A Unified Treatment of Deterministic and Random Transients in Electrical Circuits

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Abstract: - The continuity of capacitor voltages and inductor currents, well-known from the deterministic case, cannot be directly applied if the initial conditions are random. In this paper, new continuity relations for probability densities, mean values, correlation and covariance functions of the state variables are introduced. These continuity relations can be used for a global characterization of random transients.

New formulas for initial condition and forced components of transient statistical moments are also presented. These formulas offer a direct approach for random transient analysis. Deterministic transients can be regarded as particular, degenerated, random transients. On this basis, one can develop a unified treatment of deterministic and random transients in electrical circuits. This unified framework is certainly an advantage, not only in teaching activities related to transient phenomena.

Key-Words: - Electrical circuit, Initial condition, Random transients, Statistical moments, Continuity relations

1 Introduction

Students find subjects related to random phenomena as difficult and vague [1]. Particularly, random transients can be analyzed using stochastic differential equations, which are also not very attractive for the average student [2], [3], [4], [5]. Therefore, at least from a teaching point of view, a simple method extending the deterministic analysis approach to random transients is a justified attempt.

It is well known that solutions of differential equations describing deterministic transients in electrical circuits are based on the continuity of capacitor voltages and inductor currents [6], [7]. Supposing transient states released by closing or opening a switch at the moment $t = 0$, the continuity of the "initial conditions", can be written as

$$v_C(+0) = v_C(-0) = v_C(0) \qquad (1)$$

for capacitor voltages, respectively,

$$i_L(+0) = i_L(-0) = i_L(0) \qquad (2)$$

for inductor currents. In these deterministic continuity relations, the notation $+0$ signifies "just after $t = 0$" and -0 means "just before $t = 0$". The physical reasons for the above continuity relations are obvious: according to $i_C = C\dfrac{dv_C}{dt}$, in order for the capacitor voltage to change instantaneously, the capacitor current i_C, would have to be infinite. Similarly, in order for the inductor current to change instantaneously, the inductor voltage would have to be infinite.

This paper refers to random transients in linear electrical circuits, due to the uncertainty of the initial condition values or/and to the stochastic character of the input signals.

Section 2 presents some mathematical generalizations of the deterministic continuity relations, namely for probability densities, mean values, covariance and correlation functions. New formulas for initial condition and forced components of transient statistical moments as a direct approach for random transient analysis are introduced in section 3. Two detailed illustrative examples are presented in section 4. Deterministic transients are regarded as particular, degenerated cases of random transients. On this basis, one can develop a unified treatment of deterministic and random transients in electrical circuits. The last section is dedicated to conclusion.

2 Generalized Continuity Relations

In this section, the continuity relations (1) and (2) are generalized to include the case of random initial condition. The generalization is of pure mathematical nature and refers to probability density functions (p.d.f.), as well to first and second order statistical moments.

2.1 Continuity of probability densities

Obviously, the continuity of each possible value of a capacitor voltage assures the continuity of the p.d.f. of this voltage. This fact is expressed by the following relation:

$$p^+(v_C) = p^-(v_C) = p(v_C)\,. \tag{3}$$

The continuity of the p.d.f. of an inductor current can be expressed by similar equalities:

$$p^+(i_L) = p^-(i_L) = p(i_L)\,. \tag{4}$$

In the relations (3) and (4), $p^-(\cdot)$, $p^+(\cdot)$ and $p(\cdot)$ are representing the p.d.f. "just before $t = 0$", "just after $t = 0$" and at the moment t = 0, respectively. With the same physical justification, one can state continuity relations for mutual p.d.f. of two or more random variables.

If v_C and i_L are continuous random variables, $p(\cdot)$ in (3) and (4) are ordinary functions. However, if v_C and i_L are of discrete or mixed types, the corresponding p.d.f. contain Dirac impulses (generalized functions). In the particular case of deterministic initial conditions, the continuity relations (1) and (2) can be written as equivalent relations between probability densities:

$$\delta^+[v_C - v_C(+0)] = \delta^-[v_C - v_C(-0)] = \\ = \delta[v_C - v_C(0)] \tag{5}$$

$$\delta^+[i_L - i_L(+0)] = \delta^-[i_L - i_L(-0)] \\ = \delta[i_L - i_L(0)] \tag{6}$$

In (5) and (6), $\delta^-[\cdot]$, $\delta^+[\cdot]$ and $\delta[\cdot]$ are Dirac impulses positioned "just before $t = 0$", "just after $t = 0$" and at the moment t = 0, respectively. Thus, deterministic initial conditions can be considered as particular, degenerated cases of general random initial conditions.

2.2 Continuity of statistical moments

In electrical circuits, the capacitor voltages and inductor currents are state variables. Therefore it is suitable to consider the state-space description of a general electrical circuit (system) [8].

A linear continuous-time system can be described by the so called state equation

$$\dot{Z}(t) = A \cdot Z(t) + B \cdot X(t) \tag{7}$$

and the output equation

$$Y(t) = C \cdot Z(t) + D \cdot X(t)\,, \tag{8}$$

where $X(t)$, $Z(t)$ and $Y(t)$ represent the input vector, the state vector and the output vector respectively. The matrices A, B, C and D are the system parameters [8]. Together, equations (7) and (8) offer a state-space approach to analysis and design of physical continuous-time linear systems, including electrical circuits.

