Identification of a continuous linear time-varying system using Haar wavelet with unit energy

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Abstract: - In this paper, identification of a continuous-time linear time-varying (LTV) system is proposed, where Haar wavelet with unit energy is employed. For that purpose, an algebraic equation is derived by expanding the input-output data and the time-varying impulse response using normalized Haar wavelets. Unknown wavelet coefficients for the a LTV system's impulse response can be effectively estimated by solving the algebraic equation. Finally, the time-varying impulse response of a LTV system can be synthesized from the estimated wavelet coefficients.

Key-Words: Linear time-varying system, impulse response, identification, Haar wavelet

1 Introduction

The wavelet transform has been effectively applied to many fields such as neural networks, communication, and image processing [1]-[3]. More specipically, system identification utilizing the wavelet transforms has received attention in control engineering and signal processing fields. In particular, wavelet-based approaches for identification of linear time-varying (LTV) systems have been addressed in a continuous-time domain [4]. For example, Daubechies wavelet was applied as an orthogonal basis. However, since no analytic expression exists for Daubechies wavelet, high computational burden is required for the system identification. On the other hand, system identification of a continuous-time LTV state-space model by Haar wavelet was reported, requiring less computational burden than that by Daubechies wavelet [5]. In particular, some properties of Haar wavelets were established [6] and utilized for state analysis and parameter estimation of bilinear systems [7].

In this paper, a new approach for an effective estimation of the impulse response of a continuous-time LTV system is proposed, requiring relatively low computational burden. More specifically, (i) an algebraic equation is firstly derived by expanding the input-output data and the time-varying impulse response using normalized Haar wavelets, then, (ii) unknown wavelet coefficients for the a LTV system's impulse response can be estimated by solving the algebraic equation, and finally, (iii) the time-varying impulse response of a LTV system can be synthesized from the estimated quanties. The proposed approach is different from conventional ones [5]-[7] in that a computationally efficient expression for multiplication of Haar wavelets is utilized by employing Haar wavelet with unit energy (i.e., normalized), which leads to efficient recursive identification of a linear time-varying system.

This paper is organized as follows: Firstly, basic properties of normalized Haar wavelet are considered in Section 2. In Section 3, the proposed approach for identification of a causal LTV system is described. Section 4 provides some simulation results, and, finally, the conclusion is drawn in Section 5. Also, two appendices are included for the proof of some equations utilized in the proposed approach.

2 Basic properties of Haar wavelet

2.1 General properties of Haar wavelet

Orthogonal basis functions including Haar wavelet has been utilized for the system identification [5]. In particular, the amplitude of the Haar wavelet is $\pm 2^j\ (j = 0,1,2,\cdots)$ in some finite intervals and zeros elsewhere (i.e., see (1)-(2)), leading to effective reduction of the calculation [7]. If the scaling function and the prototype Haar wavelet are denoted by $h_0(t)$ and $h_1(t)$, respectively, all other wavelet bases (i.e., $h_n(t)$) can be generated from dilations and translations of $h_1(t)$, and each base is normalized with unit energy [8]:

$$h_0(t)=1,\ 0\le t<1,\qquad h_1(t)=\begin{cases}1, & 0\le t<\frac{1}{2}\\ -1, & \frac{1}{2}\le t<1\end{cases}\quad (1)$$

$$h_n(t)=2^{j/2}h_1(2^j t-q),\quad n=2^j+q\ge 2,\quad j\ge 1,\quad 0\le q<2^j \quad (2)$$

Let's define $\mathbf{h}_{(m)}(t)$ as a group of the Haar wavelets:

$$\mathbf{h}_{(m)}(t)=[h_0(t),\ \ h_1(t),\ \ \cdots\ \ h_{m-1}(t)\ ]^{\mathrm{T}},\ m=2^{\mathrm{i}},\ \ \mathrm{i}\ge 0 \quad (3)$$

Also, a digital representation of $\mathbf{h}_{(m)}(t)$ is defined by

$$\mathbf{H}_{(m)}=\left[\mathbf{h}_{(m)}(\frac{1}{2m}),\ \ \mathbf{h}_{(m)}(\frac{3}{2m}),\ \ \cdots\ \ \mathbf{h}_{(m)}(\frac{2m-1}{2m})\right]\in R^{m\times m} \quad (4)$$

When m Haar bases are taken, the largest sampling time without aliasing is $1/m$ [5]. In general, a signal usually has some finite support, and thus, without loss of generality, the signal duration can be normalized as the time interval $t\in[0,1)$ as in [5]. Accordingly, any square-integrable function $y(t)$ in the interval $0\le t<1$ can be expressed by using the orthogonal bases $\{h_0(t), h_1(t), \ldots, h_n(t),\ n=1,2,\ldots,\infty\}$ [5]: i.e.,

$$y(t)=\sum_{n=0}^{\infty}c_n h_n(t),\qquad c_n=\int_0^1 y(t)h_n(t)dt \quad (5)$$

In practice, the approximation of $y(t)$ using only m Haar wavelets is as follows:

$$y(t)\cong\sum_{n=0}^{m-1}c_n h_n(t)=\mathbf{c}_{(m)}^{\mathrm{T}}\mathbf{h}_{(m)}(t),\ \mathrm{m}=2^{\mathrm{i}},\quad \mathrm{i}\ge 0 \quad (6)$$

where

$$\mathbf{c}_{(m)}=[c_0\quad c_1\quad\cdots\quad c_{m-1}]^{\mathrm{T}}$$

2.2 Multiplication of Haar wavelet

A recursive formula for the Haar product matrix can be expressed as in [7].

$$\mathbf{h}_{(m)}(t)\mathbf{h}_{(m)}^{\mathrm{T}}(t)\cong M_{(m)}(t),\qquad M_{(1)}(t)=h_0(t) \quad (7)$$

$$M_{(m)}(t)=\begin{bmatrix}M_{(\frac{m}{2})}(t) & \mathbf{H}_{(\frac{m}{2})}diag[\mathbf{h}_b(t)]\\ diag[\mathbf{h}_b(t)]\mathbf{H}_{(\frac{m}{2})}^{\mathrm{T}} & diag[\mathbf{H}_{(\frac{m}{2})}^{-1}\mathbf{h}_a(t)]\end{bmatrix} \quad (8)$$

$$\mathbf{h}_a(t)=\left[h_0(t),\quad h_1(t),\quad\cdots\quad h_{\frac{m}{2}-1}(t)\right]^{\mathrm{T}}\in R^{(\frac{m}{2})\times 1}$$

$$\mathbf{h}_b(t)=\left[h_{\frac{m}{2}}(t),\quad h_{\frac{m}{2}+1}(t),\quad\cdots\quad h_{m-1}(t)\right]^{\mathrm{T}}\in R^{(\frac{m}{2})\times 1}$$

