cases}
\tag{4}
$$

Figure 1 shows the quadratic uniform Fluency DA function ${}_{[DA]}^{3}\psi_0(t)$.

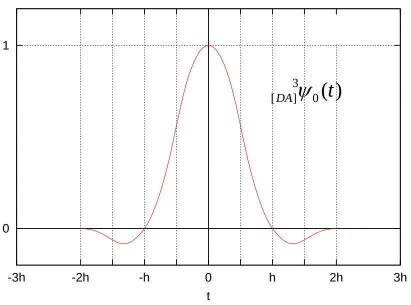

Fig. 1 Quadratic uniform Fluency DA function

It is noted that the quadratic uniform Fluency DA function ${}_{[DA]}^{3}\psi_0(t)$ is only one time continuously differentiable at $t = \pm\frac{1}{2}h, \pm h, \pm\frac{3}{2}h, \pm 2h$, which are connecting points of each piecewise polynomial .

The sampling basis $\left\{ {}_{[DA]}^{3}\psi_k \right\}_{k=-\infty}^{\infty}$ in the signal space D_3' are derived as follows.

$$
{}_{[DA]}^{3}\psi_k(t) = {}_{[DA]}^{3}\psi_0(t - kh). \quad (k = 0, \pm 1, \ldots.
$$

Thus the any signal u in D_3' is represented by

$$
u \in D'_3, u(t) = \sum_{k=-\infty}^{\infty} u(t_k) \cdot {}_{[DA]}^{3}\psi_k(t) = \sum_{k=-\infty}^{\infty} u(kh) \cdot {}_{[DA]}^{3}\psi_0(t - kh).
$$

3 Criterion for Designing Compactly Supported Non-Uniform Fluency DA Function Composed of Quadratic Piecewise Polynomials

3.1 Formulation of Compactly Supported Non-Uniform Fluency DA Function of Degree 2

The non-uniform Fluency DA function of degree 2 is designed by expanding the compactly supported uniform Fluency DA functions of degree 2.

As is understood from Eq.(4), the uniform Fluency DA function of degree 2, that is, ${}_{[DA]}^{3}\psi_0(t)$, can be generally considered to be composed of 8 piecewise polynomials in [-2h, 2h]. We formulate a compactly supported non-uniform Fluency DA function of degree 2 by $s(t)$ in this paper.

The function $s(t)$ is designed as is satisfied the following 4 conditions <1'>,<2'>,<3'> and <4'>.

<1'> It is represented by the linear combination of quadratic piecewise polynomials.

<2'> It is only one time continuously differentiable at $(-\infty, \infty)$.

<3'> It converges to 0 at the left and right second sampling points from the origin, that is, at t_{-2} and t_2.

<4'> It takes the value of 1 at the origin $t = t_0 = 0$. It takes the value of 0 at sampling points $t = t_{-2}, t_{-1}, t_1, t_2$.

Taking account of the above conditions <1'> and <3'>, the function $s(t)$ can be formulated as follows.

$$s(t) = \begin{cases} a_1 t^2 + b_1 t + c_1 \triangleq s_1(t), & t_{-2} \le t \le (t_{-2} + t_{-1})/2, \\ a_2 t^2 + b_2 t + c_2 \triangleq s_2(t), & (t_{-2} + t_{-1})/2 \le t \le t_{-1}, \\ a_3 t^2 + b_3 t + c_3 \triangleq s_3(t), & t_{-1} \le t \le (t_{-1} + t_0)/2, \\ a_4 t^2 + b_4 t + c_4 \triangleq s_4(t), & (t_{-1} + t_0)/2 \le t \le t_0, \\ a_5 t^2 + b_5 t + c_5 \triangleq s_5(t), & t_0 \le t \le (t_0 + t_1)/2, \\ a_6 t^2 + b_6 t + c_6 \triangleq s_6(t), & (t_0 + t_1)/2 \le t \le t_1, \\ a_7 t^2 + b_7 t + c_7 \triangleq s_7(t), & t_1 \le t \le (t_1 + t_2)/2, \\ a_8 t^2 + b_8 t + c_8 \triangleq s_8(t), & (t_1 + t_2)/2 \le t \le t_2, \\ 0, & otherwise. \end{cases} \quad (5)$$

Figure 2 shows general waveform of the function s(t).

