NIRE: A New Approach in Obtaining Transfer Functions for a Linear Network

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Abstract: – Nowadays, there are lots of classic and alternative method that has been used to obtain and compute transfer function for a linear network. Although, problems still occur especially in the number of nodes, loops and simultaneous equations that need to be solved by this approaches. A proposed approach called NIRE (Network-Impedance Relationship Equation) will solve these problems by developing fixed equations for obtaining transfer function, rather than to extract its nodes, loops or simultaneous equations of its network. In this approach, the only required parameters are the network's impedance connections which will be taken from two sources which are from the input terminal and output impedance. The approach has been developed in an algorithm using software called MATLAB. It can be adapted in a computer-based application and also can be introduced as an alternative method in obtaining transfer function for educational purpose.

Key-Words: – Transfer function, impedance, circuit, large-scale, linear, symbolic circuit analysis.

1 Introduction

Transfer function is a mathematical representation, in terms of spatial or temporal frequency, of the relation between the input and output of a linear, time-invariant (LTI) system. It is commonly used in the analysis of single-input single-output electronic network system where it is mainly used in signal processing, communication theory, and control theory. In its simplest form for continuous-time input signal *x*(*t*) and output *y*(*t*), the transfer function is the linear mapping of the Laplace transform of the input, *X*(*s*), to the output *Y*(*s*):

$$H(s) = \frac{Y(s)}{X(s)} \quad ...(1)$$

For a single-input single-output network, the input and output can be either in a voltage or current expression. From this combination of output over input ratio, there will be four possibility of transfer function that is [2]:

$$\text{Voltage Gain (VG)} = \frac{V_0(s)}{V_i(s)} \quad ...(2)$$

$$\text{Current Gain (IG)} = \frac{I_0(s)}{I_i(s)} \quad ...(3)$$

$$\text{Transfer Impedance (TZ)} = \frac{V_0(s)}{I_i(s)} \quad ...(4)$$

$$\text{Transfer Admittance (TY)} = \frac{I_0(s)}{V_i(s)} \quad ...(5)$$

To obtain these transfer functions, the most common and classical method used are nodal and mesh analysis. There are also some other alternative methods such as Modified Nodal Analysis [9], and Extra Element Theorem which has been extended with two and more extra element [6][7][8]. Although these alternative methods has reduced some problem regarding to the nodes and loops of the classical method, but still there are equations to be extracted from the network, which will at last be solved by simultaneous equations, which will be a complex tedious work for a large-scale network.

Fortunately, the simultaneous equations can be solved easily by using matrix conversion. Still, the number of simultaneous equations remains sufficiently large. The only solution for this problem is to remove some unnecessary or redundant components from the network, which will result in approximate analysis, not the actual, as stated in [10]. Oppositely, the transfer function obtained by using NIRE approach will results in the actual function as the entire network impedances will be taken into consideration. But still, the idea of removing unnecessary or redundant component from the network is brilliant as it can be a major advantage in a large-scale network. An interesting research, where the effect of removing those components results in an error-free analysis, would be the solution for the previous problems, thus, resulting in the simplification of circuit models [11].

A new alternative method in obtaining transfer function is introduced and named as NIRE approach. In this approach, only two equations will be extracted from the network. The first one is the total impedance equivalent equation taken from the input terminal (Z_T), and the second one is the impedances involved at output terminal (Z_o). Figure 1 shows where Z_T and Z_o are taken from.

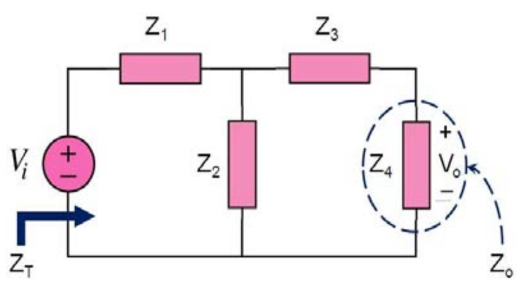

Figure 1

After that, these two equations will be manipulated by the NIRE technique to form any of the desired transfer function. As NIRE approach only uses network's impedances, none of the nodes or loops equations will be extracted from the network, thus, eliminating all problems regarding to nodes and loops equations. For the simultaneous equations problem, the NIRE technique will handle this situation by fixing the number of equations needs to be solved in order to obtain the transfer function required. As the matter of fact, NIRE technique will only have the maximum of five equations that needs to be solved.