When (7) is multiplied by a vector $\mathbf{c}_{(m)}\in R^{m\times 1}$, the following matrix $C_{(m)}\in R^{m\times m}$ can be obtained from the following recursive formula:

$$\mathbf{h}_{(m)}(t)\mathbf{h}_{(m)}^{\mathrm{T}}(t)\mathbf{c}_{(m)}=C_{(m)}\mathbf{h}_{(m)}(t) \quad (9)$$

$$C_{(m)}=\begin{bmatrix}C_{(\frac{m}{2})} & \mathbf{H}_{(\frac{m}{2})}diag[\mathbf{c}_b]\\ diag[\mathbf{c}_b]\mathbf{H}_{(\frac{m}{2})}^{-1} & diag[\mathbf{c}_a^{\mathrm{T}}\mathbf{H}_{(\frac{m}{2})}]\end{bmatrix} \quad (10)$$

$$C_{(1)}=c_0$$

$$\mathbf{c}_a=\left[c_0,\quad c_1,\quad\cdots\quad c_{\frac{m}{2}-1}\right]^{\mathrm{T}}\in R^{(\frac{m}{2})\times 1}$$

$$\mathbf{c}_b=\left[c_{\frac{m}{2}},\quad c_{\frac{m}{2}+1},\quad\cdots\quad c_{m-1}\right]^{\mathrm{T}}\in R^{(\frac{m}{2})\times 1}$$

In (7)-(10), the recursive formula is determined by not-normalized Haar wavelet [7]. The matrix $C_{(m)}$, related to the multiplication of two Haar wavelets, has an inverse term of $\mathbf{H}_{(m/2)}$ in (8) and (10). However, due to the normalized Haar wavelet, the inverse term $\mathbf{H}^{-1}{}_{(m/2)}$ in $C_{(m)}$ of (8) and (10) can be expressed by $\mathbf{H}^{\mathrm{T}}{}_{(m/2)}$ (See the Appendix A).

$$\mathbf{h}_{(m)}(t)\mathbf{h}_{(m)}^{\mathrm{T}}(t)\cong P_{(m)}(t),\qquad P_{(1)}(t)=h_0(t) \quad (11)$$

$$P_{(m)}(t)=\begin{bmatrix}P_{(\frac{m}{2})}(t) & \mathbf{H}_{(\frac{m}{2})}diag[\mathbf{h}_b(t)]\\ diag[\mathbf{h}_b(t)]\mathbf{H}_{(\frac{m}{2})}^{\mathrm{T}} & diag[\mathbf{H}_{(\frac{m}{2})}^{\mathrm{T}}\mathbf{h}_a(t)]\end{bmatrix} \quad (12)$$

$$\mathbf{h}_a(t)=\left[h_0(t),\quad h_1(t),\quad\cdots\quad h_{\frac{m}{2}-1}(t)\right]^{\mathrm{T}}\in R^{(\frac{m}{2})\times 1}$$

$$\mathbf{h}_b(t)=\left[h_{\frac{m}{2}}(t),\quad h_{\frac{m}{2}+1}(t),\quad\cdots\quad h_{m-1}(t)\right]^{\mathrm{T}}\in R^{(\frac{m}{2})\times 1}$$

Also, when the $P_{(m)}(t)$ in (11) is multiplied by a vector $\mathbf{c}_{(m)} \in R^{m\times 1}$, a matrix $T_{(m)} \in R^{m\times m}$ can be drived from the following recursive formula (See the Appendix B):

$$\mathbf{h}_{(m)}(t)\mathbf{h}^{\mathrm{T}}_{(m)}(t)\mathbf{c}_{(m)} = T_{(m)}\mathbf{h}_{(m)}(t) \tag{13}$$

$$T_{(m)} = \begin{bmatrix} T_{(\frac{m}{2})} & \mathbf{H}_{(\frac{m}{2})} diag[\mathbf{c}_b] \\ diag[\mathbf{c}_b]\mathbf{H}^{\mathrm{T}}_{(\frac{m}{2})} & diag[\mathbf{c}^{\mathrm{T}}_a \mathbf{H}_{(\frac{m}{2})}] \end{bmatrix} \tag{14}$$

$$T_{(1)} = c_0$$

$$\mathbf{c}_a = \begin{bmatrix} c_0, & c_1, & \cdots & c_{\frac{m}{2}-1} \end{bmatrix}^{\mathrm{T}} \in R^{(\frac{m}{2})\times 1}$$

$$\mathbf{c}_b = \begin{bmatrix} c_{\frac{m}{2}}, & c_{\frac{m}{2}+1}, & \cdots & c_{m-1} \end{bmatrix}^{\mathrm{T}} \in R^{(\frac{m}{2})\times 1}$$

Accordingly, $C_{(m)}$ can be obtained with lower computational burden. In particular, (13) and (14) are utilized in this paper for identification of the impulse response of an LTV system.

2 Identification of a causal LTV system

In the previous work, the identification of a linear autonomous system is performed by using a state space model [5]-[7]. However, in this paper, we consider the problem of identifying a causal LTV system expressed in an integral convolution form. Consider a continuous-time LTV system whose input- output relationship is given by

$$y(t) = \int_o^t h(t,\tau)x(\tau)d\tau, \quad t,\tau \in [0,1) \tag{15}$$

In (15), $x(\tau)$ and $h(t,\tau)$ denote the input and the impulse response of the LTV system [9]. Suppose that input and output data are given and the impulse response of a LTV system is unknown. Let the time t be fixed at an arbitrary time $t_k \in [0,1)$.

$$y(t_k) = \int_o^{t_k} h(t_k,\tau)x(\tau)d\tau \tag{16}$$

When $h(t,\tau)$ is projected onto m Haar bases, the impulse response of the LTV system can be expressed as follows [10]:

$$h(t_k,\tau) = \mathbf{a}^{\mathrm{T}}_{m,t_k}\mathbf{h}_{(m)}(\tau), \quad \mathbf{a}_{m,t_k} \in R^{m\times 1}, \ \mathbf{h}_{(m)}(\tau) \in R^{m\times 1} \tag{17}$$

In (17), $\mathbf{a}^{\mathrm{T}}_{m,t_k}$ and $\mathbf{h}_{(m)}(\tau)$ correspond respectively to Haar wavelet coefficients at $t_k \in [0,1)$ and Haar wavelet bases, which implies that $h(t,\tau)$ is not necessarily separable with respect to t and τ. In addition, the input signal $x(\tau)$ can be expanded in a similar way by Haar bases.

$$x(\tau) = \mathbf{b}^{\mathrm{T}}_{(m)}\mathbf{h}_{(m)}(\tau), \quad \mathbf{b}_{(m)} \in R^{m\times 1}, \quad \mathbf{h}_{(m)} \in R^{m\times 1} \tag{18}$$

Now, consider the problem of estimating the unknown impulse response $h(t_k,\tau)$. For that purpose, unknown coefficients $\mathbf{a}_{m,t_k}$ (or see (17)) should be estimated first. Also, the output $y(t_k)$ can be expressed from (16)-(18) as

$$y(t_k) = \int_0^{t_k} \mathbf{a}^{\mathrm{T}}_{m,t_k}\mathbf{h}_{(m)}(\tau)\, \mathbf{b}^{\mathrm{T}}_{(m)}\mathbf{h}_{(m)}(\tau)\; d\tau \tag{19}$$