The general solution of the state-space system, representing the transient state vector

$$Z(t) = Z^{IC}(t) + Z^F(t) \tag{9}$$

has two components. Thus,

$$Z^{IC}(t) = \Phi(t) \cdot Z(0) \tag{10}$$

represents the initial condition response, or the natural response of the linear time-invariant (LTI) system [6], [8]. $\Phi(t)$ in (10) is called the state transition matrix and can be obtained using the inverse Laplace, according to the formula [8]:

$$\Phi(t) = e^{At} = L^{-1}\{(sI - A)^{-1}\}\,. \tag{11}$$

For stable linear time-invariant systems, the initial condition response has always a transient character.

The second component of the transient state vector,

$$Z^F(t) = \int_0^t \Phi(t-\tau) \cdot B \cdot X(\tau) \cdot d\tau \tag{12}$$

is the forced solution, caused by the input vector, $X(t)$. According to the type of the input vector, the forced solution can have a permanent or transient character. However, if the inputs $X(t)$ have a permanent character (i.e. they are constant or periodic deterministic time functions, large sense stationary or periodically stationary random signals [1], etc.), the forced solution, also has a permanent character.

For an *n*-order system or circuit, the vector of the mean values of the state variables at moment $t = 0$, can be written in the form

$$m_Z(0) = \left\|E\{Z_i(0)\}\right\| = \left\|m_i(0)\right\|; \\ i = 1, 2, \ldots, n \tag{13}$$

where $E\{\cdot\}$ and $\|\cdot\|$ are denoting mathematical expectation and the matrix notation, respectively. In addition to the initial mean values, two second-order statistical moments at $t = 0$ are important characteristics of the state vector: the correlation matrix

$$R_Z(0,0) = \left\| E\{Z_i(0) \cdot Z_j(0)\} \right\| = \\ = \left\| r_{ij}(0) \right\|; \quad i, j = 1, 2, ..., n \tag{14}$$

and the covariance matrix

$$C_Z(0,0) = \left\| E\{Z_{Ci}(0) \cdot Z_{Cj}(0)\} \right\| = \\ = \left\| c_{ij}(0) \right\|; \quad i, j = 1, 2, ..., n. \tag{15}$$

In (14), $Z_C(\cdot) = Z(\cdot) - E\{Z(\cdot)\}$ represents a centred component of the state vector.

Using the continuity of the probability densities, one can prove the continuity of the statistical mean values of the state variables (Appendix A). Thus, one can state the continuity of the mean values, autocorrelation and covariance functions of the state variables:

$$m_Z(+0) = m_Z(-0) = m_Z(0); \tag{16}$$

$$R_Z(+0,+0) = R_Z(-0,-0) = R_Z(0,0); \tag{17}$$

$$C_Z(+0,+0) = C_Z(-0,-0) = C_Z(0,0). \tag{18}$$

The continuity relations (3), (4), (16), (17) and (18) can be used for the calculation of random transients. On the other side, deterministic transients can be regarded as degenerated random transients, where the corresponding p.d.f. are Dirac impulse functions (distributions).

3 Initial Condition and Forced Components of Statistical Moments

Usually, random transients a described by stochastic differential equations [9]. To determine the transient mean values of state variables, one can apply the one-dimensional or two-dimensional Laplace transform [10], [11], [12]. We prefer a simple, direct approach, based on the separation of initial condition and forced components of the transient statistical moments. Such a decomposition of the general transient mean values is possible due to the generalized continuity relations introduced previously (Appendix B). Actually, the transient mean of the state variables, can be expressed as

$$m_Z(t) = m_Z^F(t) + m_Z^{IC}(t) \tag{19}$$

with

$$m_Z^F(t) = \int_{-\infty}^{t} \Phi(t-u) \cdot B \cdot m_X(u) \cdot du \tag{20}$$

and

$$m_Z^{IC}(t) = \Phi(t) \cdot \{m_Z(0) - m_Z^F(0)\} \tag{21}$$

representing the forced and the initial condition (natural) components of the statistical mean, respectively. Similarly, for the covariance of the transient state vector, we obtain:

$$C_Z(t_1,t_2) = C_Z^F(t_1,t_2) + C_Z^{IC}(t_1,t_2). \tag{22}$$

In the important case when the input signals are realizations of white-noise processes, the forced component of the autocovariance is given by

$$C_Z^F(t_1,t_2) = \Phi(t_1) \cdot \\ \left\{ \int_{-\infty}^{\min(t_1,t_2)} \Phi^{-1}(u) \cdot B \cdot N \cdot B^T \cdot \left[\Phi^{-1}(u)\right]^T \cdot du \right\} \cdot \\ \Phi^T(t_2) \tag{23}$$

In (23), the notation "min(t_1,t_2)"stands for "the smallest of t_1 and t_2". The initial condition autocovariance can be calculated as

$$C_Z^{IC}(t_1,t_2) = \Phi(t_1) \cdot \left[C_Z(0,0) - C_Z^F(0,0)\right] \cdot \Phi^T(t_2). \tag{24}$$