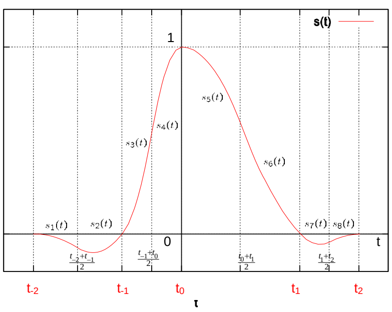

Fig.2 Quadratic Non-uniform Fluency DA Function

Taking account of the above condition <4'>, the following relations are obtained.

$$s_1(t_{-2}) = 0 \quad (6)$$

$$s_2(t_{-1}) = s_3(t_{-1}) = 0 \quad (7)$$

$$s_4(t_0) = s_5(t_0) = 1 \quad (8)$$

$$s_6(t_1) = s_7(t_1) = 0 \quad (9)$$

$$s_8(t_2) = 0. \quad (10)$$

Taking account of the above condition <2'>, the following relations are obtained.

$$s_1\left(\tfrac{t_{-2}+t_{-1}}{2}\right) = s_2\left(\tfrac{t_{-2}+t_{-1}}{2}\right) \triangleq d_{-2} \quad (11)$$

$$s_3\left(\tfrac{t_{-1}+t_0}{2}\right) = s_4\left(\tfrac{t_{-1}+t_0}{2}\right) \triangleq d_{-1} \quad (12)$$

$$s_5\left(\tfrac{t_0+t_1}{2}\right) = s_6\left(\tfrac{t_0+t_1}{2}\right) \triangleq d_1 \quad (13)$$

$$s_7\left(\tfrac{t_1+t_2}{2}\right) = s_8\left(\tfrac{t_1+t_2}{2}\right) \triangleq d_2. \quad (14)$$

$$s'_1(t_{-2}) = 0 \quad (15)$$

$$s'_1\left(\tfrac{t_{-2}+t_{-1}}{2}\right) = s'_2\left(\tfrac{t_{-2}+t_{-1}}{2}\right) \quad (16)$$

$$s'_2(t_{-1}) = s'_3(t_{-1}) \quad (17)$$

$$s'_3\left(\tfrac{t_{-1}+t_0}{2}\right) = s'_4\left(\tfrac{t_{-1}+t_0}{2}\right) \quad (18)$$

$$s'_4(t_0) = s'_5(t_0) \quad (19)$$

$$s'_5\left(\tfrac{t_0+t_1}{2}\right) = s'_6\left(\tfrac{t_0+t_1}{2}\right) \quad (20)$$

$$s'_6(t_1) = s'_7(t_1) \quad (21)$$

$$s'_7\left(\tfrac{t_1+t_2}{2}\right) = s'_8\left(\tfrac{t_1+t_2}{2}\right) \quad (22)$$

$$s'_8(t_2) = 0. \quad (23)$$

Moreover, from Eqs.(6)-(23), the following simultaneous equations Eq.(24) concerning to 24 unknown parameters $\{a_k\}_{k=1}^{8}$, $\{b_k\}_{k=1}^{8}$ and $\{c_k\}_{k=1}^{8}$ are derived.