The NIRE approach along with its technique will surely be a handful tool in obtaining transfer functions for a large-scale network that consists of a large amount of components.

2 The NIRE Approach

The main developments in symbolic network analysis have been in the realm of the frequency domain [1] which is usually been expressed in the Laplace form. Symbolic analysis is a method to calculate the behavior or characteristic of a network represented by symbols. The method is complementary to numerical analysis where the variables and the circuit elements are represented by numbers [5]. The goal of such an analysis is the generation of a fully or partially symbolic transfer function that can be expressed as:

$$H(s,X) = \frac{N(s,X)}{D(s,X)} \quad ...(6)$$

where $N(s,X)$ and $D(s,X)$ are numerator and denominator polynomials in s-domain and the symbolic network variables X. Usually, the variable X is the impedance equivalent (Z-equivalent) of the components in the network. In other words, the symbolic analysis approach will express the transfer function in a symbolic Z-equivalent form in the s-domain representation.

However, in NIRE approach, the network components are transformed into its symbolic Z-equivalent only, not in its s-domain representation. The transformation of the symbolic Z-equivalent into its s-domain representation will be compute at the end of the NIRE technique process. The transformation from network's components into its symbolic Z-equivalent form is shown in Figure 2.

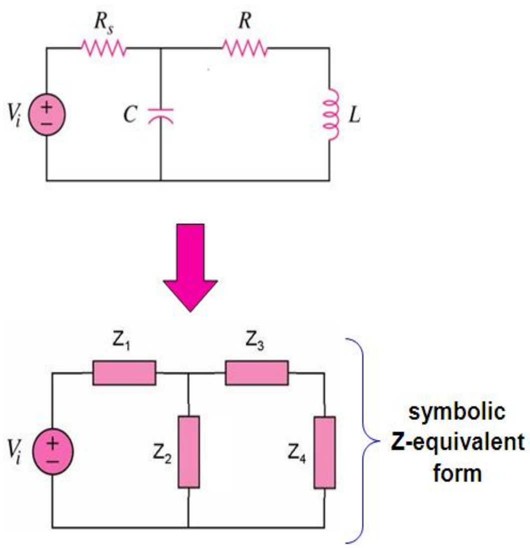

Figure 2

As discussed earlier, the two required parameters for NIRE approach are Z_T and Z_o, which can be expressed in their numerator and denominator form:

$$Z_T = \frac{N_T}{D_T} \quad ...(7)$$

$$Z_0 = \frac{N_0}{D_0} \quad ...(8)$$

where N_T and N_o are the numerator of total impedance and output impedance while D_T and D_o are the denominator of total impedance and output impedance.

Additionally, in NIRE approach, all symbolic equations and terms must be in the form of 'sum of product of the impedances', where product of the impedances is called 'element' afterwards. Figure 3 shows the correct way to express the symbolic impedance form.

$$\left.\underbrace{Z_1Z_2}_{\text{element}} + \underbrace{Z_2Z_3}_{\text{element}} + \underbrace{Z_1^2}_{\text{element}}\right\} \text{sum of product of impedances}$$

Figure 3

Since the transfer function can be divided into four types as discussed earlier, each of the type has its own NIRE technique's equations or steps to be applied to it. Due to this, the desired transfer function's type needs to be defined first before applying the NIRE technique. Those transfer function types are expressed by their numerator and denominator form:

$$\mathrm{VG} = \frac{N_{VG}}{D_{VG}} \quad ...(9)$$

$$\mathrm{IG} = \frac{N_{IG}}{D_{IG}} \quad ...(10)$$

$$\mathrm{TZ} = \frac{N_{TZ}}{D_{TZ}} \quad ...(11)$$

$$\mathrm{TY} = \frac{N_{TY}}{D_{TY}} \quad ...(12)$$

After applying the NIRE technique, it will result in a symbolic Z-equivalent transfer function which can now be transformed into its s-domain representation and to be inserted with values (if any). The methodology of NIRE approach can be summarized as in the flowchart of Figure 4.