Finally, the transient autocorrelation function can be written as the sum

$$R_Z(t_1,t_2) = R_Z^F(t_1,t_2) + R_Z^{IC}(t_1,t_2) \tag{25}$$

with

$$R_Z^F(t_1,t_2) = m_Z^F(t_1) \cdot m_Z^F(t_2) + C_Z^F(t_1,t_2) \tag{26}$$

representing the forced component. On the other side, the initial condition component of the transient autocorrelation function

$$R_Z^{IC}(t_1,t_2) = \Phi(t_1) \cdot \{R_Z(0,0) + R_Z^F(0,0) - \\ m_Z(0) \cdot \left[m_Z^F(0)\right]^T - m_Z^F(0) \cdot m_Z^T(0) - \\ -2 \cdot C_Z^F(0,0)\} \cdot \Phi^T(t_2) + \Phi(t_1) \cdot \\ \{\left[m_Z(0) - m_Z^F(0)\right] \cdot \left[m_Z^F(t_2)\right]^T\} \cdot \\ m_Z^F(t_1) \cdot \{m_Z^T(0) - \left[m_Z^F(0)\right]^T\} \cdot \Phi^T(t_2). \tag{27}$$

looks rather complicated. However, we have the possibility to calculate the transient autocorrelation based on the previously determined mean value and autocovariance [1]:

$$R_Z(t_1,t_2) = C_Z(t_1,t_2) + m_Z(t_1) \cdot m_Z(t_2). \tag{28}$$

One can observe that the initial condition components of the transient statistical moments, contain the effect of the mean value, autocovariance and autocorrelation at $t = 0$ and $t_1 = t_2 = 0$, respectively. These values can are known as a consequence of the established generalized continuity relations.

If, for simplicity, in the output equation (8) we consider, as usual the case is, the matrix $D = 0$, the transient statistical moments of the output vector can be expressed by the following formulas:

$$m_Y(t) = C \cdot m_Z(t), \tag{29}$$

$$C_Y(t_1,t_2) = C \cdot C_Z(t_1,t_2) \cdot C^T, \tag{30}$$

$$R_Y(t_1,t_2) = C \cdot R_Z(t_1,t_2) \cdot C^T . \qquad (31)$$

4 Illustrative Examples

4.1 Electrical circuit with random initial condition and deterministic input voltage

In order to illustrate the utilization of the derived continuity relations, we consider the simple case of the RC circuit represented in Fig.1, with constant input voltage: $x(t) = 1\text{V}$ for $t \geq 0$. The coefficients of state equations (7) and (8) are: $A = -a$; $B = a$; $C = 1$ and $D = 0$ with $a = 1/RC$. For numerical computations the following values are considered: $R = 10k\Omega$; $C = 1\mu F$ resulting a time constant $RC = 10ms$ and $a = 100Hz$. The single state variable (the voltage on the capacitor) is also the output signal, $z(t) = y(t)$. In this case, the state-space equations can be expressed as

$$\dot{y}(t) = -a \cdot y(t) + a \cdot x(t) \qquad (32)$$

with solution

$$y(t) = e^{-at} \cdot y(0) + 1 - e^{-at}; \quad t \geq 0 . \qquad (33)$$

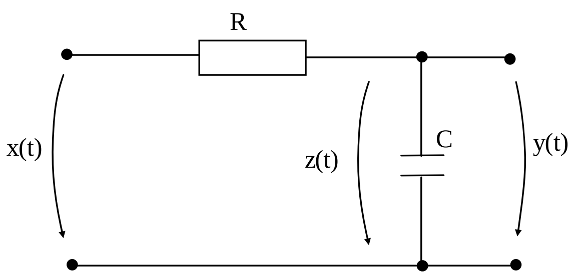

Fig.1 A simple system: the RC low-pass circuit

Contrary to the usual assumption, we consider that the initial voltage on the capacitor is unknown and has a uniform p. d. f.

$$p_Y^-(y) = p_Y(y;-0) = \begin{cases} \dfrac{1}{v_2 - v_1} & v_1 \leq y \leq v_2 \\ 0 & \text{otherwise.} \end{cases} \qquad (34)$$

We can now apply the continuity relation

$$p_Y^-(y) = p_Y^+(y) = p_Y(y) = \Pi(y; v_1, v_2) \qquad (35)$$

Since $y(0)$ is unknown, according to (33), the output voltage is a linear transformation of the random variable $Y(0)$:

$$Y(t) = \alpha \cdot Y(0) + \beta$$

where $\alpha = e^{-at}$ and $\beta = 1 - e^{-at}$. The p.d.f. of the output variable $Y(t)$ is [1]:

$$p_Y(y;t) = \Pi(y; \alpha v_1 + \beta, \alpha v_2 + \beta) = \begin{cases} \dfrac{1}{\alpha(v_2 - v_1)} & \alpha v_1 + \beta \leq y \leq \alpha v_2 + \beta \\ 0 & \text{otherwise.} \end{cases} \qquad (36)$$