Since $\mathbf{b}^{\mathrm{T}}_{(m)}\mathbf{h}_{(m)}(\tau)$ is a scalar, (19) can be written by

$$\begin{aligned} y(t_k) &= \int_0^{t_k} \mathbf{a}^{\mathrm{T}}_{m,t_k}\mathbf{h}_{(m)}(\tau)\mathbf{h}^{\mathrm{T}}_{(m)}(\tau)\mathbf{b}_{(m)}\; d\tau \\ &= \mathbf{a}^{\mathrm{T}}_{m,t_k} \int_0^{t_k} \mathbf{h}_{(m)}(\tau)\mathbf{h}^{\mathrm{T}}_{(m)}(\tau)\, \mathbf{b}_{(m)}\; d\tau \end{aligned} \tag{20}$$

Note that $\mathbf{h}_{(m)}(\tau)\mathbf{h}^{\mathrm{T}}_{(m)}(\tau)$ in (20) is a function of τ and can be described by Haar bases [6]. That is, from (13)-(14), there exists $\Theta_{(m)}$ ($\Theta_{(m)} \in R^{m\times m}$) satisfyin $\mathbf{h}_{(m)}(\tau)\mathbf{h}^{\mathrm{T}}_{(m)}(\tau)\, \mathbf{b}_{(m)} = \Theta_{(m)}\mathbf{h}_{(m)}(\tau)$. Then, (20) becomes

$$y(t_k) = \mathbf{a}^{\mathrm{T}}_{m,t_k}\Theta_{(m)} \int_0^{t_k} \mathbf{h}_{(m)}(\tau)d\tau \tag{21}$$

Since the Haar wavelet possesses finite points of discontinuity on the bounded time domain, the Haar wavelet can be integrable over the interval $[0,t_k)$ [11].

Let $\mathbf{P}_{(m)}$ be calculated from the following integration of the Haar wavelet.

$$\mathbf{P}_{(m)} = \int_0^{t_k} \mathbf{h}_{(m)}(\tau)d\tau, \quad \mathbf{P}_{(m)} \in R^{m\times 1} \tag{22}$$

By substituting (22) into (21), we have

$$y(t_k) = \mathbf{a}^{\mathrm{T}}_{m,t_k}\Theta_{(m)}\mathbf{P}_{(m)} \tag{23}$$

For a simple notation, let's denote $\Theta_{(m)}\mathbf{P}_{(m)}$ by $\mathbf{w}_{(m)} \in R^{m\times 1}$. From (17), we can see that m unknown coefficients in $\mathbf{a}_{m,t_k}$ should be estimated to identify $h(t,\tau)$, but we have only one equation (23). To solve such problem, m different inputs are applied to the LTV system, producing m outputs observed at t_k and leading to the following m equations to solve m unknown coefficients in $\mathbf{a}_{m,t_k}$.

$$\begin{aligned} y_1(t_k) &= \int_0^{t_k} h(t_k,\tau)x_1(\tau)d\tau = \mathbf{a}_{m,t_k}\Theta_{(m),1}\mathbf{P}_{(m)} = \mathbf{w}_{(m),1}^{\mathrm{T}}\mathbf{a}_{m,t_k} \\ y_2(t_k) &= \int_0^{t_k} h(t_k,\tau)x_2(\tau)d\tau = \mathbf{a}_{m,t_k}\Theta_{(m),2}\mathbf{P}_{(m)} = \mathbf{w}_{(m),2}^{\mathrm{T}}\mathbf{a}_{m,t_k} \\ &\vdots \\ y_m(t_k) &= \int_0^{t_k} h(t_k,\tau)x_m(\tau)d\tau = \mathbf{a}_{m,t_k}\Theta_{(m),m}\mathbf{P}_{(m)} = \mathbf{w}_{(m),m}^{\mathrm{T}}\mathbf{a}_{m,t_k} \end{aligned} \tag{24}$$

Furthermore, (24) can be described in the following matrix form:

$$\mathbf{Y}(t_k) = \mathbf{W}_{(m)}^{\mathrm{T}}\ \mathbf{a}_{m,t_k} \tag{25}$$

where

$$\mathbf{Y}(t_k) = [\ y_1(t_k),\quad y_2(t_k),\quad \cdots \quad y_m(t_k)\ ]^{\mathrm{T}} \in R^{m\times 1}$$
$$\mathbf{W}_{(m)} = [\ \mathbf{w}_{(m),1},\quad \mathbf{w}_{(m),2},\quad \cdots \quad \mathbf{w}_{(m),m}\ ] \in R^{m\times m}$$

Accordingly, $h(t_k,\tau)$ can be achieved from (25) and (17) if $\mathbf{W}_{(m)}$ is of full rank m. When $\mathbf{W}_{(m)}$ is not of full rank, we need to set up input data until m column vectors of $\mathbf{W}_{(m)}$ are linearly independent.

4 Simulation results

To demonstrate the performance of the proposed approach, three LTV systems with different (or not necessarily separable with respect to its arguments) impulse responses are considered.

Example 1: Consider an LTV system whose impulse response is given by

$$h(t,\tau) = \cos(2\pi(\tau^2 + t\tau)) \tag{26}$$

For this simulation, a piecewise-constant function was applied as the input to the LTV system (26) and the output was obtained from (15). Fig. 1 illustrates the true impulse responses (at $t = 0.7$ and $t = 0.3$) and their approximations estimated by the proposed Haar wavelet-based approach (here, $m = 32$).

Example 2: Consider another LTV system whose impulse response is given by

$$h(t,\tau) = \sin(10\pi(\tau + t))e^{-t(\tau+t)} \tag{27}$$

Note that while the impulse response changes rapidly with τ than that of *Example 1*, the same input data as in *Example 1* were used in this simulation. In Fig. 2, the exponentially damped sinusoids (i.e., true ones and their respective approximations at $t = 0.9$ and $t = 0.4$: here, $m = 32$) are presented, verifying that the proposed approach leads to a high-quality impulse response estimate even in case of rapidly time-varying linear systems.

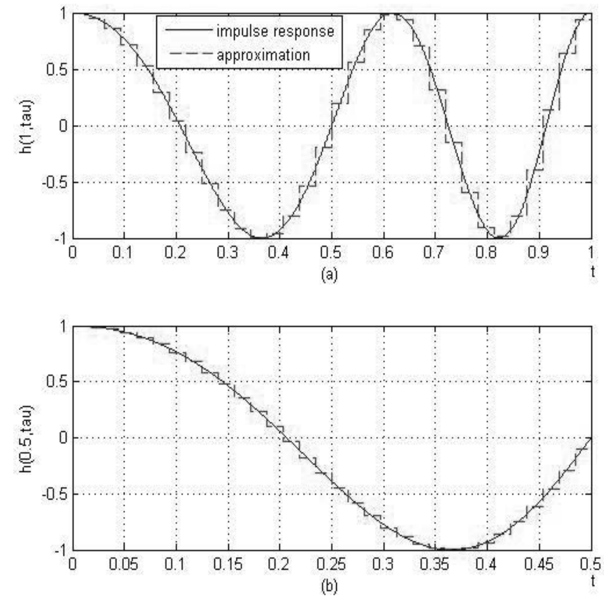

Fig. 1. The time-varying function and its approximation: (a) *t*=1 and (b) *t*=0.5