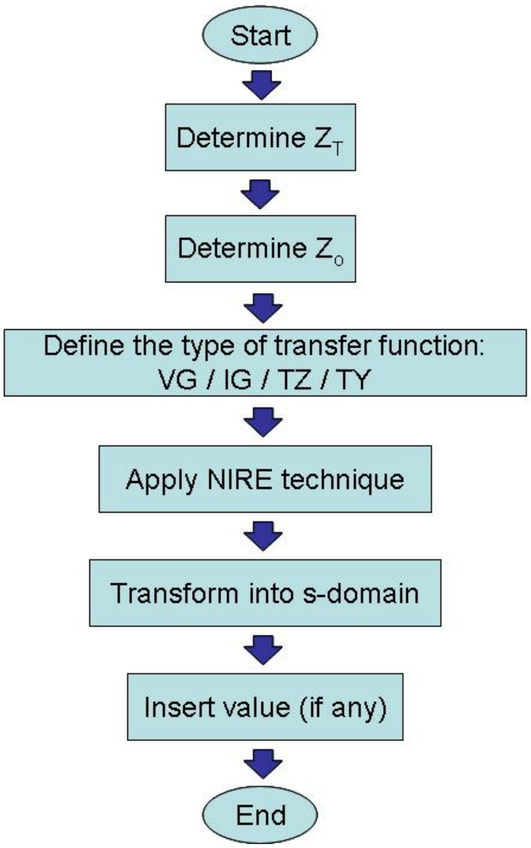

Figure 4

3 The NIRE Technique

The heart of the NIRE approach is its technique in developing relationship equations to suit any of the desired transfer function. NIRE technique will took the two required parameters which are Z_T and Z_o, and manipulate each of their numerator and denominator elements to form several relationship equations. Each equation is unique to each of the four types of transfer function. These equations will be developed gradually as they follow the fixed steps of NIRE technique.

The NIRE technique will only developed the maximum of five steps (equations) in obtaining transfer function. This means that only these steps will be taken regardless the size of the network. There will be no extra or additional steps or equations that have to be developed other than these relationship equations. This would be a major advantage if a transfer function from a large-scale network has to be obtained.

The five relationship equations that will be developed according to their steps are:

1. Denominator equation
2. Product equation
3. Sum of Matched Elements (SoME) equation
4. Re-multiplication equation (for TZ and TY only)
5. Combination equation

These five relationship equations will develop the transfer function's numerator and denominator parts. Figure 5 shows the relationship equations that have been grouped into their numerator or denominator part.

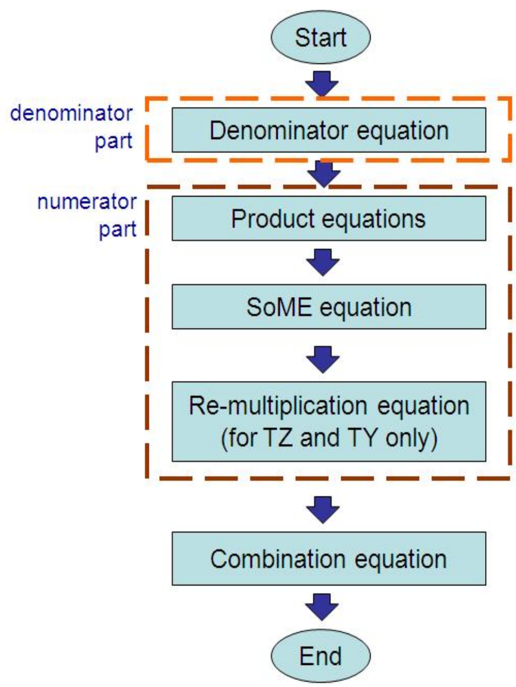

Figure 5

To ease the understanding of NIRE technique, an example as shown in Figure 6 will help to demonstrate each step taken and the relationship equations that will be developed.