This p.d.f. is represented in Fig.2 for $v_1 = -1\text{V}$, $v_2 = 2\text{V}$ at five different moments during the transient process: $t = 0ms$, $10ms$, $20ms$, $30ms$ and $40ms$. The transient p.d.f. shows that with increasing time the random effect of the initial condition disappears and the output voltage becomes deterministic. The uniform p.d.f. approaches a Dirac impulse, for $t \to \infty$

$$\lim_{t \to \infty} p_Y(y,t) = \delta(y - 1) . \qquad (37)$$

The initial condition of the mean value is

$$m_Y(0) = \int_{v_1}^{v_2} y \frac{1}{v_2 - v_1} dy = \frac{v_1 + v_2}{2} . \qquad (38)$$

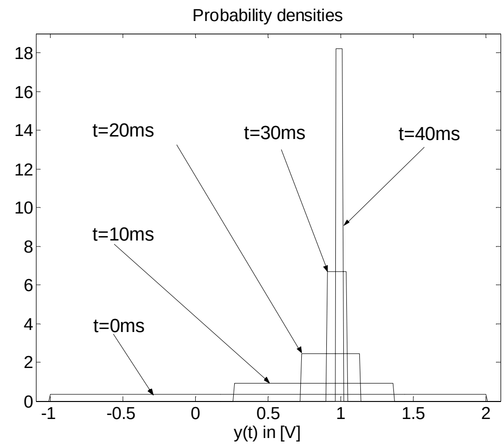

Fig.2 Probability density function of the output voltage for five distinct time values

The transient mean value has the expression

$$m_Y(t) = E\{Y(t)\} = = e^{-at} m_Y(0) + 1 - e^{-at} . \qquad (39)$$

For $v_1 = -1$ and $v_2 = 2$, the mean value of the output voltages can be written as

$$m_Y(t) = 1 - 0.5 \cdot e^{-at} \qquad (40)$$

This particular mean value is represented in Fig.3, together with four transient output voltages corresponding to $y(0) = -0.5\text{V}$; 0V; 1.5V and 2V. Obviously, the mean value can be regarded as the output voltage for $y(0) = 0.5\text{V}$ initial condition.

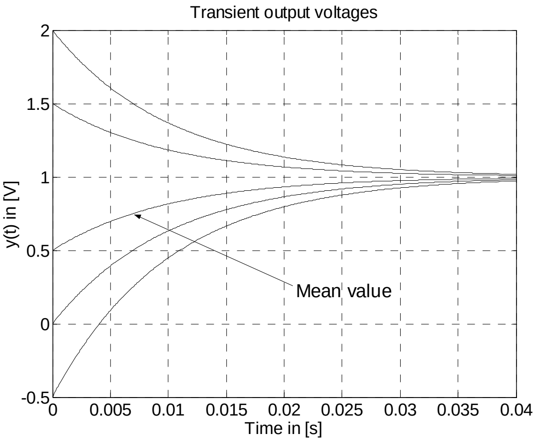

Fig.3 Particular transient voltages on the capacitor for different initial conditions

In order to determine the initial autocorrelation function we express the joint p.d.f. using the conditional p.d.f. [1]:

$$p_Y^-(y_1,y_2)=p_Y^-(y_1)\cdot p_Y^-(y_2\mid y_1)= \\ =\Pi(y_1,v_1,v_2)\cdot\delta(y_2-y_1) \qquad (41)$$

It follows, for every $t_1,t_2\le 0$,

$$R_Y(t_1,t_2)=M\{Y(t_1)Y(t_2)\}= \\ =\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y_1y_2\cdot\Pi(y_1,v_1,v_2)\cdot\delta(y_2-y_1)dy_1dy_2= \\ =\frac{v_1^2+v_1v_2+v_2^2}{3}=R_Y(0,0). \qquad (42)$$

According to the definition of the autocorrelation function we obtain the general expression:

$$\begin{aligned}R_Y(t_1,t_2)&=M\{Y(t_1)Y(t_2)\}=\\&=e^{-a(t_1+t_2)}\cdot R(0,0)+\\&+e^{-at_1}\left(1-e^{-at_2}\right)\cdot m_Y(0)+\\&+e^{-at_2}\left(1-e^{-at_1}\right)\cdot m_Y(0)+\\&+\left(1-e^{-at_1}\right)\cdot\left(1-e^{-at_2}\right)\end{aligned} \qquad (43)$$

For the particular values $v_1=-1\text{V}$, $v_2=2\text{V}$, the transient autocorrelation function equals

$$R_Y(t_1,t_2)=e^{-a(t_1+t_2)}- \\ -\frac{1}{2}\cdot e^{-at_1}-\cdot\frac{1}{2}e^{-at_2}+1 \qquad (44)$$

This last expression is represented in the Fig.4, for the domain $0\le t_1,t_2\le 40ms$. Obviously, with increasing t_1 and t_2, the autocorrelation approaches a constant value equal to the square of the constant steady-state value of the output voltage.