$$a_1 t_{-2}^2 + b_1 t_{-2} + c_1 = 0 \quad (24)$$

$$a_1(t_{-2} + t_{-1})^2 + 2b_1(t_{-2} + t_{-1}) + 4c_1 = 4d_{-2} \quad (25)$$

$$a_2(t_{-2} + t_{-1})^2 + 2b_2(t_{-2} + t_{-1}) + 4c_2 = 4d_{-2} \quad (26)$$

$$a_2 t_{-1}^2 + b_2 t_{-1} + c_2 = 0 \quad (27)$$

$$a_3 t_{-1}^2 + b_3 t_{-1} + c_3 = 0 \quad (28)$$

$$a_3(t_{-1} + t_0)^2 + 2b_3(t_{-1} + t_0) + 4c_3 = 4d_{-1} \quad (29)$$

$$a_4(t_{-1} + t_0)^2 + 2b_4(t_{-1} + t_0) + 4c_4 = 4d_{-1} \quad (30)$$

$$a_4 t_0^2 + b_4 t_0 + c_4 = 1 \quad (31)$$

$$a_5 t_0^2 + b_5 t_0 + c_5 = 1 \quad (32)$$

$$a_5(t_0 + t_1)^2 + 2b_5(t_0 + t_1) + 4c_5 = 4d_1 \quad (33)$$

$$a_6(t_0 + t_1)^2 + 2b_6(t_0 + t_1) + 4c_6 = 4d_1 \quad (34)$$

$$a_6 t_1^2 + b_6 t_1 + c_6 = 0 \quad (35)$$

$$a_7 t_1^2 + b_7 t_1 + c_7 = 0 \quad (36)$$

$$a_7(t_1 + t_2)^2 + 2b_7(t_1 + t_2) + 4c_7 = 4d_2 \quad (37)$$

$$a_8(t_1 + t_2)^2 + 2b_8(t_1 + t_2) + 4c_8 = 4d_2 \quad (38)$$

$$a_8 t_2^2 + b_8 t_2 + c_8 = 0 \quad (39)$$

$$2a_1 t_{-2} + b_1 = 0 \quad (40)$$

$$a_1(t_{-2} + t_{-1}) + b_1 = a_2(t_{-2} + t_{-1}) + b_2 \quad (41)$$

$$2a_2 t_{-1} + b_2 = 2a_3 t_{-1} + b_3 \quad (42)$$

$$a_3(t_{-1} + t_0) + b_3 = a_4(t_{-1} + t_0) + b_4 \quad (43)$$

$$2a_4 t_0 + b_4 = 2a_5 t_0 + b_5 \quad (44)$$

$$a_5(t_0+t_1)+b_5=a_6(t_0+t_1)+b_6 \tag{45}$$

$$2a_6t_1+b_6=2a_7t_1+b_7 \tag{46}$$

$$a_7(t_1+t_2)+b_7=a_8(t_1+t_2)+b_8 \tag{47}$$

$$2a_8t_2+b_8=0 \tag{48}$$

The 24 unknown parameters $\{a_k\}_{k=1}^8$, $\{b_k\}_{k=1}^8$ and $\{c_k\}_{k=1}^8$ are obtained by solving the above simultaneous equations.

From Eqs.(24),(25) and (40), a_1, b_1 and c_1 are obtained as follows:

$$\begin{aligned} a_1&=\frac{4d_{-2}}{(t_{-1}-t_{-2})^2},\\ b_1&=\frac{-8t_{-2}d_{-2}}{(t_{-1}-t_{-2})^2},\\ c_1&=\frac{4t_{-2}^2d_{-2}}{(t_{-1}-t_{-2})^2}. \end{aligned} \tag{49}$$

From Eqs.(26),(27),(41) and (40), a_2, b_2 and c_2 are obtained as follows:

$$\begin{aligned} a_2&=\frac{-12d_{-2}}{(t_{-1}-t_{-2})^2},\\ b_2&=\frac{8(t_{-2}+2t_{-1})d_{-2}}{(t_{-1}-t_{-2})^2},\\ c_2&=\frac{-4t_{-1}(2t_{-2}+t_{-1})d_{-2}}{(t_{-1}-t_{-2})^2}. \end{aligned} \tag{50}$$

From Eqs.(28),(29),(42) and (50), a_3, b_3 and c_3 are obtained as follows:

$$\begin{aligned} a_3&=\frac{4}{t_0-t_{-1}}\left(\frac{d_{-1}}{t_0-t_{-1}}+\frac{4d_{-2}}{t_{-1}-t_{-2}}\right),\\ b_3&=\frac{-8}{t_0-t_{-1}}\left\{\frac{t_{-1}d_{-1}}{t_0-t_{-1}}+\frac{d_{-2}(3t_{-1}+t_0)}{t_{-1}-t_{-2}}\right\},\\ c_3&=\frac{4t_{-1}}{t_0-t_{-1}}\left\{\frac{t_{-1}d_{-1}}{t_0-t_{-1}}+\frac{2d_{-2}(t_{-1}+t_0)}{t_{-1}-t_{-2}}\right\}. \end{aligned} \tag{51}$$

From Eqs.(30),(31),(43) and (51), a_4, b_4 and c_4 are obtained as follows:

$$\begin{aligned} a_4&=\frac{4}{t_0-t_{-1}}\left(\frac{1-3d_{-1}}{t_0-t_{-1}}-\frac{4d_{-2}}{t_{-1}-t_{-2}}\right),\\ b_4&=\frac{4}{t_0-t_{-1}}\left\{-\frac{(t_{-1}+t_0)-2d_{-1}(t_{-1}+2t_0)}{t_0-t_{-1}}+\frac{2d_{-2}(t_{-1}+3t_0)}{t_{-1}-t_{-2}}\right\},\\ c_4&=\frac{1}{t_0-t_{-1}}\left\{\frac{(t_{-1}+t_0)^2}{t_0-t_{-1}}-\frac{4t_0d_{-1}(2t_{-1}+t_0)}{t_0-t_{-1}}-\frac{8d_{-2}t_0(t_{-1}+t_0)}{t_{-1}-t_{-2}}\right\}. \end{aligned} \tag{52}$$

From Eqs.(32),(33),(45) and (54), a_5, b_5 and c_5 are obtained as follows:

$$\begin{aligned} a_5&=\frac{4}{t_1-t_0}\left(\frac{1-3d_{-1}}{t_1-t_0}-\frac{4d_2}{t_2-t_1}\right),\\ b_5&=\frac{4}{t_1-t_0}\left\{-\frac{(t_0+t_1)-2d_1(t_1+2t_0)}{t_1-t_0}+\frac{2d_2(t_1+3t_0)}{t_2-t_1}\right\},\\ c_4&=\frac{1}{t_1-t_0}\left\{\frac{(t_0+t_1)^2}{t_1-t_0}-\frac{4t_0d_1(2t_1+t_0)}{t_1-t_0}-\frac{8d_2t_0(t_1+t_0)}{t_2-t_1}\right\}. \end{aligned} \tag{53}$$