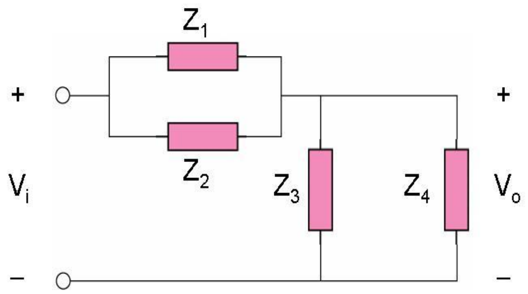

Figure 6

For the required parameters of Z_T and Z_o, the equations are:

$$Z_T = \frac{N_T}{D_T} = \frac{Z_1Z_2Z_4 + Z_1Z_2Z_3 + Z_3Z_4Z_2 + Z_3Z_4Z_1}{Z_2Z_4 + Z_2Z_3 + Z_1Z_4 + Z_1Z_3} \quad ...(13)$$

$$Z_o = \frac{N_o}{D_o} = Z_4 \quad ...(14)$$

Assume that the desired transfer function for this example is a voltage gain (VG).

3.1 Denominator equation

The first step in NIRE technique is the determination of the transfer function's denominator part. This first relationship equation is directly taken from the numerator or denominator part of Z_T.

$$D_{VG} = N_T \quad ...(15)$$
$$D_{IG} = D_T \quad ...(16)$$
$$D_{TZ} = D_T \quad ...(17)$$
$$D_{TY} = N_T \quad ...(18)$$

For the example in Figure 6, the denominator of its voltage gain is the numerator of Z_T from equation (13):

$$D_{VG} = N_T = Z_1Z_2Z_4 + Z_1Z_2Z_3 + Z_3Z_4Z_2 + Z_3Z_4Z_1 \quad ...(19)$$

The steps afterwards are for finding the transfer function's numerator part.

3.2 Product equation

The second relationship equation is called product equation where the remaining parts of numerator or denominator from the first relationship equation are multiplied with the output impedance.

$$P_{VG} = D_T \times Z_0 \quad ...(20)$$
$$P_{IG} = N_T \times \frac{1}{Z_0} \quad ...(21)$$
$$P_{TZ} = N_T \times \frac{1}{Z_0} \quad ...(22)$$
$$P_{TY} = D_T \times Z_0 \quad ...(23)$$

For the example in Figure 6, the product equation is the multiplication of its total impedance's denominator (eq. 13) with output impedance (eq. 14).

$$P_{VG} = D_T \times \frac{N_0}{D_0} = \left(Z_2Z_4 + Z_2Z_3 + Z_1Z_4 + Z_1Z_3\right) \times Z_4 = Z_2Z_4^{\,2} + Z_2Z_3Z_4 + Z_1Z_4^{\,2} + Z_1Z_3Z_4 \quad ...(24)$$

3.3 SoME equation

The third relationship equation is called Sum of Matched Elements (SoME) equation. In this equation, every single element from the second relationship equation will be compared with elements from the first relationship equation that has been obtained. After that, the matched elements will be summed up.