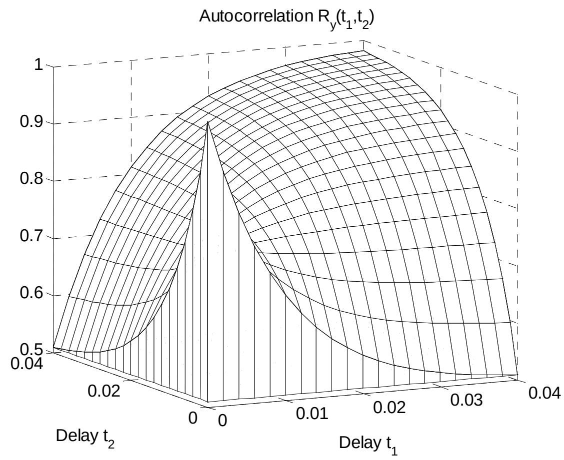

Fig.4 The transient autocorrelation function,
$R_Y(t_1,t_2);\quad 0\le t_1,t_2\le 40ms$

The initial value of the covariance function can be calculated using the centered initial p.d.f.

$$\Pi\left(y;\frac{v_1-v_2}{2},\frac{v_2-v_1}{2}\right)$$

It follows,

$$C_Y(t_1,t_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y_1y_2\cdot\Pi\left(y;\frac{v_1-v_2}{2},\frac{v_2-v_1}{2}\right)\cdot \\ \cdot\delta(y_2-y_1)dy_1dy_2= \\ =\frac{(v_2-v_1)^2}{12}=C_Y(0,0) \qquad (45)$$

Finally, we obtain the transient covariance function,

$$C_Y(t_1,t_2)=C_Y(0,0)\cdot e^{-a(t_1+t_2)}. \qquad (46)$$

For $v_1=-1$ and $v_2=2$, the covariance has the particular expression:

$$C_Y(t_1,t_2)=\frac{3}{4}\cdot e^{-a(t_1+t_2)} \qquad (47)$$

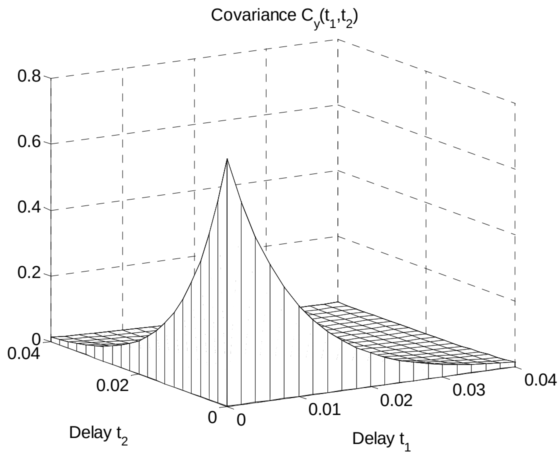

Fig.5 The transient autocovariance function,
$C_Y(t_1,t_2);\quad 0\le t_1,t_2\le 40ms$

This autocovariance function is represented in Fig.5. For increasing t_1 and t_2 the autocovariance approaches zero because the steady-state value of the output voltage is a constant. This means also that for increasing t_1 and t_2, the output process becomes purely deterministic.

Particular case

One can think about a deterministic initial condition case as obtained for $|v_2 - v_1| \to 0$, so that $v_1 = v_2 = v$ is the known initial capacitor voltage. The initial p.d.f. of the state variable (and of the output voltage) is a Dirac impulse

$$p_Y^-(y) = \delta(y - v). \quad (48)$$

Furthermore, for $t \geq 0$ the p.d.f. of the output voltage is a Dirac impulse moving, for $t \to \infty$, between $m_Y(0) = v$ and $m_Y(\infty) = 1$:

$$p_Y(y;t) = \delta[y - (\alpha \cdot v + \beta)]. \quad (49)$$

The corresponding "mean value" of the degenerated random process is actually the purely deterministic capacitor voltage:

$$m_Y(t) = \alpha \cdot v + \beta = 1 + (v - 1) \cdot e^{-a \cdot t}. \quad (50)$$

The well-known input step voltage and the output exponential signal are shown in Fig.6 for the deterministic initial condition $v = -1\text{V}$.

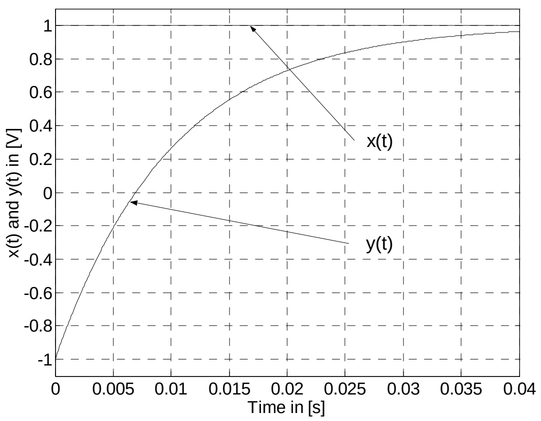

Fig.6 Input and output voltages in a deterministic case

Obviously, the autocovariance function equals zero, denoting the absence of any random fluctuation,

$$C_Y(t_1, t_2) = C_Y(0,0) = 0. \quad (51)$$

With $R_Y(0,0) = v^2$, one can easily determine the output autocorrelation function

$$R_Y(t_1, t_2) = e^{-a(t_1+t_2)} \cdot v^2 + e^{-at_1}\left(1 - e^{-at_2}\right) \cdot v + \\ + e^{-at_2}\left(1 - e^{-at_1}\right) \cdot v + \left(1 - e^{-at_1}\right) \cdot \left(1 - e^{-at_2}\right) = \\ = m_Y(t_1) \cdot m_Y(t_2). \quad (52)$$