From Eqs.(34),(35),(46) and (55), a_6, b_6 and c_6 are obtained as follows:

$$\begin{aligned} a_6&=\frac{4}{t_1-t_0}\left(\frac{d_1}{t_1-t_0}+\frac{4d_2}{t_2-t_1}\right),\\ b_6&=\frac{-8}{t_1-t_0}\left\{\frac{d_1t_1}{t_1-t_0}+\frac{d_2(3t_1+t_0)}{t_2-t_1}\right\},\\ c_6&=\frac{4}{t_1-t_0}\left(\frac{d_1}{t_1-t_0}+\frac{4d_2}{t_2-t_1}\right). \end{aligned} \tag{54}$$

From Eqs.(36),(37),(47) and (56), a_7, b_7 and c_7 are obtained as follows:

$$\begin{aligned} a_7&=\frac{-12d_2}{(t_2-t_1)^2},\\ b_7&=\frac{8(2t_1+t_2)d_2}{(t_2-t_1)^2},\\ c_7&=\frac{-4t_{-1}(t_1+2t_2)d_2}{(t_2-t_1)^2}. \end{aligned} \tag{55}$$

From Eqs.(38),(39) and (48), a_8, b_8 and c_8 are obtained as follows:

$$\begin{aligned} a_8&=\frac{4d_2}{(t_2-t_1)^2},\\ b_8&=\frac{-8t_2d_2}{(t_2-t_1)^2},\\ c_8&=\frac{4t_2^2d_2}{(t_2-t_1)^2}. \end{aligned} \tag{56}$$

As the results, it seems that 24 unknown parameters $\{a_k\}_{k=1}^8$, $\{b_k\}_{k=1}^8$ and $\{c_k\}_{k=1}^8$ can be obtaiend. However, 4 parameters d_{-2},d_{-1},d_1 and d_2 defined by Eqs. (11), (12), (13) and (14), respectively can be arbitrarily set. It is noted that Eq.(44) has not been used in the above processes. This means that the 4 parameters d_{-2},d_{-1},d_1 and d_2 are not independent. By substituting a_4,a_5,b_4

and b_5 for Eq.(44), the following relation is obtained.

$$\frac{2d_{-2}}{t_{-2}-t_{-1}} - \frac{1-2d_{-1}}{t_{-1}-t_0} = \frac{2d_2}{t_2-t_1} - \frac{1-2d_1}{t_1-t_0} \quad (57)$$

We consider a criterion for deciding any three parameters out of d_{-2}, d_{-1}, d_1 and d_2 in the next section.

3.2 Criterion for deciding Compactly Supported Non-Uniform Fluency DA Function of Degree 2

We consider how is the compactly supported non-uniform Fluency DA function of degree 2 $s(t)$ obtained in the section 3.1, in case that the sampling interval is constant, that is, $h = 1$.

As is understood from Eq.(4), the quadratic uniform Fluency DA function has the property of $\frac{d}{dt}\,{}_{[DA]}^{\;\;3}\psi_0(0) = 0$. So, in deciding the function $s(t)$, the property of $s'(t_0) = 0$ is also used. By applying the property and $t_0 = 0$ to Eq.(19), the following relation

$$b_4 = b_5 = 0 \quad (58)$$

is obtained. Moreover, $t_{k+1} - t_k = 1$ for arbitrary integer k holds good. From Eqs.(52) and (53), the following relations

$$\begin{aligned} d_{-2} &= \tfrac{1}{2} - d_{-1}, \\ d_2 &= \tfrac{1}{2} - d_1, \end{aligned} \quad (59)$$

are derived. Moreover, by using the property that the quadratic uniform Fluency DA function are symmetry, that is, ${}_{[DA]}^{\;\;3}\psi_0(-t) = {}_{[DA]}^{\;\;3}\psi_0(t)$, the relations $d_{-2} = d_2$ and $d_{-1} = d_1$ hold good. When we put $d_{-1} = d_1 \triangleq d$, the following relation

$$d_{-2} = d_2 = \tfrac{1}{2} - d. \quad (60)$$

is derived. As the results, the quadratic uniform Fluency DA function $s(t)$ is represented as follows:

$$\begin{aligned}
&s_1(t) = 2(1-2d)(t+2)^2, \quad -2 \le t \le \tfrac{-3}{2}, \\
&s_2(t) = 2(2d-1)(3t^2+8t+5), \quad \tfrac{-3}{2} \le t \le -1, \\
&s_3(t) = 4\left\{(2-3d)t^2 + (3-4d)t - (d-1)\right\}, \\
&\qquad -1 \le t \le \tfrac{-1}{2}, \\
&s_4(t) = 4(d-1)t^2 + 1, \quad \tfrac{-1}{2} \le t \le 0, \\
&s_5(t) = 4(d-1)t^2 + 1, \quad 0 \le t \le \tfrac{1}{2}, \\
&s_6(t) = 4\left\{(2-3d)t^2 - (3-4d)t - (d-1)\right\}, \\
&\qquad \tfrac{1}{2} \le t \le 1, \\
&s_7(t) = 2(2d-1)(3t^2-8t+5), \quad 1 \le t \le \tfrac{3}{2}, \\
&s_8(t) = 2(1-2d)(t-2)^2, \quad \tfrac{3}{2} \le t \le 2, \\
&0, \quad \textit{otherwise.}
\end{aligned} \quad (61)$$

[Proposition 1] Under the condition of $t_k = k, (k = 0, \pm 1, \pm 2, \ldots)$, the quadratic non-uniform Fluency DA function $s(t)$ is identical with ${}_{[DA]}^{\;\;3}\psi_0(t)$ in the criterion of $\int_{-\infty}^{\infty} \{s'(t)\}^2 dt \to \min$.