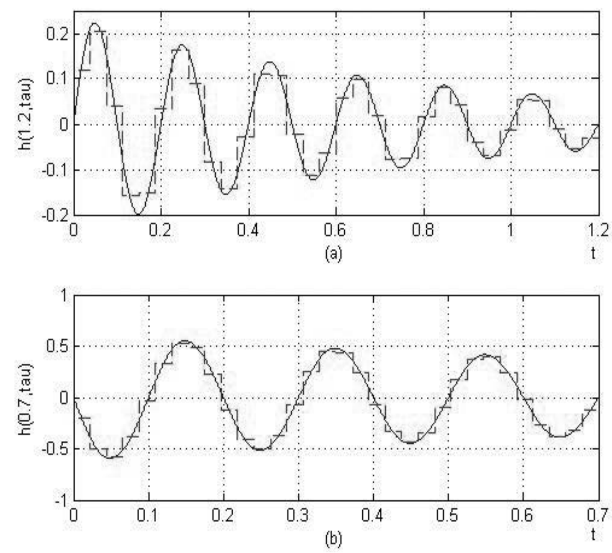

Fig. 2. A time-varying function and its approximation: (a) *t*=1.2 and (b) *t*=0.7

Example 3: Consider the problem of estimating the impulse response of an LTV system by varying the resolution. More specifically, Haar wavelets with different numbers of bases (m =8, 16, and 32) are utilized for the system identification, and an LTV system with the following impulse response is considered:

$$h(t,\tau) = \tau\, e^{-10t(\tau+t)} \tag{28}$$

As in *Example 1* and *Example 2*, the same input data are also utilized. In Fig. 3, the true impulse response at $t = 0.7$ and its approximation (obtained by the proposed approach) are shown, from which it can be seen that multiresolution analysis by Haar wavelet with larger number of bases (e.g., m =32) yields a better approximation to the true impulse response than ones by Haar wavelet with smaller number of bases (e.g., m =8, 16)).

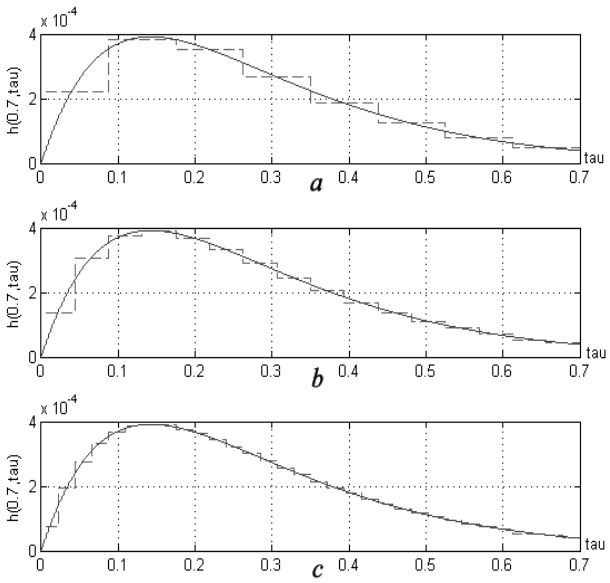

Fig. 3. Approximations and multiresolution analysis by Haar wavelet with (a) m=8, (b) m=16, and (c) m=32.

5 Conclusions

In this paper, the problem of identifying a LTV system from input and output data is considered, whereby Haar wavelet is employed to form an algebraic equation for the system identification, from which Haar wavelet coefficients for the impulse response are estimated. Also, since the Haar wavelet posses a finite value in a bounded time-domain and with unit energy, the proposed approach yields better computational efficiency than that by other wavelet or by square functions such as Walsh's. Future research includes further extension of the proposed approach to the identification of nonlinear LTV systems.

Appendix A

(11)-(12) can be proved by using the following mathematical induction:

1) $m = 2$ (i.e., $\mathrm{i} = 1$)

$$\begin{aligned}\mathbf{h}_{(2)}(t)\mathbf{h}_{(2)}^{T}(t) &= \begin{bmatrix} h_0(t) \\ h_1(t) \end{bmatrix} \begin{bmatrix} h_0(t) & h_1(t) \end{bmatrix} \\ &= \begin{bmatrix} h_0(t) & h_1(t) \\ h_1(t) & h_0(t) \end{bmatrix}\end{aligned} \tag{29}$$

When m is 2, $P_{(2)}(t)$ is derived from (12) as follows:

$$P_{(2)}(t) = \begin{bmatrix} h_0(t) & h_1(t) \\ h_1(t) & h_0(t) \end{bmatrix} \tag{30}$$

Since the right side of (29) and (30) is exactly same, (11)-(12) is true when m is 2.

2) $m = 2^k$ (i.e., $\mathrm{i} = k$)
We suppose that (11)-(12) is true when m is 2^k .

$$\mathbf{h}_{(2^k)}(t)\mathbf{h}_{(2^k)}^{T}(t) = P_{(2^k)}(t) \tag{31}$$

$$\mathbf{h}_a(t) = \begin{bmatrix} h_0(t) & h_1(t) & \cdots & h_{2^{k-1}-1}(t) \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{h}_b(t) = \begin{bmatrix} h_{2^{k-1}}(t) & h_{2^{k-1}+1}(t) & \cdots & h_{2^k-1}(t) \end{bmatrix}^{\mathrm{T}}$$

3) $m = 2^{k+1}$ (i.e., $\mathrm{i} = k+1$)
When m is 2^{k+1} , the left side of (11) is represented by following form:

$$\begin{aligned}\mathbf{h}_{(2^{k+1})}(t)\mathbf{h}_{(2^{k+1})}^{T}(t) &= \begin{bmatrix} \mathbf{h}_a(t) \\ \mathbf{h}_b(t) \end{bmatrix} \begin{bmatrix} \mathbf{h}_a^{\mathrm{T}}(t) & \mathbf{h}_b^{\mathrm{T}}(t) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{h}_a(t)\mathbf{h}_a^{\mathrm{T}}(t) & \mathbf{h}_a(t)\mathbf{h}_b^{\mathrm{T}}(t) \\ \mathbf{h}_b(t)\mathbf{h}_a^{\mathrm{T}}(t) & \mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t) \end{bmatrix}\end{aligned} \tag{32}$$

$$\mathbf{h}_a(t) = \begin{bmatrix} h_0(t) & h_1(t) & \cdots & h_{2^k-1}(t) \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{h}_b(t) = \begin{bmatrix} h_{2^k}(t) & h_{2^k+1}(t) & \cdots & h_{2^{k+1}-1}(t) \end{bmatrix}^{\mathrm{T}}$$