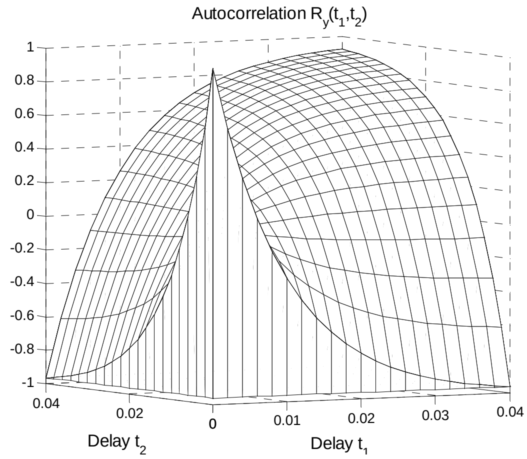

Fig.7 The output autocorrelation for the deterministic case, $v = -1\text{V}$

The autocorrelation (52) is represented in Fig. 7. As a product of two mean values considered at different moments, the autocorrelation function brings no supplementary information, so its calculation is useless. Moreover, $\mathrm{R_y}(t_1, t_2)$ is not so easy to interpret as the output signal shown in Fig.6.

One can observe from this example, that the transient p.d.f., mean value, correlation and covariance functions offer a global description of all possible transients of the state variables or output signals in an electrical circuit. This global characterization is a valuable alternative in the case when, due to the random initial condition, one can not specify a particular deterministic transient process. On the other side, purely deterministic transients can be regarded as degenerated random processes. In such cases, the "mean value" of the degenerated random process, actually the deterministic signals of the circuit, offer a complete description of the transient state. The autocovariance function equals zero while the autocorrelation, containing only redundant information, is useless. However, the possibility of unified treatment of random and deterministic transients brings new insight in transient phenomena characterization.

4.2 Electrical circuit with random input voltage and deterministic initial condition

Once again we consider the simple system represented in Fig.1. This time, the initial condition, $y(0) = z(0) = 0$, is deterministic, but the input signal is a non-centered white-noise with mean value and autocovariance given by:

$$m_x(t) = U\,; \quad C_x(t_1, t_2) = N \cdot \delta(t_1 - t_1). \quad (53)$$

In this case, the state transition matrix is simply

$$\mathbf{\Phi}(t) = \mathbf{e}^{\mathbf{A}t} = e^{-at} = \exp(-at)\,. \tag{54}$$

Using the relations (20), (21), (19) and (29), we obtain successively:

$$m_z^F(t) = \int_{-\infty}^{t} \exp[-a(t-p)] \cdot a \cdot U \cdot dp = U\,;$$

$$m_z^{IC}(t) = \exp(-at)\cdot[0-U] = -U\cdot\exp(-at)\,;$$

$$m_y(t) = m_z(t) = U[1-\exp(-at)]\,. \tag{55}$$

With $C_z(0,0) = 0$ and relations (23), (24), (22) and (30) it results:

$$C_z^F(t_1,t_2) = \frac{N\cdot a}{2}\cdot\exp(-a|t_2-t_1|)\,;$$

$$C_z^{IC}(t_1,t_2) = -\frac{N\cdot a}{2}\cdot\exp[-a|t_2-t_1|]\cdot$$
$$\cdot\exp[-a(t_1+t_2)];$$

$$C_y(t_1,t_2) = C_z(t_1,t_2) = \frac{N\cdot a}{2}\cdot\exp(-a|t_2-t_1|)\cdot$$
$$\cdot\{1-\exp[-a(t_1+t_2)\}. \tag{56}$$

The transient autocovariance function (56) is represented in Fig.8 for $N = 0.04$. The transient behavior of the function is obvious. With increasing time, the output noise does not disappear and becomes stationary.

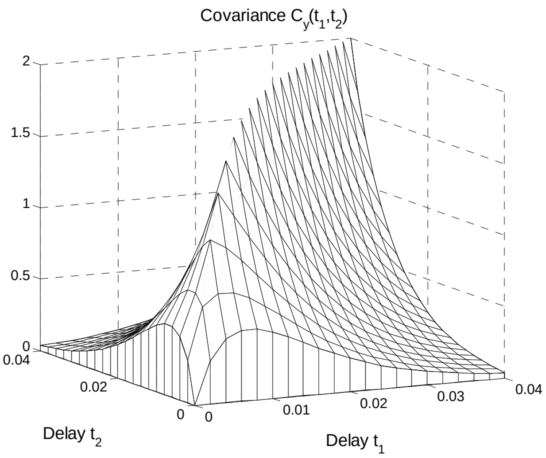

Fig.8 The transient autocovariance function, in the case of white-noise input

To determine the transient autocorrelation, one can easily utilize the relations (28), (55) and (56):

$$R_y(t_1,t_2) = R_z(t_1,t_2) = U^2\cdot\{1-\exp(-a\cdot t_1) -$$
$$-\exp(-a\cdot t_2) + \exp[-a\cdot(t_1+t_2)]\} +$$
$$+\frac{N\cdot a}{2}\cdot\exp(-a\cdot|t_2-t_1|)\cdot$$
$$\{1-\exp[-a\cdot(t_1+t_2)]\}. \tag{57}$$