Let defined the SoME equation in mathematical expression. First are the product equation, P and the denominator equation, D that will be compared. These equations can be written as:

$$P = \sum_{i=1}^{j} p_i \quad ...(25)$$

$$D = \sum_{i=1}^{k} d_i \quad ...(26)$$

where p_i are product's elements and d_i are denominator's elements. Both of these elements can be grouped into sets, namely E_P and E_D.

$$E_P = \{p_1, p_2, \ldots, p_j\} \quad ...(27)$$

$$E_D = \{d_1, d_2, \ldots, d_k\} \quad ...(28)$$

The intersection of set E_P and set E_D results in another set that contains the matched elements of those two sets, namely M.

$$M = E_P \cap E_D = \{m_1, m_2, \ldots, m_n\} \quad ...(29)$$

A function that sums up all the elements in M is introduced and this function makes up the third relationship equation that can be expressed as:

$$f_{SoME}(P, D) = \sum_{i=1}^{n} m_i \quad ...(30)$$

For all the four possible transfer functions, each third relationship equation can be written as:

$$N_{VG} = f_{SoME}(P_{VG}, D_{VG}) \quad ...(31)$$

$$N_{IG} = f_{SoME}(P_{IG}, D_{IG}) \quad ...(32)$$

$$N_{TZ} = f_{SoME}(P_{TZ}, D_{TZ}) \quad ...(33)$$

$$N_{TY} = f_{SoME}(P_{TY}, D_{TY}) \quad ...(34)$$

As for the example in Figure 6, the elements in product equation (eq. 24) and denominator equation (eq. 19) will compared, matched and summed by the SoME equation.

$$\begin{aligned} N_{VG} &= f_{SoME}(P_{VG}, D_{VG}) \\ &= f_{SoME}([\mathrm{Z}_2\mathrm{Z}_4{}^2, \mathrm{Z}_2\mathrm{Z}_3\mathrm{Z}_4, \mathrm{Z}_1\mathrm{Z}_4{}^2, \mathrm{Z}_1\mathrm{Z}_3\mathrm{Z}_4], \\ &[\mathrm{Z}_1\mathrm{Z}_2\mathrm{Z}_4, \mathrm{Z}_1\mathrm{Z}_2\mathrm{Z}_3, \mathrm{Z}_3\mathrm{Z}_4\mathrm{Z}_2, \mathrm{Z}_3\mathrm{Z}_4\mathrm{Z}_1]) \\ &= \mathrm{Z}_3\mathrm{Z}_4\mathrm{Z}_2 + \mathrm{Z}_3\mathrm{Z}_4\mathrm{Z}_1 \quad ...(35) \end{aligned}$$

3.4 Re-multiplication equation

The fourth relationship equation in the NIRE technique steps is called re-multiplication equation. This step will only be taken if the desired transfer function's type is either transfer impedance (TZ) or transfer admittance (TY). For voltage gain (VG) or current gain (IG), this step will be skipped. This fourth relationship equation can be written as:

$$N_{TZ} = N_{TZ} \times Z_0 \quad ...(36)$$

$$N_{TY} = N_{TY} \times \frac{1}{Z_0} \quad ...(37)$$

As the desired transfer function of the example in Figure 6 is a voltage gain, this re-multiplication equation will be skipped and the process continues to the next NIRE's relationship equation.

3.5 Combination equation

Lastly, for the fifth step, the denominator result obtained from the first relationship equation is combined with the numerator result obtained from the fourth relationship equation to form the complete desired transfer function as in equation (9), (10), (11) and (12). This relationship equation is called combination equation.

For the example in Figure 6, its combination equation is developed from equation (35) and equation (19).

$$\frac{V_0}{V_i} = \frac{N_{VG}}{D_{VG}}$$
$$= \frac{Z_3 Z_4 Z_2 + Z_3 Z_4 Z_1}{Z_1 Z_2 Z_4 + Z_1 Z_2 Z_3 + Z_3 Z_4 Z_2 + Z_3 Z_4 Z_1} \quad ...(38)$$

This transfer function will be transform into its s-domain representation and to be inserted with values. If the original network of the example is as shown in Figure 7, the voltage gain of the network in s-domain will be as in equation (39).