(Proof) By substituting $s(t)$ of Eq.(61) for the relation $\int_{-\infty}^{\infty} \{s'(t)\}^2 dt$, the following relation

$$\begin{aligned}
&\int_{-\infty}^{\infty} \{s'(t)\}^2 dt \\
&= \int_{-2}^{-3/2} \left\{s_1'(t)\right\}^2 dt + \int_{-3/2}^{-1} \left\{s_2'(t)\right\}^2 dt + \int_{-1}^{-1/2} \left\{s_3'(t)\right\}^2 dt \\
&+ \int_{-1/2}^{0} \left\{s_4'(t)\right\}^2 dt + \int_{0}^{1/2} \left\{s_5'(t)\right\}^2 dt + \int_{1/2}^{1} \left\{s_6'(t)\right\}^2 dt \\
&+ \int_{1}^{3/2} \left\{s_7'(t)\right\}^2 dt + \int_{3/2}^{2} \left\{s_8'(t)\right\}^2 dt \\
&= \tfrac{64}{3} d^2 - 24d + 10 \\
&= \tfrac{64}{3}\left(d - \tfrac{9}{16}\right)^2 + \tfrac{13}{4}
\end{aligned} \quad (62)$$

is obtained. As the results, the parameter 'd' which minimizes Eq.(62) is obtained as $d = \frac{9}{16}$. When we substitute $d = \frac{9}{16}$ for 'd' of Eq.(61), the uniform Fluency DA function of degree 2 $s(t)$ is identical with ${}_{[DA]}^{\;\;3}\psi_0(t)$ described in subsection 2.2.

(Q.E.D.)

As is described in section 2.2, the compactly supported uniform Fluency DA function of degree 2 is useful for generating analog signals as continuous signals from digital signals as discrete signals.

We use the criterion of $\int_{-\infty}^{\infty}\{s'(t)\}^2 dt \to \min.$ to design the compactly supported non-uniform Fluency DA function of degree 2 in this paper.

4 Design of Compactly Supported Non-Uniform Fluency DA Function of Degree 2 based on Geometric Criterion of Waveform

In the previous section, the criterion for designing non-uniform Fluency DA function of degree2 is discussed. We use the criterion of $\int_{-\infty}^{\infty}\{s'(t)\}^2 dt \to \min.$ to decide any three parameters out of d_{-2}, d_{-1}, d_1 and d_2.

4.1 Non-Uniform Fluency DA Function of Degree 2 with $s'(t_0)=0$

The non-uniform Fluency DA function of degree 2 is formulated by $s(t)$ in section 3.1. The parameters d_{-2}, d_{-1}, d_1 and d_2 are not fixed. So, taking account of the condition $s'(t_0)=0$, the following relations are obtained from Eq.(57).

$$
\begin{aligned}
d_{-2} &= \tfrac{t_{-1}-t_{-2}}{t_0-t_{-1}}\left(\tfrac{1}{2}-d_{-1}\right),\\
d_2 &= \tfrac{t_1-t_2}{t_0-t_1}\left(\tfrac{1}{2}-d_1\right).
\end{aligned}
\qquad (63)
$$

The non-uniform Fluency DA function of degree 2 $s(t)$ is reduced by substituting d_{-2} and d_2 of Eq.(63) for Eqs.(49)-(56) as follows.