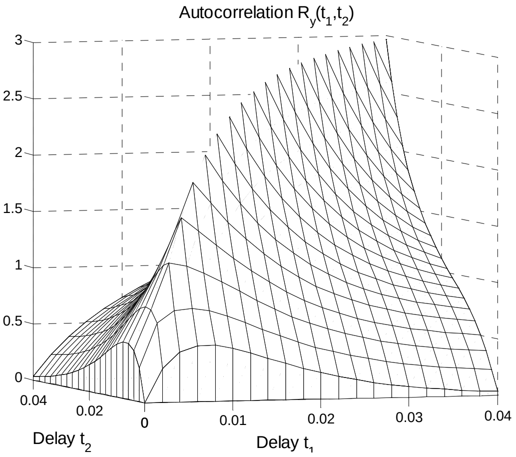

Fig.9 The transient autocorrelation function in the case of white-noise input

The function $R_Y(t_1,t_2)$, is represented in Fig.9 for $U = 1$ and $N = 0.04$. It has the same shape as $C_Y(t_1,t_2)$, but includes also the effect of the non-zero mean value. $R_Y(t_1,t_2)$ and $C_Y(t_1,t_2)$ clearly show that, with increasing time, the output voltage approaches a stationary correlated not centered signal. The stationary autocorrelation and autocovariance can be obtained from (56) and (57) for $t_1 \to \infty$, $t_2 \to \infty$ with finite $t_2 - t_1 = \tau$:

$$R_y(\tau) = U^2 + \frac{N\cdot a}{2}\cdot\exp(-a|\tau|)\,; \tag{58}$$

$$C_y(\tau) = \frac{N\cdot a}{2}\cdot\exp(-a|\tau|)\,. \tag{59}$$

These functions are represented in Fig.10. They are different only due to the mean value $U = 1\text{V}$ of the stationary output signal.

Particular case

Within this example one can obtain a pure deterministic transient process diminishing the noise component of the input signal. Thus, the deterministic step function $x(t) = U$ can be seen as the mean value of a degenerated random signal obtained for $N \to 0$. According to (56), if $N = 0$, $C_y(t_1,t_2) = 0$, denoting a deterministic process. The autocorrelation (57) becomes

$$R_y(t_1,t_2) = U^2\cdot\{1-\exp(-a\cdot t_1)-\exp(-a\cdot t_2) +$$
$$+\exp[-a(t_1+t_2)]\} = m_y(t_1)\cdot m_y(t_2) \tag{60}$$

where, according to (50) with $\nu = 0$ and $U \neq 1$, the “mean value” of the degenerated output random process has the expression

$$m_y(t) = U\cdot[1-\exp(-at)]\,. \tag{61}$$

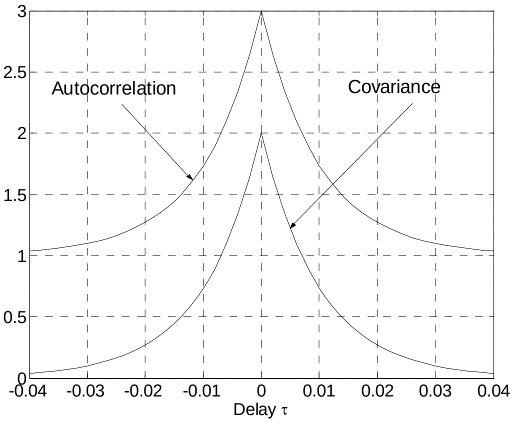

Fig.10 Stationary autocorrelation and autocovariance functions, $R_Y(\tau)$ and $C_Y(\tau); -40ms \le \tau \le 40ms$

Once again, we see that the deterministic transient is complete described by the "mean value" (61), i.e. by the changing capacitor voltage. The output autocovariance equals zero while the output autocorrelation contains only redundant information. However, from a theoretical point of view is important to realize the possibility of a unified treatment of deterministic and random transients, so bringing new insight in transient phenomena characterization.

5 Conclusion

This paper presents continuity relations for probability densities, mean values, correlation and covariance functions of state variables in electrical circuits. The introduced relations are mathematical generalizations of the well known initial condition continuity relations from the deterministic case.

New formulas for initial condition (natural) and forced components of the most important statistical moments (mean value, autocovariance and autocorrelation functions) are also presented. Two detailed computation examples, completed with graphical representation of relevant functions illustrate the use of the generalized continuity relations and of the natural and forced components of the statistical moments, for a global random transients characterization.

Deterministic transients can be regarded as particular, degenerated random transients. On this basis one can develop a unified analysis approach of deterministic and random transients in electrical circuits. This unified framework brings new insight in transient phenomena characterization and is certainly an advantage not only in teaching activities related to transient analysis.