$$\frac{V_0(s)}{V_i(s)} = \frac{1}{s+1} \quad ...(39)$$

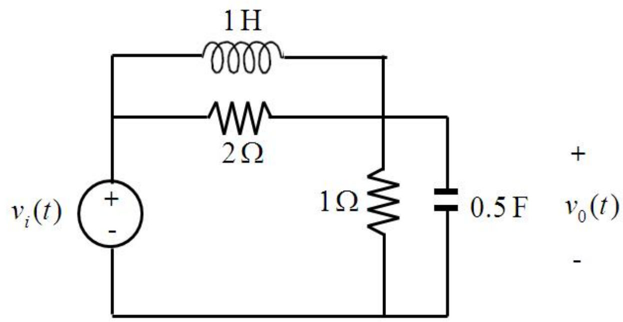

Figure 7

All the steps taken in NIRE technique can be summarized as shown in Table 1.

Table 1: The NIRE technique steps

Steps	Equation	Relationship Equation			
		Voltage Gain (VG)	Current Gain (IG)	Transfer Impedance (TZ)	Transfer Admittance (TY)
1	Denominator	$D_{VG} = N_T$	$D_{IG} = D_T$	$D_{TZ} = D_T$	$D_{TY} = N_T$
2	Product	$P_{VG} = D_T \times Z_0$	$P_{IG} = N_T \times \frac{1}{Z_0}$	$P_{TZ} = N_T \times \frac{1}{Z_0}$	$P_{TY} = D_T \times Z_0$
3	SoME	$N_{VG} = f_{SoME}(P_{VG}, D_{VG})$	$N_{IG} = f_{SoME}(P_{IG}, D_{IG})$	$N_{TZ} = f_{SoME}(P_{TZ}, D_{TZ})$	$N_{TY} = f_{SoME}(P_{TY}, D_{TY})$
4	Re-multiplication	-	-	$N_{TZ} = N_{TZ} \times Z_0$	$N_{TY} = N_{TY} \times \frac{1}{Z_0}$
5	Combination	$VG = \frac{N_{VG}}{D_{VG}}$	$IG = \frac{N_{IG}}{D_{IG}}$	$TZ = \frac{N_{TZ}}{D_{TZ}}$	$TY = \frac{N_{TY}}{D_{TY}}$

3 Results & Discussion

As NIRE approach do not simplified or modified a circuit network, the transfer function obtained will be exactly the same as the one obtained from any classical analysis method such as Nodal Analysis approach. The differences between these two approaches are the numbers of equations need to be solved in order to obtain the desired transfer function. NIRE approach has completely eliminates all nodes equations, loops equations and simultaneous equations that need to be solved by replacing them with a fixed number of relationship equations.

Let took an *n*-th order resistive ladder network for an example. As the order becomes larger, the number of resistors involved in the network will double themselves. Figure 8 shows some resistive ladder network as it becomes larger and the location of input and output terminal for its voltage gain determination.

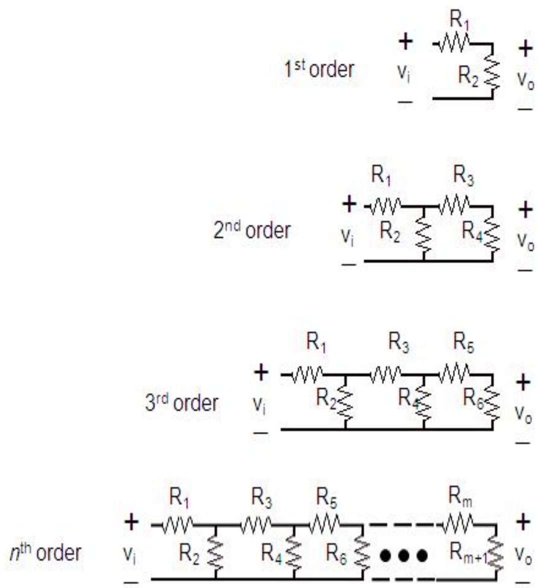

Figure 8

If 100th order is selected and Nodal Analysis approach is used to determine the voltage gain of the network, the number of developed equations due to voltage nodes is 100 and the number of simultaneous equations needed to be solved is 100. It is clearly that the number of equations would be twice the number of its order. If NIRE approach was used, the number of developed equations that needed to be solved is only 6, which are:

1. Z_T determination
2. Z_o determination
3. Denominator equation
4. Product equation
5. SoME equation
6. Combination equation

relationship equations (3–6)

The developed equations in NIRE approach will remains 6 regardless the number of the network's order. These statistics can be illustrated into graph as in Figure 9.