$$
\begin{aligned}
&s_1(t)=\tfrac{2(1-2d_{-1})}{(t_0-t_{-1})(t_{-1}-t_{-2})}(t-t_{-2})^2, [t_{-2}, \tfrac{t_{-2}+t_{-1}}{2}]\\
&s_2(t)=\tfrac{-2(1-2d_{-1})}{(t_0-t_{-1})(t_{-1}-t_{-2})}(3t-t_{-1}-2t_{-2})(t-t_{-1}),\\
&\qquad [\tfrac{t_{-2}+t_{-1}}{2}, t_{-1}]\\
&s_3(t)=\tfrac{4}{(t_0-t_{-1})^2}[(-3d_{-1}+2)t^2+\{2(2t_{-1}+t_0)d_{-1}-\\
&\qquad (3t_{-1}+t_0)\}t-2t_{-1}t_0d_{-1}-t_{-1}^2(d_{-1}-1)+t_{-1}t_0]\\
&\qquad [t_{-1}, \tfrac{t_{-1}+t_0}{2}]\\
&s_4(t)=\tfrac{4(-1+d_{-1})}{(t_0-t_{-1})^2}(t-t_0)^2+1, [\tfrac{t_{-1}+t_0}{2}, t_0]\\
&s_5(t)=\tfrac{4(-1+d_1)}{(t_0-t_{-1})^2}(t-t_0)^2+1, [t_0, \tfrac{t_0+t_1}{2}]\\
&s_6(t)=\tfrac{4}{(t_1-t_0)^2}[(-3d_1+2)t^2+\{2(2t_1+t_0)d_1-\\
&\qquad (3t_1+t_0)\}t-2t_0t_1d_1-t_1^2(d_1-1)+t_0t_1]\\
&\qquad [\tfrac{t_0+t_1}{2}, t_1]\\
&s_7(t)=\tfrac{-2(1-2d_1)}{(t_2-t_{11})(t_1-t_0)}(3t-t_1-2t_2)(t-t_1),\\
&\qquad [t_1, \tfrac{t_1+t_2}{2}]\\
&s_8(t)=\tfrac{2(1-2d_1)}{(t_2-t_1)(t_1-t_0)}(t-t_2)^2, [\tfrac{t_1+t_2}{2}, t_2]\\
&0, \text{otherwise.}
\end{aligned}
\qquad (64)
$$

As is understood from Eq.(64), the 2 parameters d_{-1} and d_1 are not fixed.

4.2 Non-Uniform Fluency DA Function of Degree 2 with $s'(t_0)=0$ and $\int_{-\infty}^{\infty}\{s'(t)\}^2 dt \to \min.$

The non-uniform Fluency DA function of degree 2 is decided by using the criterion of

$$\int_{-\infty}^{\infty}\{s'(t)\}^2 dt \to \min.$$

By substituting Eq.(64) for $\int_{-\infty}^{\infty}\{s'(t)\}^2 dt$, we get

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\{s'(t)\}^2 dt=\int_{-t_{-2}}^{t_2}\{s'(t)\}^2 dt\\
&=\tfrac{8}{3(t_0-t_{-1})^2}\left\{4(t_0-t_{-2})d_{-1}^2+(4t_{-2}+t_{-1}-5t_0)d_{-1}-t_{-2}-t_{-1}+2t_0\right\}\\
&+\tfrac{8}{3(t_1-t_0)^2}\left\{4(t_2-t_0)d_1^2-(4t_2-t_1+5t_0)d_{-1}+t_2+t_1-2t_0\right\}.
\end{aligned}
\qquad (65)
$$

Based on the criterion of $\int_{-\infty}^{\infty}\{s'(t)\}^2 dt \to \min.$,the parameters d_{-1} and d_1 are fixed as follows:

$$d_{-1} = \frac{-4t_{-2}-t_{-1}+5t_0}{8(t_0-t_{-2})},$$
$$d_1 = \frac{4t_2+t_1-5t_0}{8(t_2-t_0)}. \qquad (66)$$

Based on the geometric criterion of waveform, all of the four parameters d_{-2}, d_{-1}, d_1 and d_2 are fixed from Eqs.(63) and (66), Therefore, the compactly supported non-uniform Fluency DA functions of degree 2 $s(t)$ is derived as follows:

$$s_1(t) = \frac{-1}{2(t_0-t_{-2})(t_{-1}-t_{-2})}(t-t_{-2})^2, [t_{-2}, \tfrac{t_{-2}+t_{-1}}{2}]$$
$$s_2(t) = \frac{1}{2(t_0-t_{-2})(t_{-1}-t_{-2})}(3t-t_{-1}-2t_{-2})(t-t_{-1}), \quad [\tfrac{t_{-2}+t_{-1}}{2}, t_{-1}]$$
$$s_3(t) = \frac{t_{-1}-t}{2(t_0-t_{-1})^2(t_0-t_{-2})}\{t_{-1}^2 - 4t_{-2}(t_{-1}-t) - t_0(t+2t_0) + t_{-1}(5t_0-3t)\}, \quad [\tfrac{t_{-1}+t_0}{2}, t_0]$$
$$s_4(t) = \frac{4t_{-2}-t_{-1}-3t_0}{2(t_0-t_{-1})^2(t_0-t_{-2})}(t-t_0)^2 + 1, [\tfrac{t_{-1}+t_0}{2}, t_0]$$
$$s_5(t) = \frac{-4t_2+t_1+3t_0}{2(t_1-t_0)^2(t_2-t_0)}(t-t_0)^2 + 1, [t_0, \tfrac{t_0+t_1}{2}]$$
$$s_6(t) = \frac{t_1-t}{2(t_1-t_0)^2(t_2-t_0)}\{t_1^2 - 4t_2(t_1-t) - t_0(t+2t_0) + t_1(5t_0-3t)\}, \quad [\tfrac{t_0+t_1}{2}, t_1]$$
$$s_7(t) = \frac{1}{2(t_2-t_0)(t_2-t_1)}(3t-t_1-2t_2)(t-t_1), \quad [t_1 \tfrac{t_1+t_2}{2}]$$
$$s_8(t) = \frac{-1}{2(t_2-t_0)(t_2-t_1)}(t-t_2)^2, [\tfrac{t_1+t_2}{2}, t_2],$$
$$0, \quad \text{otherwise.} \qquad (67)$$

Figures 3 demonstrates the non-uniform Fluency DA Function of degree 2 $s(t)$ for sampling points $t_{-2} = -5, t_{-1} = -2, t_0 = 0, t_1 = 4,$.