ⅰ) Multiplication of $\mathbf{h}_a(t)$ and $\mathbf{h}_a^{\mathrm{T}}(t)$

In (32), $\mathbf{h}_a(t)$ is the same for $\mathbf{h}_{(2^k)}(t)$ in (31). Therefore, $\mathbf{h}_a(t)\mathbf{h}_a^{\mathrm{T}}(t)$ and $\mathbf{h}_{(2^k)}(t)\mathbf{h}_{(2^k)}^{T}(t)$ are the same so that $\mathbf{h}_a(t)\mathbf{h}_a^{\mathrm{T}}(t)$ is represented by assumption of 2) as following:

$$\mathbf{h}_a(t)\mathbf{h}_a^{\mathrm{T}}(t) = P_{(2^k)}(t) \tag{33}$$

ii) Multiplication of $\mathbf{h}_a(t)$ and $\mathbf{h}_b^{\mathrm{T}}(t)$

The multiplication of $\mathbf{h}_a(t)$ and $\mathbf{h}_b^{\mathrm{T}}(t)$ is represented as following:

$$\begin{aligned}\mathbf{h}_a(t)\mathbf{h}_b^{\mathrm{T}}(t) &= \begin{bmatrix} h_0(t) \\ \vdots \\ h_{2^k-1}(t) \end{bmatrix} \begin{bmatrix} h_{2^k}(t) & \cdots & h_{2^{k+1}-1}(t) \end{bmatrix} \\ &= \begin{bmatrix} h_0(t)h_{2^k}(t) & \cdots & h_0(t)h_{2^{k+1}-1}(t) \\ \vdots & \ddots & \vdots \\ h_{2^k-1}(t)h_{2^k}(t) & \cdots & h_{2^k-1}(t)h_{2^{k+1}-1}(t) \end{bmatrix}\end{aligned} \tag{34}$$

The multiplication of $\mathbf{h}_a(t)$ and $\mathbf{h}_b^{\mathrm{T}}(t)$ is the same as the multiplication between $\mathbf{h}_b^{\mathrm{T}}(t)$ and the sampling of $\mathbf{h}_a(t)$.

$$\begin{aligned}&\mathbf{h}_a(t)\mathbf{h}_b^{\mathrm{T}}(t) \\ &= \begin{bmatrix} h_0(t)h_{2^k}(t) & \cdots & h_0(t)h_{2^{k+1}-1}(t) \\ \vdots & \ddots & \vdots \\ h_{2^k-1}(t)h_{2^k}(t) & \cdots & h_{2^k-1}(t)h_{2^{k+1}-1}(t) \end{bmatrix} \\ &= \begin{bmatrix} h_0(\frac{1}{2^{k+1}})h_{2^k}(t) & h_0(\frac{3}{2^{k+1}})h_{2^k+1}(t) & \cdots & h_0(\frac{2^{k+1}-1}{2^{k+1}})h_{2^{k+1}-1}(t) \\ h_1(\frac{1}{2^{k+1}})h_{2^k}(t) & h_1(\frac{3}{2^{k+1}})h_{2^k+1}(t) & \cdots & h_1(\frac{2^{k+1}-1}{2^{k+1}})h_{2^{k+1}-1}(t) \\ \vdots & \vdots & \ddots & \\ h_{2^k-1}(\frac{1}{2^{k+1}})h_{2^k}(t) & h_{2^k-1}(\frac{3}{2^{k+1}})h_{2^k+1}(t) & \cdots & h_{2^k-1}(\frac{2^{k+1}-1}{2^{k+1}})h_{2^{k+1}-1}(t) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{h}_{(2^k)}(\frac{1}{2^{k+1}})h_{2^k}(t) & \mathbf{h}_{(2^k)}(\frac{3}{2^{k+1}})h_{2^k+1}(t) & \cdots & \mathbf{h}_{(2^k)}(\frac{2^{k+1}-1}{2^{k+1}})h_{2^{k+1}-1}(t) \end{bmatrix}\end{aligned} \tag{35}$$

The right hand side of (35) is rewritten by the digital representation $\mathbf{H}_{(2^k)}$ and $\mathbf{h}_b(t)$:

$$\begin{aligned}&\mathbf{h}_a(t)\mathbf{h}_b^{T}(t) \\ &= \begin{bmatrix} \mathbf{h}_{(2^k)}(\frac{1}{2^{k+1}}) & \cdots & \mathbf{h}_{(2^k)}(\frac{2^{k+1}-1}{2^{k+1}}) \end{bmatrix} diag[\mathbf{h}_b(t)] \\ &= \mathbf{H}_{(2^k)} diag[\mathbf{h}_b(t)]\end{aligned} \tag{36}$$

iii) Multiplication of $\mathbf{h}_b(t)$ and $\mathbf{h}_a^{\mathrm{T}}(t)$

The multiplication of $\mathbf{h}_b(t)$ and $\mathbf{h}_a^{\mathrm{T}}(t)$ is represented as following

$$\begin{aligned}\mathbf{h}_b(t)\mathbf{h}_a^{\mathrm{T}}(t) &= \begin{bmatrix} h_{2^k}(t) \\ \vdots \\ h_{2^{k+1}-1}(t) \end{bmatrix} \begin{bmatrix} h_0(t) & \cdots & h_{2^k-1}(t) \end{bmatrix} \\ &= \begin{bmatrix} h_{2^k}(t)h_0(t) & \cdots & h_{2^k}(t)h_{2^k-1}(t) \\ \vdots & \ddots & \vdots \\ h_{2^{k+1}-1}(t)h_0(t) & \cdots & h_{2^{k+1}-1}(t)h_{2^k-1}(t) \end{bmatrix}\end{aligned} \tag{37}$$

In similar way, the multiplication of $\mathbf{h}_b(t)$ and $\mathbf{h}_a^{\mathrm{T}}(t)$ is same as multiplication of $\mathbf{h}_b(t)$ and the sampling of $\mathbf{h}_a^{\mathrm{T}}(t)$.

$$\begin{aligned}&\mathbf{h}_b(t)\mathbf{h}_a^{\mathrm{T}}(t) \\ &= \begin{bmatrix} h_{2^k}(t)h_0(\frac{1}{2^{k+1}}) & h_{2^k}(t)h_1(\frac{1}{2^{k+1}}) & \cdots & h_{2^k}(t)h_{2^k-1}(\frac{1}{2^{k+1}}) \\ h_{2^k+1}(t)h_0(\frac{3}{2^{k+1}}) & h_{2^k+1}(\frac{3}{2^{k+1}})h_1(t) & \cdots & h_{2^k+1}(\frac{3}{2^{k+1}})h_{2^k-1}(t) \\ \vdots & \vdots & \ddots & \\ h_{2^{k+1}-1}(\frac{2^{k+1}-1}{2^{k+1}})h_0(t) & h_{2^{k+1}-1}(\frac{2^{k+1}-1}{2^{k+1}})h_1(t) & \cdots & h_{2^{k+1}-1}(\frac{2^{k+1}-1}{2^{k+1}})h_{2^k-1}(t) \end{bmatrix}\end{aligned} \tag{38}$$