Appendix A

The continuity of statistical moments is a direct consequence of the continuity of the probability densities. For example, one can write

$$\int_{-\infty}^{\infty} z_i \cdot p^+(z_i) \cdot dz_i = \int_{-\infty}^{\infty} z_i \cdot p^-(z_i) \cdot dz_i = = \int_{-\infty}^{\infty} z_i \cdot p(z_i) \cdot dz_i \quad \text{(A1)}$$

or, [1]

$$E\{Z_i(0^+)\} = E\{Z_i(0^-)\} = = E\{Z_i(0)\}; \quad i = 1, 2, ..., n. \quad \text{(A2)}$$

From (A2) the continuity of the mean values vector follows:

$$m_Z(+0) = m_Z(-0) = m_Z(0). \quad \text{(A3)}$$

Using the continuity of joint densities of any two variables from the state vector,

$$p^+(z_i, z_j) = p^-(z_i, z_j) = p(z_i, z_j) \quad \text{(A4)}$$
$$i, j = 1, 2, ..., n$$

we conclude the equality of second order expected values

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_i z_j \cdot p^+(z_i, z_j)\, dz_i dz_j =$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_i z_j \cdot p^-(z_i, z_j)\, dz_i dz_j = \quad \text{(A5)}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_i z_j \cdot p(z_i, z_j)\, dz_i dz_j .$$

and, finally, the continuity of the correlation matrix:

$$R_Z(+0,+0) = R_Z(-0,-0) = R_Z(0,0). \quad \text{(A6)}$$

Using the continuity of joint p.d.f., the continuity of the covariance matrix can also be put into evidence. However, the continuity of the covariance can be proved from the continuity of the correlation and the mean functions. Actually, taking (A3) and (A6) into account and the following equalities for $t_1 = t_2 = 0$,

$$\begin{aligned} &c_{ij}(t_1, t_2) = M\{\lfloor Z_{Ci}(t_1)\rfloor \cdot \lfloor Z_{Cj}(t_2)\rfloor\} = \\ &M\{Z_i(t_1) \cdot Z_j(t_2)\} - \\ &- M\{Z_i(t_1)\} \cdot M\{Z_j(t_2)\} = \\ &r_{ij}(t_1, t_2) - m_i(t_1) \cdot m_j(t_2); \quad i, j = 1, 2, ..., n. \end{aligned} \quad \text{(A7)}$$

the continuity of the autocovariance matrix follows:

$$C_Z(+0,+0) = C_Z(-0,-0) = C_Z(0,0). \quad \text{(A8)}$$

Appendix B

The general solution of the state equation (7) can be written as follows:

$$Z(t) = \Phi(t)\cdot Z(0) - \Phi(t)\int_{-\infty}^{0}\Phi^{-1}(u)\cdot B\cdot X(u)\cdot du +$$

$$+\Phi(t)\int_{-\infty}^{t}\Phi^{-1}(u)\cdot B\cdot X(u)\cdot du \qquad \text{(B1)}$$

Taking into account that $\Phi(t)\cdot\Phi^{-1}(u) = \Phi(t-u)$, $\Phi(0)=1$, $m_z^F(t) = \int_{-\infty}^{t}\Phi(t-u)\cdot B\cdot m_x(u)\cdot du$ and $\int_{-\infty}^{0}\Phi^{-1}(u)\cdot B\cdot m_x(u)\cdot du = m_z^F(0)$, we consider the expected value from (B1) and the relations (19), (20) and (21) are evident.

For simplicity, let us consider input signals of centered white-noise type:

$$m_x = 0;\; C_x(t_1,t_2) = N\cdot\delta(t_2 - t_1). \qquad \text{(B2)}$$

According to the definition of autocovariance function and using (B1), one can write

$$C_z(t_1,t_2) = E\{[Z(t_1) - E\{Z(t_1)\}]\cdot[Z(t_2) -$$
$$- E\{Z(t_2)\}]^T\} = E\{[\Phi(t_1)\cdot Z(0) -$$
$$-\Phi(t_1)\cdot\int_{-\infty}^{0}\Phi^{-1}(u)\cdot B\cdot X_0(u)\cdot du +$$
$$+\Phi(t_1)\cdot\int_{-\infty}^{t_1}\Phi^{-1}(u)\cdot B\cdot X_0(u)\cdot du - \Phi(t_1)\cdot m_z(0)]\cdot$$
$$\cdot[\Phi(t_2)\cdot Z(0) - \Phi(t_2)\cdot\int_{-\infty}^{0}\Phi^{-1}(v)\cdot B\cdot X_0(v)\cdot dv +$$
$$+\Phi(t_2)\cdot\int_{-\infty}^{t_2}\Phi^{-1}(v)\cdot B\cdot X_0(v)\cdot dv -$$
$$-\Phi(t_2)\cdot m_z(0)]^T \qquad \text{(B3)}$$

If in (B3) we consider the statistical independence of the random variables

$Z(0)$ and $\int_{-\infty}^{t}\Phi(t-u)\cdot B\cdot X_0(u)\cdot du$ as well as the relation [9]

$$E\{\int_{t_0}^{t_1}\int_{t_0}^{t_2}\Phi^{-1}(u)\cdot B\cdot X_0(u)\cdot du\cdot dv^T\cdot X_0^T(v)\cdot B^T\cdot$$
$$[\Phi^{-1}(v)]^T\} = \int_{t_0}^{\min(t_1,t_2)}\Phi(u)\cdot B\cdot N\cdot B^T\cdot[\Phi^{-1}(u)]^T\cdot du$$

the decomposition (22) of the autocovariance, the forced component (23) and the initial condition component (24) result by identification.

The relations (25), (26) and (27) for the autocorrelation function can be obtained in a similar way.

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