Since the non-uniform Fluency DA function of degree 2 $s(t)$ at $t = t_0$ is obtained in Eq.(67), the interpolation of any signal u with non-uniform sampling points $\{t_k\}_{k=-\infty}^{\infty}$ in $D_3{}'$ is also formulated.

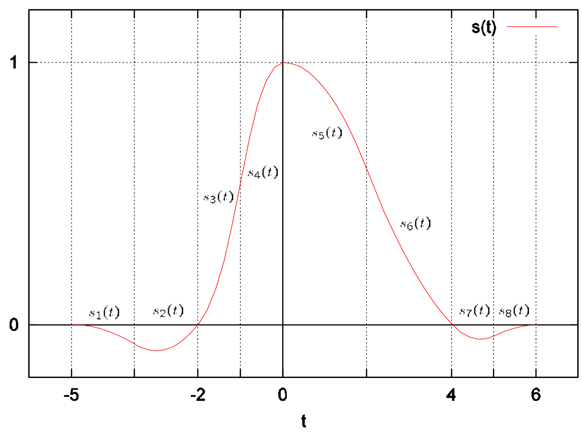

Fig.3 An example of the non-uniform fluency DA function s(t)

[Proposition 2] Let ${}_{[DA]}s_k(t)$ denotes the non-uniform Fluency DA function of degree 2 at each non-uniform sampling point $t = t_k, (k = 0, \pm 1, \ldots)$.

Then any signal u in $D_3{}'$ is expressed as

$$u \in D'_3, u(t) = \sum_{k=-\infty}^{\infty} u(t_k) \cdot {}_{[DA]}s_k(t),$$

where each Fluency DA function ${}_{[DA]}s_k(t)$ is expressed by eight quadratic piecewise polynomials $\{s_{k,l}(t)\}_{l=1}^{8}$ as follows.

$$s_{k,1}(t) = \frac{-1}{2(t_k-t_{k-2})(t_{k-1}-t_{k-2})}(t-t_{k-2})^2, [t_{k-2}, \tfrac{t_{k-2}+t_{k-1}}{2}]$$
$$s_{k,2}(t) = \frac{1}{2(t_k-t_{k-2})(t_{k-1}-t_{k-2})}(3t-t_{k-1}-2t_{k-2})(t-t_{k-1}), \quad [\tfrac{t_{k-2}+t_{k-1}}{2}, t_{k-1}]$$
$$s_{k,3}(t) = \frac{t_{k-1}-t}{2(t_k-t_{k-1})^2(t_k-t_{k-2})}\{t_{k-1}^2 - 4t_{k-2}(t_{k-1}-t) - t_k(t+2t_k) + t_{k-1}(5t_k-3t)\}, \quad [\tfrac{t_{k-1}+t_k}{2}, t_k]$$
$$s_{k,4}(t) = \frac{4t_{k-2}-t_{k-1}-3t_k}{2(t_k-t_{k-1})^2(t_k-t_{k-2})}(t-t_k)^2 + 1, [\tfrac{t_{k-1}+t_k}{2}, t_k]$$
$$s_{k,5}(t) = \frac{-4t_{k+2}+t_{k+1}+3t_k}{2(t_{k+1}-t_k)^2(t_{k+2}-t_k)}(t-t_k)^2 + 1, [t_k, \tfrac{t_k+t_{k+1}}{2}]$$

$$s_{k,6}(t) = \frac{t_{k+1}-t}{2(t_{k+1}-t_k)^2(t_{k+2}-t_k)}\{t_{k+1}^2 - 4t_{k+2}(t_{k+1}-t) - t_k(t+2t_k) + t_{k+1}(5t_k-3t)\}, \quad [\tfrac{t_k+t_{k+1}}{2}, t_{k+1}]$$
$$s_{k,7}(t) = \frac{1}{2(t_{k+2}-t_k)(t_{k+2}-t_{k+1})}(3t-t_{k+1}-2t_{k+2})(t-t_{k+1}), \quad [t_{k+1} \tfrac{t_{k+1}+t_{k+2}}{2}]$$
$$s_{k,8}(t) = \frac{-1}{2(t_{k+2}-t_k)(t_{k+2}-t_{k+1})}(t-t_{k+2})^2, [\tfrac{t_{k+1}+t_{k+2}}{2}, t_{k+2}]$$
$$0, \quad \text{otherwise.}$$

By using Proposition 2, any signal u with non-uniform sampling points $\{t_k\}_{k=-\infty}^{\infty}$ in $D_3{}'$ cab be interpolated.