The right hand side of (38) is represented by $\mathbf{h}_b(t)$ and $\mathbf{H}_{(2^k)}$ as following:

$$\begin{aligned}\mathbf{h}_b(t)\mathbf{h}_a^{\mathrm{T}}(t) &= diag[\mathbf{h}_b(t)] \begin{bmatrix} \mathbf{h}_{(2^k)}^{\mathrm{T}}(\frac{1}{2^{k+1}}) \\ \mathbf{h}_{(2^k)}^{\mathrm{T}}(\frac{3}{2^{k+1}}) \\ \vdots \\ \mathbf{h}_{(2^k)}^{\mathrm{T}}(\frac{2^{k+1}-1}{2^{k+1}}) \end{bmatrix} \\ &= diag[\mathbf{h}_b(t)]\mathbf{H}_{(2^k)}^{\mathrm{T}}\end{aligned} \tag{39}$$

iv) Multiplication of $\mathbf{h}_b(t)$ and $\mathbf{h}_b^{\mathrm{T}}(t)$

The multiplication of $\mathbf{h}_b(t)$ and $\mathbf{h}_b^{\mathrm{T}}(t)$ is represented as following:

$$\mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t) = \begin{bmatrix} h_{2^k}^2(t) & 0 & \cdots & 0 \\ 0 & h_{2^k+1}^2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & h_{2^{k+1}-1}^2(t) \end{bmatrix} \tag{40}$$

For simple derivation, let $h_n^2(t)$ be $2^k\rho_n(t)$, and the function $\rho_n(t)$ is defined as following:

$$\rho_n(t) = h_0(2^k t - q),\ \ n = 2^k + q,\ q = 0,1,2,\ldots,2^k - 1 \quad (41)$$

Therefore, (40) is represented as following:

$$\mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t) = \begin{bmatrix} 2^k \rho_{2^k}(t) & 0 & \cdots & 0 \\ 0 & 2^k \rho_{2^k+1}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 2^k \rho_{2^{k+1}-1}(t) \end{bmatrix} \quad (42)$$

Also, (42) can be expressed by diagnol matrix.

$$\mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t) = diag[\boldsymbol{\rho}_{(2^k)}],$$
$$\boldsymbol{\rho}_{(2^k)} = \left[2^k \rho_{2^k}(t) \ \cdots \ 2^k \rho_{2^{k+1}-1}(t)\right]^{\mathrm{T}} \quad (43)$$

By properties of the Haar wavelet transform, $\boldsymbol{\rho}_{(2^k)}$ can be expressed by the linear combination of $\mathbf{h}_a(t)$ [5].

$$\boldsymbol{\rho}_{(2^k)}(t) = \mathbf{F}\mathbf{h}_a(t) = \begin{bmatrix} 2^k \rho_{2^k}(t) \\ 2^k \rho_{2^k+1}(t) \\ \vdots \\ 2^k \rho_{2^{k+1}-1}(t) \end{bmatrix} = \mathbf{F} \begin{bmatrix} h_0(t) \\ h_1(t) \\ \vdots \\ h_{2^k-1}(t) \end{bmatrix} \quad (44)$$

To find the matrix $\mathbf{F} \in R^{2^k \times 2^k}$, $\boldsymbol{\rho}_{(2^k)}(t)$ and $\mathbf{h}_a(t)$ are sampled by the same sampling rate as following:

$$\begin{bmatrix} 2^k \rho_{2^k}(\frac{1}{2^{k+1}}) & 2^k \rho_{2^k}(\frac{3}{2^{k+1}}) & \cdots & 2^k \rho_{2^k}(\frac{2^{k+1}-1}{2^{k+1}}) \\ 2^k \rho_{2^k+1}(\frac{1}{2^{k+1}}) & 2^k \rho_{2^k+1}(\frac{3}{2^{k+1}}) & \cdots & 2^k \rho_{2^k+1}(\frac{2^{k+1}-1}{2^{k+1}}) \\ \vdots & \vdots & \ddots & \vdots \\ 2^k \rho_{2^{k+1}-1}(\frac{1}{2^{k+1}}) & 2^k \rho_{2^{k+1}-1}(\frac{3}{2^{k+1}}) & \cdots & 2^k \rho_{2^{k+1}-1}(\frac{2^{k+1}-1}{2^{k+1}}) \end{bmatrix}$$
$$= \begin{bmatrix} 2^k & 0 & 0 & \cdots & 0 \\ 0 & 2^k & 0 & \cdots & 0 \\ 0 & 0 & 2^k & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 2^k \end{bmatrix} = \mathbf{F}\left[\mathbf{h}_{(2^k)}(\frac{1}{2^{k+1}}) \ \cdots \ \mathbf{h}_{(2^k)}(\frac{2^{k+1}-1}{2^{k+1}})\right] \quad (45)$$
$$= \mathbf{F}\mathbf{H}_{(2^k)}$$

Since $\mathbf{H}_{(2^k)}$ is always invertable, the inverse matrix is obtained as following:

$$\mathbf{H}_{(2^k)}^{-1} = (diag[\mathbf{r}]\mathbf{H}_{(2^k),an})^{-1} \quad (46)$$

$$\mathbf{r} = \left[1 \ \ 1 \ \ \sqrt{2} \ \ \sqrt{2} \ \ 2 \ \ 2 \ \ 2 \ \ 2 \ \ \cdots \ \ 2^{k/2} \ \ \cdots \ \ 2^{k/2}\right]$$

$\mathbf{H}_{(2^k),an}$ is digital representation of abnormalized Haar wavelet and has inverse matrix [7].

$$\begin{aligned} \mathbf{H}_{(2^k)}^{-1} &= (diag[\mathbf{r}]\mathbf{H}_{(2^k),an})^{-1} \\ &= \frac{1}{2^k}\mathbf{H}_{(2^k),an}^{T} diag[\mathbf{r}_1] diag[\mathbf{r}_2] \\ &= \frac{1}{2^k}\mathbf{H}_{(2^k),an}^{T} diag[\mathbf{r}] \\ &= \frac{1}{2^k}\mathbf{H}_{(2^k)}^{\mathrm{T}} \end{aligned} \quad (47)$$

$$\mathbf{r}_1 = \left[1 \ \ 1 \ \ 2 \ \ 2 \ \ 4 \ \ 4 \ \ 4 \ \ 4 \ \ \cdots \ \ 2^k \ \ \cdots \ \ 2^k\right]$$
$$\mathbf{r}_2 = \left[1 \ \ 1 \ \ \frac{1}{\sqrt{2}} \ \ \frac{1}{\sqrt{2}} \ \ \frac{1}{2} \ \ \frac{1}{2} \ \ \frac{1}{2} \ \ \frac{1}{2} \ \ \cdots \ \ \frac{1}{2^{\frac{k}{2}}} \ \ \cdots \ \ \frac{1}{2^{\frac{k}{2}}}\right]$$

Therefore, the matrix $\mathbf{F}$ is obtained as following:

$$\begin{bmatrix} 2^k & 0 & 0 & \cdots & 0 \\ 0 & 2^k & 0 & \cdots & 0 \\ 0 & 0 & 2^k & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 2^k \end{bmatrix} = \mathbf{F}\mathbf{H}_{(2^k)}$$
$$\Leftrightarrow 2^k \mathbf{I}\mathbf{H}_{(2^k)}^{-1} = \mathbf{F}, \quad \mathbf{I}(\in R^{2^k \times 2^k}) \text{ is identity matrix} \quad (46)$$
$$\Leftrightarrow 2^k \mathbf{H}_{(2^k)}^{-1} = \mathbf{H}_{(2^k)}^{\mathrm{T}} = \mathbf{F}$$