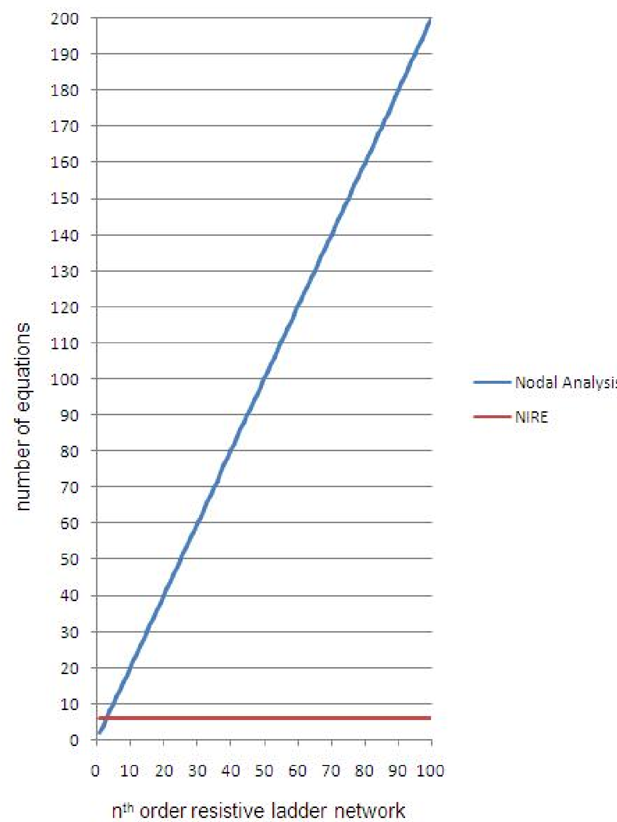

Figure 9

Although NIRE approach has an advantage for a large-scale network, it has a disadvantage where the equation's complexity increased as the network becomes larger. This occurs because all the components are transformed into its Z-equivalent form. If a network has 100 components in it, which means there are 100 impedances involved in Z_T and Z_o determination, denominator equation, product equation, SoME equation and combination equation. This would be a very complex and tedious work to be done especially if the equations are solved by manual or hand calculation. However, this problem can be overcome by implementing the NIRE approach in a computer-based software such as MATLAB. The manipulation of N_T, D_T and Z_o will also be easily done as MATLAB has a simplification command *numden* in its symbolic math toolbox to extract the numerator and denominator part of Z_T.

4 Conclusion

As a conclusion, NIRE approach can be a good alternative method in obtaining transfer function for a large-scale network. The developed equations that need to be solved have been fixed regardless the size of the network. Furthermore, the component's value would be inserted at the end of the process resulting in an error-free analysis method for obtaining the actual transfer function. Unfortunately, the developed equations in NIRE approach will become complex when the number of components involved becomes larger. However, this problem can be overcome by implementing the NIRE approach, technique and relationship equations as an algorithm to by execute by a computer-based calculation program.

This new approach can also be a good circuit analysis tools where voltage or current value for a certain component in a network can be determined easily as Z_T has been determine earlier and only Z_o are changeable to satisfy the analysis requirement. As transfer function is quite similar to two-port network's parameters, NIRE approach can also be adapted in two-port network method. A research in two-port network system [12] has an interesting result as it claims that by using a hierarchical symbolic analysis, the run time of a computational program would be faster as the network is transformed into several block and interconnected with each other. This method will be implemented in NIRE approach as for future development. A network's simplification model would also be a good future development as discussed in Section 1.

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