Figures 4 demonstrate an interpolation by using Proposition 2.

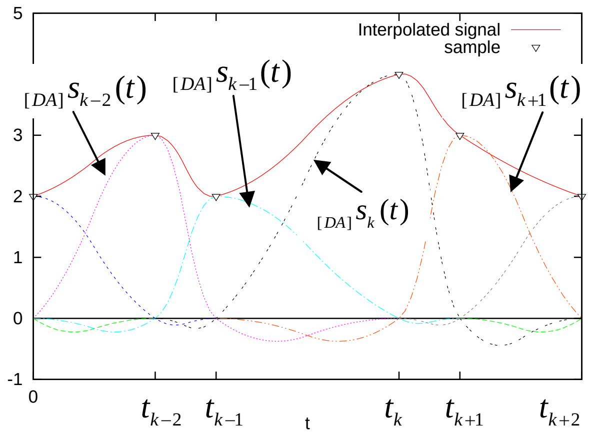

Fig. 4 Interpolation by using the non-uniform fluency DA function

5 Discussions

This section discusses interpolation results by using the non-uniform Fluency DA function which is derived in subsection 3.2. In order to evaluate their effectiveness, interpolation results by using cubic splines are also demonstrated.

In the field of computer graphics, cubic splines have been often used to interpolate a sequence of sampled-values with non-uniform sampling points. This is because a cubic spline $u(t)$ interpolates the sampled-values with the most smooth in the sense of

$$\int_{-\infty}^{\infty} \frac{\{u''(t)\}^2}{\left[1+\{u'(t)\}^2\right]^{5/2}} dt \le \int_{-\infty}^{\infty} \{u''(t)\}^2 dt \to \min. \qquad (68)$$

The curve smoothness $\int_{-\infty}^{\infty} \{u''(t)\}^2 dt$ in Eq.(68) is an upper bound of square of curve curvature .

Figures 5 and 6 demonstrate interpolation results for a kind of step function and for a typical case, respectively. Interpolated curves by using the non-uniform Fluency DA functions are drawn with solid lines, and then those by using cubic splines are drawn with dotted lines.

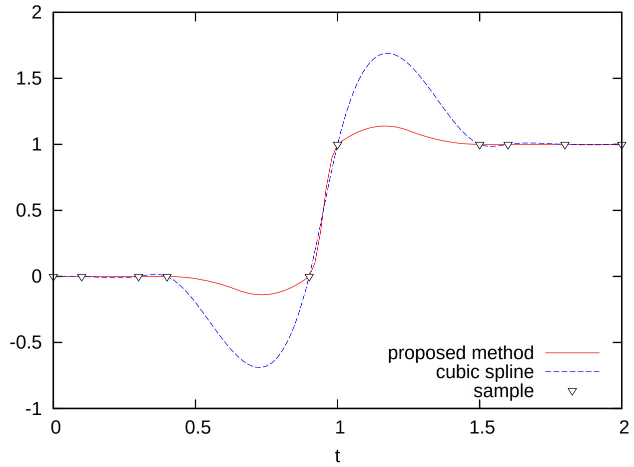

Fig. 5 Interpolation results for step function

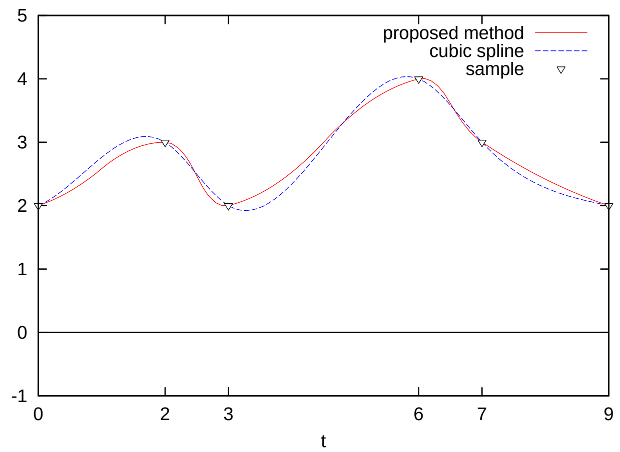

Fig. 6 Interpolation results for a typical case

As are understood from these results, we draw interpolated curves with less overshoot or undershoot.

Furthermore, we evaluate the interpolated results from the view of curve length. For fig. 5, the ratio of curve length by cubic splines to curve length by proposed method is approximately 1.59. Furthermore, for fig. 6, the ratio is approximately 1.03. From these results, interpolated curves by non-uniform fluency DA functions pass through sampled-values shorter than those by cubic splines.

6 Conclusions

This paper proposed fluency DA functions as sampling functions for non-uniform sampling points, each of which is composed with piecewise polynomials of degree 2. Each of them was designed based on geometric criterion of curve. The *fluency DA functions* of degree 2 generate smooth and undulate signals from a sequence of sampled-values.

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