Therefore, $\boldsymbol{\rho}_{(2^k)}(t)$ is expressed as

$$\boldsymbol{\rho}_{(2^k)}(t) \cong \mathbf{H}_{(2^k)}^{\mathrm{T}}\mathbf{h}_a(t) \quad (47)$$

Also, $\mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t)$ is represented as following:

$$\mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t) = diag[\boldsymbol{\rho}_{(2^k)}] \cong diag[\mathbf{H}_{(2^k)}^{\mathrm{T}}\mathbf{h}_a(t)] \quad (48)$$

From (33)-(48), (32) is expressed as follows:

$$\begin{aligned} &\mathbf{h}_{(2^{k+1})}(t)\mathbf{h}_{(2^{k+1})}^{T}(t) \\ &= \begin{bmatrix} \mathbf{h}_a(t)\mathbf{h}_a^{\mathrm{T}}(t) & \mathbf{h}_a(t)\mathbf{h}_b^{\mathrm{T}}(t) \\ \mathbf{h}_b(t)\mathbf{h}_a^{\mathrm{T}}(t) & \mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t) \end{bmatrix} \\ &\cong \begin{bmatrix} P_{(2^k)}(t) & \mathbf{H}_{(2^k)} diag[\mathbf{h}_b(t)] \\ diag[\mathbf{h}_b(t)]\mathbf{H}_{(2^k)}^{\mathrm{T}} & diag[\mathbf{H}_{(2^k)}^{\mathrm{T}}\mathbf{h}_a(t)] \end{bmatrix} \end{aligned} \quad (49)$$
$$\mathbf{h}_a(t) = \left[h_0(t) \quad h_1(t) \quad \cdots \quad h_{2^k-1}(t)\right]^{\mathrm{T}}$$

$$\mathbf{h}_b(t) = \begin{bmatrix} h_{2^k}(t) & h_{2^k+1}(t) & \cdots & h_{2^{k+1}-1}(t) \end{bmatrix}^{\mathrm{T}}$$

Therefore, (11)-(12) are proved.

Appendix B

Proof of (13)-(14) can be obtained by using the following mathematical induction:

1) $m = 2$ (i.e., $\mathrm{i} = 1$)

$$\mathbf{h}_{(2)}(t)\mathbf{h}_{(2)}^T(t)\mathbf{c}_{(2)} = \begin{bmatrix} h_0(t) \\ h_1(t) \end{bmatrix} \begin{bmatrix} h_0(t) & h_1(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} c_0 & c_1 \\ c_1 & c_0 \end{bmatrix} \begin{bmatrix} h_0(t) \\ h_1(t) \end{bmatrix} \tag{50}$$

The right hand side of (50) is equal to that of (13)-(14) when m is 2.

$$T_{(2)} = \begin{bmatrix} c_0 & c_1 \\ c_1 & c_0 \end{bmatrix}, T_{(1)} = c_0,\ \mathbf{c}_a = c_0, \mathbf{c}_b = c_1 \tag{51}$$

Therefore, (13)-(14) is true when m is 2.

2) $m = 2^k$ (i.e., $\mathrm{i} = k$):

We suppose that (13)-(14) is true when m is 2^k.

$$\mathbf{h}_{(2^k)}(t)\mathbf{h}_{(2^k)}^T(t)\mathbf{c}_{(2^k)} = T_{(2^k)}\mathbf{h}_{(2^k)}(t) \tag{52}$$

$$\mathbf{c}_a(t) = \begin{bmatrix} c_0(t) & c_1(t) & \cdots & c_{2^{k-1}-1}(t) \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{c}_b(t) = \begin{bmatrix} c_{2^{k-1}}(t) & c_{2^{k-1}+1}(t) & \cdots & c_{2^k-1}(t) \end{bmatrix}^{\mathrm{T}}$$

3) $m = 2^{k+1}$ (i.e., $\mathrm{i} = k+1$):
When m is 2^{k+1}, the left side of (11) is represented by following form:

$$\begin{aligned} &\mathbf{h}_{(2^{k+1})}(t)\mathbf{h}_{(2^{k+1})}^T(t)\mathbf{c}_{(2^{k+1})} \\ &= \begin{bmatrix} \mathbf{h}_a(T) \\ \mathbf{h}_b(t) \end{bmatrix} \begin{bmatrix} \mathbf{h}_a^{\mathrm{T}}(t) & \mathbf{h}_b^{\mathrm{T}}(t) \end{bmatrix} \begin{bmatrix} \mathbf{c}_a \\ \mathbf{c}_b \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{h}_a(t)\mathbf{h}_a^{\mathrm{T}}(t)\mathbf{c}_a + \mathbf{h}_a(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b \\ \mathbf{h}_b(t)\mathbf{h}_a^{\mathrm{T}}(t)\mathbf{c}_a + \mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b \end{bmatrix} \end{aligned} \tag{53}$$

$$\mathbf{c}_a(t) = \begin{bmatrix} c_0(t) & c_1(t) & \cdots & c_{2^k-1}(t) \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{c}_b(t) = \begin{bmatrix} c_{2^k}(t) & c_{2^k+1}(t) & \cdots & c_{2^{k+1}-1}(t) \end{bmatrix}^{\mathrm{T}}$$

ⅰ) Analysis of $\mathbf{h}_a(t)\mathbf{h}_a^{\mathrm{T}}(t)\mathbf{c}_a$

For similar reason in ⅰ) of Appendix A, $\mathbf{h}_a(t)\mathbf{h}_a^{\mathrm{T}}(t)\mathbf{c}_a$ is represented by following:

$$\mathbf{h}_a(t)\mathbf{h}_a^{\mathrm{T}}(t)\mathbf{c}_a = T_{(2^k)}\mathbf{h}_{(2^k)}(t) \tag{54}$$

ⅱ) Analysis of $\mathbf{h}_a(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b$

For similar way in ⅱ) of Appendix A, $\mathbf{h}_a(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b$ is represented by following:

$$\begin{aligned} &\mathbf{h}_a(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b \\ &= \begin{bmatrix} \mathbf{h}_{(2^k)}(\frac{1}{2^{k+1}})h_{2^k} & \cdots & \mathbf{h}_{(2^k)}(\frac{2^{k+1}-1}{2^{k+1}})h_{2^{k+1}-1} \end{bmatrix}\mathbf{c}_b \\ &= \begin{bmatrix} \mathbf{h}_{(2^k)}(\frac{1}{2^{k+1}}) & \cdots & \mathbf{h}_{(2^k)}(\frac{2^{k+1}-1}{2^{k+1}}) \end{bmatrix} diag[\mathbf{c}_b]\mathbf{h}_b(t) \\ &= \mathbf{H}_{(2^k)} diag[\mathbf{c}_b]\mathbf{h}_b(t) \end{aligned} \tag{55}$$

ⅲ) Analysis of $\mathbf{h}_b(t)\mathbf{h}_a^{\mathrm{T}}(t)\mathbf{c}_a$:

$$\begin{aligned} &\mathbf{h}_b(t)\mathbf{h}_a^{\mathrm{T}}(t)\mathbf{c}_a \\ &= \begin{bmatrix} h_{2^k}(t) \\ \vdots \\ h_{2^{k+1}-1}(t) \end{bmatrix} \begin{bmatrix} h_0(t) & \cdots & h_{2^k-1}(t) \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_{2^k-1} \end{bmatrix} \\ &= \begin{bmatrix} h_{2^k}(t)h_0(t)c_0 + \cdots + h_{2^k}(t)h_{2^k-1}(t)c_{2^k-1} \\ \vdots \\ h_{2^{k+1}-1}(t)h_0(t)c_0 + \cdots + h_{2^{k+1}-1}(t)h_{2^k-1}(t)c_{2^k-1} \end{bmatrix} \end{aligned} \tag{56}$$

Also, (56) can be expressed as

$$\begin{aligned} &\mathbf{h}_b(t)\mathbf{h}_a^{\mathrm{T}}(t)\mathbf{c}_a \\ &= \begin{bmatrix} h_{2^k}(t)h_0(\frac{1}{2^{k+1}})c_0 + \cdots + h_{2^k}(t)h_{2^k-1}(\frac{1}{2^{k+1}})c_{2^k-1} \\ \vdots \\ h_{2^{k+1}-1}(t)h_0(\frac{2^{k+1}-1}{2^{k+1}})c_0 + \cdots + h_{2^{k+1}-1}(t)h_{2^k-1}(\frac{2^{k+1}-1}{2^{k+1}})c_{2^k-1} \end{bmatrix} \\ &= \begin{bmatrix} h_0(\frac{1}{2^{k+1}}) & \cdots & h_{2^k-1}(\frac{1}{2^{k+1}}) \\ \vdots & \ddots & \vdots \\ h_0(\frac{2^{k+1}-1}{2^{k+1}}) & \cdots & h_{2^k-1}(\frac{2^{k+1}-1}{2^{k+1}}) \end{bmatrix} \begin{bmatrix} h_{2^k}(t)c_0 \\ \vdots \\ h_{2^{k+1}-1}(t)c_{2^k-1} \end{bmatrix} \\ &= \begin{bmatrix} h_0(\frac{1}{2^{k+1}}) & \cdots & h_{2^k-1}(\frac{1}{2^{k+1}}) \\ \vdots & \ddots & \vdots \\ h_0(\frac{2^{k+1}-1}{2^{k+1}}) & \cdots & h_{2^k-1}(\frac{2^{k+1}-1}{2^{k+1}}) \end{bmatrix} diag[\mathbf{c}_a]\mathbf{h}_b(t) \\ &= \mathbf{H}_{(2^k)}^{\mathrm{T}} diag[\mathbf{c}_a]\mathbf{h}_b(t) \end{aligned} \tag{57}$$

iv) Analysis of $\mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b$:

From (42), $\mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b$ can be expressed as follows:

$$\begin{aligned}
&\mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b \\
&= \begin{bmatrix} 2^k\rho_{2^k}(t) & 0 & \cdots & 0 \\ 0 & 2^k\rho_{2^k+1}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 2^k\rho_{2^{k+1}-1}(t) \end{bmatrix}\mathbf{c}_b \quad (58)\\
&= diag[\mathbf{c}_b]\begin{bmatrix} 2^k\rho_{2^k}(t) \\ 2^k\rho_{2^k+1}(t) \\ \vdots \\ 2^k\rho_{2^{k+1}-1}(t) \end{bmatrix}
\end{aligned}$$

By (44), (58) is represented as

$$\mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b = diag\,[\mathbf{c}_b]\begin{bmatrix} 2^k\rho_{2^k}(t) \\ 2^k\rho_{2^k+1}(t) \\ \vdots \\ 2^k\rho_{2^{k+1}-1}(t) \end{bmatrix} = \mathbf{F}\mathbf{h}_a(t) \quad (59)$$

In the same way as (45), (59) can be written as

$$diag[\mathbf{c}_b]\begin{bmatrix} 2^k & 0 & 0 & \cdots & 0 \\ 0 & 2^k & 0 & \cdots & 0 \\ 0 & 0 & 2^k & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 2^k \end{bmatrix} = \mathbf{F}\mathbf{H}_{(2^k)} \quad (60)$$

$$\Leftrightarrow diag[\mathbf{c}_b]2^k\mathbf{I}\mathbf{H}^{-1}_{(2^k)} = \mathbf{F}$$

$\mathbf{I}(\in R^{2^k\times 2^k})$ is identity matrix

$$\Leftrightarrow diag[\mathbf{c}_b]2^k\mathbf{H}^{-1}_{(2^k)} = diag[\mathbf{c}_b]\mathbf{H}^{\mathrm{T}}_{(2^k)} = \mathbf{F}$$

Therefore, $\mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b$ can be expressed as following:

$$\mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b = diag[\mathbf{c}_b]\mathbf{H}^{\mathrm{T}}_{(2^k)}\mathbf{h}_a(t) \quad (61)$$

From (54)-(61), (53) can be represented by

$$\begin{aligned}
&\mathbf{h}_{(2^{k+1})}(t)\mathbf{h}^T_{(2^{k+1})}(t)\mathbf{c}_{(2^{k+1})} \\
&= \begin{bmatrix} \mathbf{h}_a(t)\mathbf{h}_a^{\mathrm{T}}(t)\mathbf{c}_a + \mathbf{h}_a(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b \\ \mathbf{h}_b(t)\mathbf{h}_a^{\mathrm{T}}(t)\mathbf{c}_a + \mathbf{h}_b(t)\mathbf{h}_b^{\mathrm{T}}(t)\mathbf{c}_b \end{bmatrix} \\
&= \begin{bmatrix} T_{(2^k)}\mathbf{h}_a(t) + \mathbf{H}_{(2^k)}diag[\mathbf{c}_b]\mathbf{h}_b(t) \\ diag[c_a^{\mathrm{T}}\mathbf{H}_{(2^k)}]\mathbf{h}_b(t) + diag[\mathbf{c}_b]\mathbf{H}^{\mathrm{T}}_{(2^k)}\mathbf{h}_a(t) \end{bmatrix} \quad (62)\\
&= \begin{bmatrix} T_{(2^k)} & \mathbf{H}_{(2^k)}diag[\mathbf{c}_b] \\ diag[\mathbf{c}_b]\mathbf{H}^{\mathrm{T}}_{(2^k)} & diag[c_a^{\mathrm{T}}\mathbf{H}_{(2^k)}] \end{bmatrix}\begin{bmatrix} \mathbf{h}_a(t) \\ \mathbf{h}_b(t) \end{bmatrix}
\end{aligned}$$

$$\mathbf{c}_a(t) = \begin{bmatrix} c_0(t) & c_1(t) & \cdots & c_{2^k-1}(t) \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{c}_b(t) = \begin{bmatrix} c_{2^k}(t) & c_{2^k+1}(t) & \cdots & c_{2^{k+1}-1}(t) \end{bmatrix}^{\mathrm{T}}$$

Therefore, (13)-(14) is proved by mathematical induction.

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